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On Linear Relations Generated by Nonnegative Operator Function and Degenerate Elliptic Differential-Operator Expression

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In terms of boundary values, we describe a spectrum of the linear relations indicated in the paper title. We study the invertible restrictions of maximal relation and show that the operators inverse to these restrictions are integral. The criterion of holomorphicity of the family of these operators is determined. Using the results obtained, we show that the minimal relation is symmetric in Hilbert space and describe all generalized resolvents of this relation.

Key words: linear relation, symmetric relation, spectrum, generalized resolvent, holomorphic operator function, differential elliptic-type expression, Green function, Banach space, Hilbert space.

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1. Introduction

In the present paper, we study the linear relations generated by a weight nonnegative operator function and a differential expression with variable unbounded positively definite operator coefficient degenerating on one of the ends of the interval. For the case when there is no operator weight, the spaces of boundary values (SBV) for maximal operator generated by this differential operator expression were constructed in [1–5]. The SBV allows to describe various classes of restrictions of maximal operator. (The results of papers [1–3] can be found in monograph [6].)

Differential expressions with operator weight generate linear relations that, in general, are not operators. In the present paper we construct the SBV for a maximal relation. We study various restrictions of maximal relation and describe the spectrum of these restrictions by using SBV. We prove that if the relation $(L(\lambda)-\lambda E)^{-1}$ is a bounded everywhere defined operator, then it is an integral operator. In this case we determine the criterion of holomorphicity for the operator function $\lambda \to (L(\lambda)-\lambda E)^{-1}$ (here $L(\lambda)$ is a restriction of maximal relation, $\lambda \in \mathbb{C}$, E is the identity operator). To simplify the proofs the main theorems are proved with abstract spaces of boundary values being used. A description of the generalized resolvents of minimal relation is based on the obtained results. Notice that the formula of generalized resolvents of minimal relation generated by nonnegative operator function and differential expression with bounded operator coefficients was obtained in [7, 8]. Our formula differs from that given in [7, 8], because we consider a differential elliptic-type expression with unbounded operator coefficient.

One of the difficulties in the studying of operators and relations generated by differential operator expression of elliptic-type is the constructing of the Green function in one of the boundary value problems. We construct this function in Sect. 3.

2. Main Assumptions, Notation

Let H be a separable Hilbert space with the scalar product $(\cdot,\cdot)$ and the norm $\|\cdot\|$. On a compact interval $[0,b]$, we consider the differential expression

$$l[y] = -y'' + t^{\alpha}\mathcal{A}_1(t)y,$$

where $\alpha \geqslant 0$, and the operator function $\mathcal{A}_1(t)$ satisfies the following conditions: 1) $\mathcal{A}_1(t)$ is a positively definite selfadjoint operator in H for any fixed $t \in [0,b]$; 2) the operators $\mathcal{A}_1(t)$ have the constant domain $\mathcal{D}(\mathcal{A}_1(t)) = \mathcal{D}(\mathcal{A}_1)$; 3) $\mathcal{A}_1(t)x$ is a function strongly continuously differentiable on $[0,b]$ for any $x \in \mathcal{D}(\mathcal{A}_1)$.

We fix a point $t_0 \in [0,b]$. Let $\{\hat{H}_\tau\}$, $-1 \leqslant \tau \leqslant 1$, be a Hilbert scale of the spaces [6, Ch. 2; 9, Ch. 1] generated by $\mathcal{A}_1(t_0)$. Notice that the definition of the Hilbert scale implies $\hat{H}_0 = H$. It follows from the properties of $\mathcal{A}_1(t)$ that the scale $\{\hat{H}_\tau\}$ does not depend on the choice of point $t_0 \in [0,b]$ in the sense below. If $t_0' \in [0,b]$ is any other point and $\{\hat{H}_\tau'\}$ is a scale of the spaces generated by operator $\mathcal{A}_1(t_0')$, then the sets $\hat{H}_\tau$ and $\hat{H}_\tau'$ coincide and their norms are equivalent. For fixed $t \in [0,b]$, the operator $\mathcal{A}_1(t)$ is a continuous one-to-one mapping of $\hat{H}_{+1}$ onto H. Then its adjoint operator $\mathcal{A}_1^+(t)$ is a continuous one-to-one mapping of H onto $\hat{H}_{-1}$, and $\mathcal{A}_1^+(t)$ is an extension of $\mathcal{A}_1(t)$ [6, Ch. 2; 9, Ch. 1]. Further, we denote $l^+[y] = -y'' + t^{\alpha}\mathcal{A}_1^+(t)y$.

Let $A(t)$ be a function strongly measurable on $[0,b]$ whose values are bounded selfadjoint operators in H. Suppose the norm $\|A(t)\|$ is integrable on $[0,b]$. Moreover, we assume that the inequality

$$(A(t)x,x) \geqslant 0 \tag{1}$$

holds for any $x \in H$ and for almost all $t \in [0, b]$. Generally, it is assumed that a set of points $t \in [0, b]$ satisfying (1) depends on x.

We claim that there exists a set $\mathcal{I}_0 \subset [0, b]$ of measure zero such that the set $\mathcal{I} = [0, b] \setminus \mathcal{I}_0$ has the following property: for all $t \in \mathcal{I}$ and for all $x \in H$ inequality (1) holds. Indeed, due to separability of the space H there exists a countable set $\{x_n\}$ $(n \in \mathbb{N})$ dense in H. Let $\mathcal{I}_n$ be a set of $t \in [0, b]$ such that inequality (1) holds, where x is replaced by x_n. We denote $\mathcal{I}_{0,n} = [0, b] \setminus \mathcal{I}_n$, $\mathcal{I}_0 = \bigcup_n \mathcal{I}_{0,n}$. Then the measure of the set $\mathcal{I}_0$ is equal to zero, and for all $t \in \mathcal{I} = [0, b] \setminus \mathcal{I}_0$ and for all $n \in \mathbb{N}$ inequality (1) holds, where x is replaced by x_n. Since the operator $A(t)$ is bounded and the set $\{x_n\}$ is dense in H, we obtain the desired statement. So, inequality (1) holds on some set $\mathcal{I}$ such that $\mathcal{I}$ does not depend on $x \in H$, and the measure of the set $[0, b] \setminus \mathcal{I}$ is equal to zero.

Since the norm $\|A(t)\|$ is integrable on $[0, b]$, we have $\left\|A^{1/p}(t)\right\| \in L_p(0, b)$. On the set of functions continuous on the interval $[0, b]$ and ranging in H, we introduce the norm

$$\|y\|_p = \left(\int_0^b \left\| A^{1/p}(t) y(t) \right\|^p dt \right)^{1/p}, \quad 1 \leqslant p < \infty.$$

Identifying the functions y such that $\|y\|_p = 0$ with zero, then performing the completion, we obtain a Banach space denoted by $\mathrm{B} = L_p(H, A(t); 0, b)$. The elements of B are the classes of functions identified with each other in the norm $\|\cdot\|_p$. In what follows, $\tilde{y}$ denotes a class of functions with representative y. To avoid a complicated terminology we say that the function y belongs to B.

Let $G_0(t)$ be a set of elements $x \in H$ such that $A(t)x = 0$, $H(t) = H \ominus G_0(t)$, and $A_0(t)$ be a restriction of $A(t)$ to $H(t)$. Then the operator $A_0(t)$ acting in $H(t)$ has the inverse $A_0^{-1}(t)$ (which, in general, is unbounded). By $\{H_\xi(t)\}$, $-\infty < \xi < \infty$, we denote a Hilbert scale of spaces generated by operator $A_0^{-1}(t)$. As known from [6, Ch. 2; 9, Ch. 1], the operator $A_0(t)$ can be extended to the operator $\tilde{A}_0(t) = \tilde{A}_{0,\alpha}(t)$ that continuously and bijectively maps $H_{-\alpha}(t)$ onto $H_{1-\alpha}(t)$, $0 \leqslant \alpha \leqslant 1$. Further, in $\tilde{A}_{0,\alpha}(t)$ we will omit the symbol α characterizing the domain of operator $\tilde{A}_{0,\alpha}(t)$). By $\tilde{A}(t)$ we denote the operator that is defined on $H_{-\alpha}(t) \oplus G_0(t)$ and is equal to $\tilde{A}_0(t)$ on $H_{-\alpha}(t)$ and to zero on $G_0(t)$. Obviously, the operator $\tilde{A}(t)$ is an extension of $A(t)$.

The description of the space B for $p > 1$ was given in [8] and the case of $p = 2$ was considered in [10]. The space B consists of elements (i.e., function classes) with representatives of the form $\tilde{A}_0^{-1/p}(t) P(t) h(t)$, where $P(t)$ is an orthogonal projection of H onto $H(t)$, $h(t) \in L_p(H; 0, b)$. Without changing considerably the proof given in [8], we obtain the above statements for $p = 1$. The space $L_1(H, A(t); 0, b)$ is used only when constructing the Green function in Sect. 3.

For $p>1$, the dual space of B is the space $\mathrm{B}^* = L_q(H, A(t); 0, b)(p^{-1}+q^{-1}=1)$ (see [8]). A sesquilinear form (i.e., the form that is linear in the first argument and antilinear in the second one) determined by duality between B and B^* is denoted by $\langle\cdot,\cdot\rangle$, and the action of the functional $\tilde{g} \in \mathrm{B}^*$ on the element $\tilde{f} \in \mathrm{B}$ is given by the equality

$$\langle\tilde{f},\tilde{g}\rangle = \int\limits_0^b (\tilde{A}(t)f(t), g(t))dt,$$

which is independent of the choice of representatives $f \in \tilde{f}$, $g \in \tilde{g}$.

2. The Green Function

In this section, we construct the Green function $G(t,s,\lambda)$ of the Neumann problem for the expression $l^+[y] - \lambda A(t)y$. The construction is based on the Green function $G(t,s)$ (see [5]).

By [5], the operator function $G(t,s)$ is called the Green function of the Neumann problem for the expression $l[y]$, i.e., of the problem

$$l[y] = -y'' + t^\alpha \mathcal{A}_1(t)y = g(t), \tag{2}$$

$$y'(0) = y'(b) = 0, \tag{3}$$

if the integral $y(t) = \int_0^b G(t,s)g(s)ds$ is a strong solution (see [11]) of equation (2) and it satisfies conditions (3) for any strongly continuous function $g(t)$ in the space $\hat{H}_{+1}$. By [11], the function $y(t)$ ($t \in [0,b]$) is called a strong solution of equation (2) if $y(t) \in \mathcal{D}(\mathcal{A}_1)$ for any t, and $y(t)$ is twice differentiable in H, and $y(t)$ satisfies (2). It was proved in [5] that for sufficiently large k there exists a Green function $G_k(t,s)$ of the Neumann problem for the expression

$$l_k[y] = -y'' + t^\alpha \mathcal{A}_1(t)y + k^2 t^\alpha y.$$

Lemma 1. *There exists a Green function of problem* (2), (3).

P r o o f. By $\mathcal{L}'$ ($\mathcal{L}'_k$) denote an operator generated by the expression $l[y]$ ($l_k[y]$) on the functions $y(t)$ that are strongly continuous in $\hat{H}_{+1}$ on $[0,b]$, twice differentiable in H on $[0,b]$ and they satisfy boundary conditions of the Neumann problem (3). Let $\mathcal{L}$ ($\mathcal{L}_k$) be a closure of $\mathcal{L}'$ as well as of $\mathcal{L}'_k$ in the space $L_2(H;0,b)$. It was proved in [5] that for sufficiently large k the operator $\mathcal{L}_k^{-1}$ exists, it is continuous in $L_2(H;0,b)$ and is an integral operator with the kernel $G_k(t,s)$. Since $\mathcal{L}$ differs from $\mathcal{L}_k$ by a bounded selfadjoint operator and $\mathcal{L}_k$ is selfadjoint, we see that $\mathcal{L}$ is also selfadjoint. Obviously, $\mathcal{L}$ is nonnegative. We claim that the operator $\mathcal{L}^{-1}$ exists and it is bounded in $L_2(H;0,b)$.

Indeed, let $\{y_n\}$ be a sequence of functions y_n from the domain of $\mathcal{L}'$ such that

$$(\mathcal{L}'y_n, y_n)_{L_2(H;0,b)} \to 0$$

as $n \to \infty$ and $\|y_n\|_{L_2(H;0,b)} = 1$. Therefore,

$$\begin{aligned}(\mathcal{L}'y_n, y_n)_{L_2(H;0,b)} &= \int_0^b \left\|y_n'(t)\right\|^2 dt + \int_0^b t^\alpha (\mathcal{A}_1(t)y_n(t), y_n(t))dt \\ &\geqslant \int_0^b \left\|y_n'(t)\right\|^2 dt + c_1 \int_0^b t^\alpha (y_n(t), y_n(t))dt \to 0 \quad \text{as} \;\; n \to \infty,\end{aligned}$$

where $c_1 > 0$ does not depend on t. (Here and further, the symbols $c_1, c_2, \dots$ denote positive constants that are different in various inequalities.) Hence,

$$\int_0^b \left\|y_n'(t)\right\|^2 dt \to 0$$

and

$$\int_0^b t^\alpha (y_n(t), y_n(t))dt = (\alpha+1)^{-1} b^{\alpha+1} \left\|y_n(b)\right\|^2 - \int_0^b t^{\alpha+1} \mathrm{Re}(y_n'(t), y_n(t))dt \to 0$$

as $n \to \infty$. (Here the formula of integration by parts is used.) This yields that $\|y_n(b)\| \to 0$. Therefore, as $n \to \infty$,

$$y_n(t) = y_n(b) - \int_t^b y_n'(t)dt \to 0$$

uniformly on $[0, b]$. The above contradicts the equality $\|y_n\|_{L_2(H;0,b)} = 1$. Thus the existence and boundedness of the operator $\mathcal{L}^{-1}$ are proved. Consequently, the operator $\mathcal{L}$ is positively definite in $L_2(H; 0, b)$.

We denote $\mathbf{G}_k = \mathcal{L}_k^{-1}$. As noted above, $\mathbf{G}_k$ is an integral operator with the kernel $G_k(t, s)$. By $\mathcal{T}$ denote the operator of multiplication on t^α in $L_2(H; 0, b)$. Suppose $\mathbf{G}_\mathcal{T} = \mathcal{T}^{1/2}\mathbf{G}_k\mathcal{T}^{1/2}$. The operator $\mathbf{G}_\mathcal{T}$ is selfadjoint. Moreover, $\mathbf{G}_\mathcal{T}$ is an integral operator with the kernel $t^{\alpha/2}G_k(t, s)s^{\alpha/2}$. We will prove that the operator $k^2\mathbf{G}_\mathcal{T} - E$ has an everywhere defined inverse operator in the space $L_2(H; 0, b)$.

Let $v_n \in L_2(H; 0, b)$, where $n \in \mathbb{N}$. We denote $\mathcal{T}^{1/2}v_n = u_n$, $\mathbf{G}_k u_n = w_n$. Then $\mathcal{L}_k w_n = u_n$ and $\mathcal{T}^{-1/2}\mathcal{L}_k w_n = v_n$. It follows from the equality $\mathcal{L}_k =$

$\mathcal{L} + k^2\mathcal{T}$ that $\mathcal{L}w_n$ belongs to the domain of operator $\mathcal{T}^{-1/2}$, and $\mathcal{T}^{-1/2}\mathcal{L}w_n = v_n - k^2\mathcal{T}^{1/2}w_n$. Hence, by direct calculation we obtain

$$\begin{aligned}(v_n, (E - k^2\mathbf{G}_\mathcal{T})v_n) &= (v_n, v_n) - k^2(\mathcal{T}^{1/2}\mathbf{G}_k\mathcal{T}^{1/2}v_n, v_n)\\ &= (\mathcal{T}^{-1/2}(\mathcal{L} + k^2\mathcal{T})w_n, \mathcal{T}^{-1/2}(\mathcal{L} + k^2\mathcal{T})w_n) - k^2(w_n, (\mathcal{L} + k^2\mathcal{T})w_n)\\ &= k^2(\mathcal{L}w_n, w_n) + (\mathcal{T}^{-1/2}\mathcal{L}w_n, \mathcal{T}^{-1/2}\mathcal{L}w_n)\\ &= k^2(\mathcal{L}w_n, w_n) + (v_n - k^2\mathcal{T}^{1/2}w_n, v_n - k^2\mathcal{T}^{1/2}w_n)\end{aligned}$$

(in this equality, $(\cdot,\cdot)$ is a scalar product in $L_2(H;0,b)$). Suppose $(v_n, (E - k^2\mathbf{G}_\mathcal{T})v_n) \to 0$ as $n \to \infty$. It follows from the last equalities that

$$(\mathcal{L}w_n, w_n) \to 0, \quad (v_n - \mathcal{T}^{1/2}w_n, v_n - \mathcal{T}^{1/2}w_n) \to 0.$$

Since $\mathcal{L}$ is a positive definite operator, we have $w_n \to 0$ in $L_2(H;0,b))$. Therefore, $v_n \to 0$ in $L_2(H;0,b)$ as $n \to \infty$. Thus the operator $(k^2\mathbf{G}_\mathcal{T} - E)^{-1}$ exists and it is everywhere defined.

In the space $L_2(H;0,b)$, we consider the integral equation

$$\mathcal{K}(t,s)x = t^{\alpha/2}G_k(t,s)x + k^2\int_0^b t^{\alpha/2}G_k(t,\tau)\tau^{\alpha/2}\mathcal{K}(\tau,s)x\,d\tau \tag{4}$$

with the unknown function $\mathcal{K}(t,s)x$, where $x \in H$. Since the operator $k^2\mathbf{G}_\mathcal{T} - E$ has the everywhere defined inverse operator, we see that the equation (4) is solvable. In [5], it was proved that

$$\|G_k(t,s)\| \leqslant c_1, \tag{5}$$

where c_1 does not depend on s, t. Using (4), (5), we obtain

$$\int_0^b \|\mathcal{K}(t,s)x\|^2\,dt \leqslant \left\|(k^2\mathbf{G}_\mathcal{T} - E)^{-1}\right\|^2 \int_0^b \left\|t^{\alpha/2}G_k(t,s)x\right\|^2 dt \leqslant c_2\,\|x\|^2\,, \tag{6}$$

where c_2 does not depend on s. Using (4)–(6), we get

$$\|\mathcal{K}(t,s)\| \leqslant c_3, \tag{7}$$

where c_3 does not depend on t, s.

We define the function $G(t,s)$ by the formula

$$G(t,s)x = G_k(t,s)x + k^2\int_0^b G_k(t,\tau)\tau^{\alpha/2}\mathcal{K}(\tau,s)x\,d\tau. \tag{8}$$

It follows from (4), (8) that $t^{1/2}G(t,s) = \mathcal{K}(t,s)$. Hence, taking into account (8), we obtain

$$G(t,s)x = G_k(t,s)x + k^2 \int_0^b G_k(t,\tau)\tau^\alpha G(\tau,s)x d\tau. \tag{9}$$

The function $G(t,s)$ satisfies the boundary conditions

$$G_t'(0,s) = G_t'(b,s) = 0, \quad s \neq 0, s \neq b;$$

$$G_t'(0,0) = -E, \;\; G_t'(b,0) = G_t'(0,b) = 0, \;\; G_t'(b,b) = E. \tag{10}$$

These equalities follow from (9) and from the fact that the function $G_k(t,s)$ satisfies the same conditions (see [5]).

Formulas (5), (7), (8) imply

$$\|G(t,s)\| \leqslant c_1, \tag{11}$$

where c_1 does not depend on t, s. In [5], the operator $G_k(t,s)$ is proved to extend to $\hat{G}_k(t,s)$ in $\hat{H}_{-1}$ such that it is a continuous mapping of each space $\hat{H}_\tau$, $-1 \leqslant \tau \leqslant 1$, of the scale $\{\hat{H}_\tau\}$ into itself. The operator function $\hat{G}_k(t,s)$ is uniformly bounded on $[0,b] \times [0,b]$ with respect to the norm in each space $\hat{H}_\tau$. By the construction, the operator function $G(t,s)$ possesses the same properties.

Suppose the function $g(t)$ is strongly continuous in $\hat{H}_{+1}$. We denote

$$z(t) = \int_0^b G(t,s)g(s)ds, \quad z_k(t) = \int_0^b G_k(t,s)g(s)ds.$$

It follows from (9), (10) that $z(t)$ takes the values in $D(\mathcal{A}_1)$, it is twice strongly differentiable in H, and $z'(0) = z'(b) = 0$. Since the function $z_k(t)$ is a strong solution of the equation $l_k[y] = g$, we see that (9) implies the equality $l_k[z] = l_k[z_k] + k^2 t^\alpha z = g + k^2 t^\alpha z$. Hence $l[z] = g$. Lemma 1 is proved.

We notice some more properties of the function $G(t,s)$. Let $\mathbf{G}$ be an operator defined by the formula $\mathbf{G}v = \int_0^b G(t,s)v(s)ds$ in $L_2(H;0,b)$. Then $\mathcal{L}^{-1} = \mathbf{G}$. Since the operator $\mathcal{L}$ is selfadjoint, we have $G^*(t,s) = G(s,t)$. The function $G(t,s)$ is strongly continuous with respect to t for each fixed $s \in [0,b]$ what follows from (7), (8) and the fact that the function $G_k(t,s)$ possesses the same property (see [4, 5]).

Lemma 2. *Suppose $h(t) \in L_1(H;0,b)$. Then the function*

$$y(t) = \int_0^b G(t,s)h(s)ds \tag{12}$$

has the following properties:

(a) y is continuous on $[0,b]$ in the space H and strongly differentiable on $[0,b]$ in the space $\hat{H}_{-1}$;

(b) y' is absolutely continuous in the space $\hat{H}_{-1}$;

(c) y satisfies the equation

$$l^+[y] = -y'' + t^\alpha \mathcal{A}_1^+(t)y = h(t) \tag{13}$$

and boundary conditions (3).

P r o o f. We take a sequence of functions $h_n(t)$ such that the sequence $\{h_n(t)\}$ converges to $h(t)$ in $L_1(H;0,b)$ as $n \to \infty$ and the functions $h_n(t)$ are strongly continuous in the space $\hat{H}_{+1}$. Then, by Lemma 1, the functions $y_n(t) = \int_0^b G(t,s)h_n(s)ds$ are strong solutions of the problem (13), (3), where $h(t)$ is replaced by $h_n(t)$. Thus the equality

$$-y_n''(t) + t^\alpha \mathcal{A}_1^+(t)y_n(t) = h_n(t) \tag{14}$$

holds. From (11), (12), it follows that the sequence $\{y_n(t)\}$ converges to $y(t)$ uniformly in H. Therefore the sequence $\{\mathcal{A}_1^+(t)y_n(t)\}$ uniformly converges to $\mathcal{A}_1^+(t)y(t)$ in the space $\hat{H}_{-1}$. Then (14) implies the convergence of the sequence $\{y_n''(t)\}$ in $L_1(\hat{H}_{-1};0,b)$. From this and (3) it follows that $\{y_n'(t)\}$ converges uniformly in $\hat{H}_{-1}$. Now all assertions of Lemma 2 are obtained from the above in a standard way. The proof of Lemma 2 is complete.

Lemma 3. *For any function $h(t) \in L_1(H;0,b)$ and any elements $x_1, x_2 \in H$ there exists a unique solution $y(t)$ of equation* (13) *such that $y(t)$ has the properties (a), (b) of Lemma 2 and satisfies the boundary conditions*

$$y'(0) = -x_1, \quad y'(b) = x_2. \tag{15}$$

This solution has the form $y(t) = G(t,0)x_1 + G(t,b)x_2 + \int_0^b G(t,s)h(s)ds$.

P r o o f. First, we notice that the invertibility of operator $\mathcal{L}$ yields the uniqueness of solution. Further, as follows from [5], the function $z_k(t) = G_k(t,0)x_1 + G_k(t,b)x_2$ has the properties (a), (b) and satisfies the equation $l_k^+[y] = 0$ and conditions (15). Hence, taking into account (9), (10), we obtain that the function $z(t) = G(t,0)x_1 + G(t,b)x_2$ has the properties (a), (b), it is a solution of the equation $l^+[y] = 0$ and it satisfies the conditions (15). Now, applying Lemma 2, we complete the proof of Lemma 3.

To construct the Green function $G(t,s,\lambda)$ we consider the equation

$$l^+[y] - \lambda A(t)y = -y'' + t^\alpha \mathcal{A}_1^+(t)y(t) - \lambda A(t)y(t) = \tilde{A}(t)f(t). \tag{16}$$

Let $G(t,s,\lambda)$ be an operator function whose values are bounded operators in H. We say that $G(t,s,\lambda)$ is the Green function of problem (16), (3) if for any function $f \in L_1(H, A(t); 0, b)$ the integral

$$z_1(t) = \int_0^b G(t,s,\lambda)\tilde{A}(s)f(s)ds$$

possesses the properties (a), (b) of Lemma 2 and satisfies equation (16) and the boundary conditions (3).

As shown in the proof of Lemma 1, the operator $\mathcal{L}$ is positively definite in $L_2(H;0,b)$. From the equality $\mathbf{G} = \mathcal{L}^{-1}$ it follows that $\mathbf{G}$ is a positively definite operator. Consequently, the kernel $A^{1/2}(t)G(t,s)A^{1/2}(s)$ determines the bounded nonnegative operator

$$\mathbf{G}_A v = \int_0^b A^{1/2}(t)G(t,s)A^{1/2}(s)v(s)ds \quad (v \in L_2(H;0,b))$$

in the space $L_2(H;0,b)$.

By $\rho_0(G_A)$ we denote a set $\lambda \in \mathbb{C}$ such that the operator $\lambda G_A - E$ has a bounded everywhere defined inverse operator. The set $\rho_0(G_A)$ contains all nonreal numbers, the negative ones and zero. Further, we will assume that $\lambda \in \rho_0(G_A)$.

Theorem 1. *For any $\lambda \in \rho_0(G_A)$, there exists a Green function $G(t,s,\lambda)$ of problem* (16), (3).

P r o o f. We consider the integral equation

$$K(t,s,\lambda)x = A^{1/2}(t)G(t,s)x + \lambda \int_0^b A^{1/2}(t)G(t,\tau)A^{1/2}(\tau)K(\tau,s,\lambda)xd\tau \qquad (17)$$

with the unknown function $K(t,s,\lambda)x$, where $x \in H$. Equation (17) can be solved in $L_2(H;0,b)$ for $\lambda \in \rho_0(G_A)$.

We introduce the function $G(t,s,\lambda)$ by the equality

$$G(t,s,\lambda)x = G(t,s)x + \lambda \int_0^b G(t,\tau)A^{1/2}(\tau)K(\tau,s,\lambda)xd\tau. \qquad (18)$$

For fixed $s \in [0,b]$, the function $A^{1/2}(t)K(t,s,\lambda)x$ $(x \in H)$ belongs to $L_1(H;0,b)$. Consequently, $G(t,s,\lambda)$ is a strongly continuous function with respect to t in the space H. It follows from (17), (18) that

$$A^{1/2}(t)G(t,s,\lambda) = K(t,s,\lambda).$$

Hence, using (18), we get

$$G(t,s,\lambda)x = G(t,s)x + \lambda \int_0^b G(t,\tau)A(\tau)G(\tau,s,\lambda)x d\tau. \tag{19}$$

Moreover, by (10), it follows that

$$G'_t(0,s,\lambda)x = G'_t(b,s,\lambda)x = 0, \quad s \neq 0, s \neq b;$$

$$G'_t(0,0,\lambda)x = -x,\ G'_t(b,0,\lambda)x = G'_t(0,b,\lambda)x = 0,\ G'_t(b,b,\lambda)x = x. \tag{20}$$

Further proof is done analogously to that of [12], where the case of $\alpha = 0$ was considered. In particular, similarly as in [12], we obtain that for any element $d_1 \in H_{-1}(s) \oplus G_0(s)$ the equality

$$G^*(s,t,\bar{\lambda})\tilde{A}(s)d_1 = G(t,s,\lambda)\tilde{A}(s)d_1$$

holds. Therefore,

$$G^*(s,t,\bar{\lambda})\tilde{A}(s)f(s) = G(t,s,\lambda)\tilde{A}(s)f(s) \tag{21}$$

for any function $f \in L_1(H, A(t); 0, b)$.

For $\lambda \in \rho_0(G_A)$, the function $G(t,s,\lambda)$ is bounded with respect to the first argument. Therefore the function $G^*(t,s,\lambda)$ has the same property. From this and from (21) there follows the equality

$$\int_0^b G(t,s,\lambda)\tilde{A}(s)f(s)ds = \int_0^b G^*(s,t,\bar{\lambda})\tilde{A}(s)f(s)ds$$

and the existence of integrals in it. Using (19), (20) and the properties of function $G(t,s)$, we complete the proof.

In [12], the Green function for the expression $l^+[y] - \lambda A(t)$ was constructed in the case of $\alpha = 0$. If $\alpha = 0$ and $\lambda = 0$, then $G(t,s,0) = G(t,s)$ coincides with the Green function constructed in [13].

By $U(t,\lambda)$, denote the operator one-row matrix $U(t,\lambda) = (U_1(t,\lambda), U_2(t,\lambda))$, where

$$U_1(t,\lambda) = G(t,0,\lambda), \quad U_2(t,\lambda) = G(t,b,\lambda). \tag{22}$$

Lemma 4. *Let $\lambda \in \rho_0(G_A)$. For any elements $x_1, x_2 \in H$ and any function $f \in L_1(H, A(t); 0, b)$ there exists a unique function y having the properties (a), (b) of Lemma 2 and satisfying equation* (16) *and boundary conditions* (15). *This function has the form*

$$y(t) = U_1(t,\lambda)x_1 + U_2(t,\lambda)x_2 + F(t), \tag{23}$$

where

$$F(t) = \int_0^b G(t,s,\lambda)\tilde{A}(s)f(s)ds = \int_0^b G^*(s,t,\bar{\lambda})\tilde{A}(s)f(s)ds. \tag{24}$$

P r o o f. It follows from Lemma 3 and equalities (19), (20), (22) that the function $y_0(t) = U_1(t,\lambda)x_1 + U_2(t,\lambda)x_2$ has the properties (a), (b) of Lemma 2, and $y_0(t)$ satisfies the boundary conditions (15) and the equation

$$-y'' + t^{\alpha}\mathcal{A}_1^+(t)y - \lambda A(t)y = 0. \tag{25}$$

Hence, taking into account Theorem 1, we obtain that (23) has all the properties indicated in the lemma. To prove that problem (25), (15) has a unique solution is to prove the uniqueness of the solution of problem (16), (15). Let the function u_0, having the properties (a), (b), be a solution of equation (25) with homogeneous conditions (3). We put $u(t) = \lambda \int_0^b G(t,s)A(s)y_0(s)ds$. Using Lemma 3, we get $u_0(t) = u(t)$. Hence,

$$A^{1/2}(t)u_0(t) = \lambda \int_0^b A^{1/2}(t)G(t,s)A(s)y_0(s)ds.$$

Since $\lambda \in \rho_0(G_A)$, we have $A^{1/2}(t)u_0(t) = 0$ for almost all $t \in [0,b]$. Therefore, $u_0(t) = u(t) = 0$ for all $t \in [0,b]$. So, the uniqueness of the solution of problem (25), (15) is established. Lemma 4 is proved.

R e m a r k 1. Suppose the function y has the properties (a), (b), and it satisfies equation (16) and boundary conditions (15), where $x_1, x_2 \in H$. Then $y'(t) \in H$ for all $t \in [0,b]$.

Indeed, the function y is a solution of nondegenerate equation on each interval $[\alpha, b]$ $(\alpha > 0)$. Consequently, $y'(t) \in H$ for all $t \in [\alpha, b]$ (see [12]). Hence, taking into account (15), we obtain the desired statement.

Lemma 5. *Suppose F is defined by equality* (24)*; then the operator $\tilde{f} \to F = F(t,\tilde{f},\lambda)$ is a continuous mapping of the space* B *into the space $C(H;0,b)$.*

P r o o f coincides with that of the analogous lemma in [12].

Corollary 1. *The operator $\tilde{f} \to \tilde{F} = \tilde{F}(t,\tilde{f},\lambda)$ is continuous in* B.

4. Maximal and Minimal Relations

In this section, the maximal and minimal relations generated by expression $l^+[y]$ and operator function $A(t)$ in the space $\mathrm{B} = L_p(H, A(t); 0, b)$ are defined and the properties of these relations are studied. Everywhere below we will assume that $p > 1$.

Terminology concerning linear relations can be found, for example, in [6, 14, 15]. The linear relation T in the Banach space $\mathcal{B}$ is understood as a linear manifold $T \subset \mathcal{B} \times \mathcal{B}$. Further the following notations are used: $\{\cdot,\cdot\}$ is an ordered pair; Ker$T$ is a set of ordered pairs of the form $\{z, 0\} \in T$; $\ker T$ is a set of elements z such that $\{z, 0\} \in T$; $\mathcal{D}(T)$ is a domain of T; $\mathcal{R}(T)$ is a range of values; $\rho(T)$ is a resolvent set of the relation T, i.e., a set of points $\lambda \in \mathbb{C}$ such that the relation $(T - \lambda E)^{-1}$ is a bounded everywhere defined operator; $\sigma_c(T)$ ($\sigma_r(T)$) is a continuous spectrum (residual spectrum) of the relation T, i.e., a set of points $\lambda \in \mathbb{C}$ such that the relation $(T - \lambda E)^{-1}$ is a densely defined and unbounded (not densely defined) operator; $\sigma_p(T)$ is the point spectrum of T, i.e., a set of points $\lambda \in \mathbb{C}$ such that the relation $(T - \lambda E)^{-1}$ is not an operator. Since all relations considered are linear, the word "linear" will often be omitted.

By D' we denote a set of functions $y(t) \in$ B satisfying the following conditions: i) y is strongly continuous on $[0, b]$ in the space H and strongly differentiable in the space $\hat{H}_{-1}$, and $y'(t) \in H$ for all $t \in [0, b]$; (ii) y' is absolutely continuous in $\hat{H}_{-1}$; iii) $l^+[y](t) \in H_{1/q}(t)$ for almost all t, and the function $\tilde{A}_0^{-1}(t)l^+[y] \in$ B ($p^{-1} + q^{-1} = 1$). To each class of functions identified with $y \in D'$ in B we assign the class of functions identified with $\tilde{A}_0^{-1}(t)l^+[y]$ in B. In general, this correspondence is not an operator as the function y may be identified with zero in B and $\tilde{A}_0^{-1}(t)l^+[y]$ may be nonzero. Thus, in the space B we obtain a linear relation L'. Denote its closure by L and call it a maximal relation. We define the minimal relation L_0 as a restriction of L to the set of elements $\tilde{y} \in$ B that have representatives $y \in D'$ with the property $y(0) = y'(0) = y(b) = y'(b) = 0$.

Let Q_0 be a set of elements $x \in H \times H$ for which the equality $A(t)U(t, \lambda)x = 0$ holds almost everywhere. Using Theorem 1, we get

$$U(t, 0)x = U(t, \lambda)x - \lambda \int_0^b G^*(s, t, \bar{\lambda})\tilde{A}(s)U(s, 0)x ds, \tag{26}$$

$$U(t, \lambda)x = U(t, 0)x + \lambda \int_0^b G^*(s, t)\tilde{A}(s)U(s, \lambda)x ds. \tag{27}$$

By (26), (27), it follows that Q_0 does not depend on λ. By Q we denote an orthogonal complement of Q_0 in $H \times H$. In Q we introduce the norm

$$\|x\|_r = \left(\int_0^b \left\| A^{1/r}(s)U(s, 0)x \right\|^r ds \right)^{1/r} \leqslant k \|x\|, \quad r > 1, \ \ x \in Q. \tag{28}$$

We denote the completion of Q with respect to the norm $\|\cdot\|_r$ by $Q_-(r)$. It follows from (26), (27) that the replacement of $U(s, 0)$ by $U(s, \lambda)$ in (28) leads to the same set $Q_-(r)$ with the equivalent norm. Let the symbol $\tilde{U}(s,\lambda)x$ ($x \in Q_-(r)$) denote a class of functions to which the sequence $\{\tilde{U}(t,\lambda)x_n\}$ ($x_n \in Q$) converges whenever $\{x_n\}$ converges to x in the space $Q_-(r)$.

We introduce the operator $V_r(\lambda): Q_-(r) \to L_r(H, A(t); 0, b)$ by the formula $V_r(\lambda)x = \tilde{U}(t,\lambda)x$. It follows from (28) that the operator $V_r(\lambda)$ is continuous, the range $\mathcal{R}(V_r(\lambda))$ is closed, and $\ker V_r(\lambda) = \{0\}$. Hence the range of the adjoint operator $V_r^*(\lambda) : L_{r_1}(H, A(t); 0, b) \to Q_-^*(r) \subset Q^* = Q$ coincides with $Q_-^*(r)$ (here $r^{-1} + r_1^{-1} = 1$). We find the form of $V_r^*(\lambda)$. For any elements $x \in Q$ and $\tilde{f} \in L_{r_1}(H, A(t); 0, b)$, we have

$$\begin{aligned}\langle \tilde{f}, V_r(\lambda)x\rangle &= \int_0^b (\tilde{A}(s)f(s), U(s,\lambda)x)ds \\ &= (\int_0^b U^*(s,\lambda)\tilde{A}(s)f(s)ds, x) = (V_r^*(\lambda)\tilde{f}, x).\end{aligned} \tag{29}$$

Here $(V_r^*(\lambda)\tilde{f}, x)$ is a scalar product of the elements $V_r^*(\lambda)\tilde{f} \in Q_-^*(r) \subset Q$ and $x \in Q$ in Q. For $x_+ \in Q_-^*(r)$, this scalar product (x_+, x) is extended by continuity to the sesquilinear form (x_+, x_-) determined by the duality between $Q_-^*(r)$ and $Q_-(r)$. Taking into account (29) and that Q can be densely embedded in $Q_-(r)$, we obtain

$$V_r^*(\lambda)\tilde{f} = \int_0^b U^*(s,\lambda)\tilde{A}(s)f(s)ds. \tag{30}$$

Further, to avoid complicated notation, we denote $Q_- = Q_-(p)$, $\tilde{Q}_+ = Q_-^*(q)$, where $p^{-1} + q^{-1} = 1$. Thus the following lemma is proved.

Lemma 6. *The operator $V_q^*(\bar{\lambda})$ is a continuous mapping of* B *onto $\tilde{Q}_+$.*

Lemma 7. *For any $\lambda \in \rho_0(G_A)$, the relation $L - \lambda E$ consists of the pairs $\{\tilde{y}, \tilde{f}\} \in \mathrm{B} \times \mathrm{B}$ such that*

$$\tilde{y} = \tilde{U}(t,\lambda)x + \tilde{F}, \tag{31}$$

where $x \in Q_-$ and $\tilde{F}$ are a class of functions identified in B *with the function* (24).

P r o o f. It follows from Theorem 1, Lemma 1 and the definition of the space Q_- that a pair $\{\tilde{y}, \tilde{f}\} \in \mathrm{B} \times \mathrm{B}$ satisfying (31) belongs to $L - \lambda E$. Now let $\{\tilde{y}, \tilde{f}\} \in L - \lambda E$. Then there exists a sequence of pairs $\{\tilde{y}_n, \tilde{f}_n\} \in L' - \lambda E$ converging to the pair $\{\tilde{y}, \tilde{f}\}$ in $\mathrm{B} \times \mathrm{B}$. Using Lemma 4, we obtain that the function y_n can be represented in the form

$$y_n(t) = U(t,\lambda)x_n + \int_0^b G^*(s,t,\bar{\lambda})\tilde{A}(s)f_n(s)ds, \tag{32}$$

where $x_n \in Q$. From the convergence of the sequence of pairs $\{\tilde{y}_n, \tilde{f}_n\}$ in $\mathrm{B}\times\mathrm{B}$ there follows the convergence of the sequence $\{\tilde{U}(t,\lambda)x_n\}$ in B. When passing to (32) to the limit as $n\to\infty$, we find that $\tilde{y}$ admits the form (31). The proof of Lemma 7 is complete.

Corollary 2. *The operator $V_p(\lambda)$ is a continuous one-to-one mapping of Q_- onto* $\ker(L-\lambda E)$.

R e m a r k 2. In equality (31), the element $x\in Q_-$ and the function F are uniquely determined by the pair $\{\tilde{y},\tilde{f}\}\in L-\lambda E$. The pair $\{\tilde{y},\tilde{f}\}\in L'-\lambda E$ if and only if $x\in Q$ and in this case $x=\{-y'(0), y'(b)\}$.

R e m a r k 3. It follows from (22), (24), (30) that $V_q^*(\bar{\lambda})\tilde{f}=\{F(0),F(b)\}$.

R e m a r k 4. When $p=2$ and there is no operator weight (i.e., $A(t)=E$), the equality $Q_-=\hat{H}_{-3/2(\alpha+2)}\times\hat{H}_{-3/4}$ is valid (see [5]).

Lemma 8. *For any $\lambda\in\rho_0(G_A)$ the relation $L_0-\lambda E$ is closed.*

P r o o f. Suppose the sequence of pairs $\{\tilde{y}_n,\tilde{f}_n\}\in L_0-\lambda E$ converges to the pair $\{\tilde{y},\tilde{f}\}$ in the space $\mathrm{B}\times\mathrm{B}$. It follows from the definition of L_0 and Remark 2 that we can choose representatives y_n, f_n of the classes of functions $\tilde{y}_n$, $\tilde{f}_n$ such that they satisfy (32), where $x_n\in Q$ and $y_n(0)=y_n(b)=y_n'(0)=y_n'(b)=0$. Using Remark 3 and Lemma 4, we get $x_n=0$ and $V_q^*(\bar{\lambda})\tilde{f}_n=0$. Passing to the limit as $n\to\infty$ in the last equality and in (32), we obtain that $x=0$ and $V_q^*(\bar{\lambda})\tilde{f}=0$ in (31). Therefore $\{\tilde{y},\tilde{f}\}\in L_0-\lambda E$. Lemma 8 is proved.

R e m a r k 5. It follows from the proof of Lemma 8 that $\mathcal{R}(L_0-\lambda E)=\ker V_q^*(\bar{\lambda})$.

5. Spectrum of Restrictions of the Maximal Relation L

In this section, we introduce an abstract space of boundary values (SBV). By means of SBV we describe the spectrum of restrictions of the relation L and study the bounded operators $(L(\lambda)-\lambda E)^{-1}$, where $L_0\subset L(\lambda)\subset L$.

Suppose B_1, B_2, $\tilde{B}_1$, $\tilde{B}_2$ are Banach spaces, $T\subset \mathrm{B}_1\times\mathrm{B}_2$ is a closed relation, and $\delta\colon T\to\tilde{B}_1\times\tilde{B}_2$ is a linear operator. We denote $\delta_i=P_i\delta$, $i=1,2$, where P_i is the projection $\tilde{B}_1\times\tilde{B}_2$ onto $\tilde{B}_i$, i.e., $P_i\{x_1,x_2\}=x_i$ (the similar notation will be used in the analogous cases below). The following definition is given in [16] for operators and in [17] for relations.

Definition. *The quadruple $(\tilde{B}_1, \tilde{B}_2, \delta_1, \delta_2)$ is called a space of boundary values (SBV) for a closed relation T if the operator δ is a continuous mapping of T onto $\tilde{B}_1 \times \tilde{B}_2$, and the restriction of the operator δ_1 to* Ker$T$ *is a one-to-one mapping of* Ker$T$ *onto $\tilde{B}_1$.*

We define an operator $\Phi_\delta : \tilde{B}_1 \to \tilde{B}_2$ by the equality $\Phi_\delta = \delta_2\beta$, where $\beta = (\delta_1|_{\mathrm{Ker}T})^{-1}$ is the operator inverse to the restriction of δ_1 to Ker$T$. We denote $T_0 = \ker\delta$, $T_1 = \ker\delta_1$. Then $T_0 \subset T_1 \subset T$, $\mathcal{R}(T_1) = \mathcal{R}(T)$, and the relation T_1^{-1} is an operator (see [16, 17]).

From the definition of SBV, it follows that between the relations $\theta \subset \tilde{B}_1 \times \tilde{B}_2$ and $\tilde{T}$ with the property $T_0 \subset \tilde{T} \subset T$ there is a one-to-one correspondence determined by the equality $\delta\tilde{T} = \theta$. In this case we denote $\tilde{T} = T_\theta$.

Lemma 9. $\overline{T_\theta} = T_{\overline{\theta}}$.

Corollary 3. *The relation T_θ is closed if and only if θ is closed.*

R e m a r k 6. By the continuity of operator Φ_δ the relation θ is closed if and only if the relation $\theta - \Phi_\delta$ is closed.

Lemma 10. *Let $\mathcal{R}(T) = \mathrm{B}_2$. Then the following statements are valid:*
1) the range $\mathcal{R}(T_\theta)$ is closed if and only if the range $R(\theta - \Phi_\delta)$ is closed;
2) $\dim \mathrm{B}_2/\overline{\mathcal{R}(T_\theta)} = \dim \tilde{B}_2/\overline{\mathcal{R}(\theta - \Phi_\delta)}$*;*
3) $\dim\ker(T_\theta) = \dim\ker(\theta - \Phi_\delta)$.

The proves of Lemmas 9, 10 are based on the following statement, that might be known.

Lemma 11. *Suppose $\mathcal{B}_1$, $\mathcal{B}_2$ are Banach spaces, $\Delta : \mathcal{B}_1 \to \mathcal{B}_2$ is a bounded linear operator with the range $\mathcal{R}(\Delta) = \mathcal{B}_2$, $X \subset \mathcal{B}_1$ is a linear manifold such that* $\ker\Delta \subset X$. *Then $\Delta\overline{X} = \overline{\Delta X}$ and* $\dim \mathcal{B}_1/\overline{X} = \dim \mathcal{B}_2/\overline{\Delta X}$.

P r o o f of Lemma 11. The continuity of operator Δ implies $\Delta\overline{X} \subset \overline{\Delta X}$. We prove the inverse inclusion. Let $\mathcal{B}_1^{(0)} = \mathcal{B}_1/\ker\Delta$ be a quotient space and π be a canonical mapping of $\mathcal{B}_1$ onto $\mathcal{B}_1^0$. We define an operator Δ_0 by the equality $\Delta = \Delta_0\pi$. Then Δ_0 is a continuous one-to-one mapping of $\mathcal{B}_1^0$ onto $\mathcal{B}_2$. Let $a \in \overline{\Delta X}$, $a_n \in \Delta X$, where $n \in \mathbb{N}$. If a sequence $\{a_n\}$ converges to a, then the sequence $\{\Delta_0^{-1}a_n\}$ converges to $\Delta_0^{-1}a$ in the space $\mathcal{B}_1^{(0)}$. Since $\ker\Delta \subset X$, we see that all elements of the classes of adjacency $\Delta_0^{-1}a_n$ belong to X. Let $b \in \Delta_0^{-1}a$. Then we can choose a sequence $\{b_n\}$ such that $b_n \in \Delta_0^{-1}a_n \subset X$ and $\{b_n\}$ converges to b. Therefore $b \in \overline{X}$. Since $\Delta b = a$, we have $\overline{\Delta X} \subset \Delta\overline{X}$. The equality $\Delta\overline{X} = \overline{\Delta X}$ is proved.

Let Δ_1 be an operator defined by the equality $\Delta_1\pi_1 = \pi_2\Delta$, where π_1, π_2 are canonical mappings of $\mathcal{B}_1$, $\mathcal{B}_2$ onto quotient spaces $\mathcal{B}_1/\overline{X}$, $\mathcal{B}_2/\overline{\Delta X}$, respectively. Since Δ_1 is a continuous one-to-one mapping of $\mathcal{B}_1/\overline{X}$ onto $\mathcal{B}_2/\overline{\Delta X}$, we have $\dim \mathcal{B}_1/\overline{X} = \dim \mathcal{B}_2/\overline{\Delta X}$. The proof of Lemma 11 is complete.

P r o o f of Lemma 9. In Lemma 11 we take $\mathcal{B}_1 = \mathrm{B}_1 \times \mathrm{B}_2$, $\mathcal{B}_2 = \tilde{B}_1 \times \tilde{B}_2$, $\Delta = \delta$, $X = T_\theta$. Then $\Delta X = \delta T_\theta = \theta$, and $\Delta \overline{X} = \delta \overline{T_\theta} = \overline{\theta}$. Hence, $\overline{T_\theta} = T_{\overline{\theta}}$. Lemma 9 is proved.

P r o o f of Lemma 10. We define an operator $W : \mathrm{B}_2 \to \tilde{B}_2$ by the equality $Wf = \delta_2\{T_1^{-1}f, f\}$, where $f \in \mathrm{B}_2$. From the continuity of T_1^{-1} (see [16, 17]) and the properties of operators δ_1, δ_2 it follows that W is a continuous mapping of B_2 onto $\tilde{B}_2$. Moreover, using the definition of the relations T_0, T_1, we get $\ker W = \mathcal{R}(T_0)$. Any pair $\{y, f\} \in T$ is uniquely represented in the form $\{y, f\} = m_0 + m$, where $m_0 \in \mathrm{Ker}T$, $m \in T_1$, namely, $\{y, f\} = \{y - T_1^{-1}f, 0\} + \{T_1^{-1}f, f\}$. Hence, $(\delta_2 - \Phi_\delta \delta_1)\{y, f\} = \delta_2\{T_1^{-1}f, f\} = Wf$. Therefore, $W\mathcal{R}(T_\theta) = \mathcal{R}(\theta - \Phi_\delta)$. In Lemma 11 we take $\mathcal{B}_1 = \mathrm{B}_2$, $\mathcal{B}_2 = \tilde{B}_2$, $\Delta = W$, $X = \mathcal{R}(T_\theta)$. Then we obtain the first and the second statements of Lemma 10. An element $u \in T$ has the form $u = u_0 + v$, where $u_0 \in \mathrm{Ker}T$, $v \in T_0$, if and only if $\delta_2 u - \Phi_\delta \delta_1 u = 0$. Hence the restriction of the operator δ_1 to $\mathrm{Ker}T_\theta$ is a one-to-one mapping of $\mathrm{Ker}T_\theta$ onto $\ker(\theta - \Phi_\delta)$. From the above the third statement of the lemma follows. Lemma 10 is proved.

Let $\mathrm{B}_1 = \mathrm{B}_2 = \mathrm{B}'$ and let the quadruple $(\tilde{B}_1, \tilde{B}_2, \delta_1, \delta_2)$ be an SBV for a closed relation $T \subset \mathrm{B}' \times \mathrm{B}'$. A pair $\{y_1, y_2\} \in T$ if and only if the pair $\{y_1, y_2 - \lambda y_1\} \in T - \lambda E$. For any pair $\{y_1, y_2 - \lambda y_1\} \in T - \lambda E$ we put $\delta(\lambda)\{y_1, y_2 - \lambda y_1\} = \delta\{y_1, y_2\}$. As proved in [17], $\lambda \in \rho(T_1)$ if and only if the quadruple $(\tilde{B}_1, \tilde{B}_2, \delta_1(\lambda), \delta_2(\lambda))$ is an SBV for the relation $T - \lambda E$. As above, we denote $\Phi_{\delta(\lambda)} = \delta_2(\lambda)(\delta_1(\lambda)\mid_{\mathrm{Ker}(T-\lambda E)})^{-1} : \tilde{B}_1 \to \tilde{B}_2$. Lemma 10 implies the following assertion.

Theorem 2. *Let $\lambda \in \rho(T_1)$. Then the following statements are valid:*

1) the range $\mathcal{R}(T_\theta - \lambda E)$ is closed if and only if the range $\mathcal{R}(\theta - \Phi_{\delta(\lambda)})$ is closed;

2) $\dim \mathrm{B}'/\overline{\mathcal{R}(T_\theta - \lambda E)} = \dim \tilde{B}_2/\overline{\mathcal{R}(\theta - \Phi_{\delta(\lambda)})}$;

3) $\dim\ker(T_\theta - \lambda E) = \dim\ker(\theta - \Phi_{\delta(\lambda)})$.

Corollary 4. *Suppose that the relation θ is closed. A point $\lambda \in \rho(T_1)$ belongs to the point spectrum $\sigma_p(T_\theta)$ of the relation T_θ if and only if $\ker(\theta - \Phi_{\delta(\lambda)}) \neq \{0\}$. A point $\lambda \in \rho(T_1)$ belongs to the residual spectrum $\sigma_r(T_\theta)$ (to the continuous spectrum $\sigma_c(T_\theta)$) if and only if the relation $(\theta - \Phi_{\delta(\lambda)})^{-1}$ is a non-densely defined (densely defined and unbounded) operator. A point $\lambda \in \rho(T_1)$ belongs to the resolvent set $\rho(T_\theta)$ if and only if $(\theta - \Phi_{\delta(\lambda)})^{-1}$ is a bounded everywhere defined operator.*

Notice that for abstract SBV introduced in [20, 21] the statements similar to Cor. 4 were obtained in [18, 19].

In view of Lemma 10 and Theorem 2, we recall the following definitions (see [22] for relations and [23, Ch. 4] for operators). Let $S \subset \mathrm{B}_1 \times \mathrm{B}_2$ be a closed linear relation. The quantity $\chi(S) = \dim\ker S - \dim \mathrm{B}_2/\overline{\mathcal{R}(S)}$ is called an index of S if one of the subspaces $\ker S$ or $\mathrm{B}_2/\overline{\mathcal{R}(S)}$ is finite-dimensional. The relation S is called normal solvable if $\mathcal{R}(S)$ is closed; it is called semi-Fredholm if it is normal solvable and $\ker S$ or $\mathrm{B}_2/\overline{\mathcal{R}(S)}$ is finite-dimensional; it is called a Fredholm relation if it is semi-Fredholm, and the subspaces $\ker S$ and $\mathrm{B}_2/\overline{\mathcal{R}(S)}$ are finite-dimensional; it is called regular solvable if it is a Fredholm relation and $\chi(S)=0$; it is called solvable if $\mathcal{R}(S)=\mathrm{B}_2$ and $\ker S=\{0\}$. Theorem 2 implies that the relations $T_\theta-\lambda E$ and $\theta-\Phi_{\delta(\lambda)}$ simultaneously possess or do not possess the properties listed in this definition.

We apply the obtained results to the relation L generated by the expression $l^+[y]$ and the operator function $A(t)$.

We define the boundary operators $\gamma_1: L\to Q_-$, $\gamma_2: L\to\tilde{Q}_+$ for the relation L in the following way. Let a pair $\{\tilde{y},\tilde{f}\}\in L$. Then $\tilde{y}$ has form (31) for $\lambda=0$. By (31), to each pair $\{\tilde{y},\tilde{f}\}\in L$ we assign a pair of boundary values by the formulas

$$\gamma_1\{\tilde{y},\tilde{f}\}=x,\quad \gamma_2\{\tilde{y},\tilde{f}\}=V_q^*(0)\tilde{f}=\int_0^b U^*(s,0)\tilde{A}(s)f(s)ds. \tag{33}$$

It follows from Remark 2 that the pair $\{\gamma_1\{\tilde{y},\tilde{f}\},\gamma_2\{\tilde{y},\tilde{f}\}\}$ of boundary values is uniquely determined for each pair $\{\tilde{y},\tilde{f}\}\in L$. By Lemmas 6, 7 and Corollary 2, for each $\lambda\in\rho_0(G_A)$ the quadruple $(Q_-,\tilde{Q}_+,\gamma_1,\gamma_2)$ is the space of boundary values for the relation L. As above, by γ we denote the operator defined by the equality $\gamma\{\tilde{y},\tilde{f}\}=\{\gamma_1\{\tilde{y},\tilde{f}\},\gamma_2\{\tilde{y},\tilde{f}\}\}$. The operator γ is a continuous mapping of L onto $Q_-\times\tilde{Q}_+$. It follows from Remark 3 and the proof of Lemma 8 that $\ker\gamma=L_0$. Analogously as above, for any pair $\{y_1,y_2\}\in L$ we put $\gamma(\lambda)\{y_1,y_2-\lambda y_1\}=\gamma\{y_1,y_2\}$. Using Lemma 7, we get $\rho_0(G_A)\subset\rho(L_1)$, where $L_1=\ker\gamma_1$. Hence, for all $\lambda\in\rho_0(G_A)$ the quadruple $(Q_-,\tilde{Q}_+,\gamma_1(\lambda),\gamma_2(\lambda))$ is an SBV for the relation $L-\lambda E$. By $\Phi(\lambda)$ we denote the corresponding operator $\Phi_{\gamma(\lambda)}$. Using (33), we obtain

$$\Phi(\lambda)=\lambda\int_0^b U^*(s,0)\tilde{A}(s)U(s,\lambda)ds.$$

Let $\theta\subset Q_-\times\tilde{Q}_+$ be a linear relation and $L_\theta\subset L$ be a linear relation such that $\gamma L_\theta=\theta$. From Theorem 2, we get the following statement.

Theorem 3. *Let $\lambda\in\rho_0(G_A)$. Then the following statements are valid:*

1) the range $\mathcal{R}(L_\theta-\lambda E)$ is closed if and only if the range $\mathcal{R}(\theta-\Phi(\lambda))$ is closed;

2) $\dim \mathrm{B}/\overline{\mathcal{R}(L_\theta-\lambda E)}=\dim\tilde{Q}_+/\overline{\mathcal{R}(\theta-\Phi(\lambda)}$*;*

3) $\dim\ker(L_\theta-\lambda E)=\dim\ker(\theta-\Phi(\lambda))$.

Corollary 4 with T_θ replaced by L_θ and $\Phi_{\delta(\lambda)}$ replaced by $\Phi(\lambda)$ holds for the relation L.

Suppose $\theta(\lambda) \subset Q_- \times \tilde{Q}_+$ and $L_{\theta(\lambda)} \subset L$ are the families of linear relations such that $\gamma L_{\theta(\lambda)} = \theta(\lambda)$. By Lemma 4, the relation $R(\lambda) = (L_{\theta(\lambda)} - \lambda E)^{-1}$ is a bounded everywhere defined operator if and only if $(\theta(\lambda) - \Phi(\lambda))^{-1}$ is bounded everywhere defined.

The following two theorems can be proved in view of Lemma 7 and Corollary 4 by analogy with the corresponding assertions in [12], where the case of $p = 2$, $\alpha = 0$, was considered.

Theorem 4. *Suppose $R(\lambda) = (L_{\theta(\lambda)} - \lambda E)^{-1}$ (or $(\theta(\lambda) - \Phi(\lambda))^{-1}$) is a bounded everywhere defined operator. Then $R(\lambda)$ is an integral operator of the form*

$$R(\lambda)\tilde{f} = \int_0^b (\tilde{U}(t,\lambda)(\theta(\lambda) - \Phi(\lambda))^{-1} U^*(s,\bar{\lambda}) + G^*(s,t,\bar{\lambda}))\tilde{A}(s))f(s)ds. \qquad (34)$$

Theorem 5. *Suppose the relation $R(\lambda_0)$ (or $(\theta(\lambda_0) - \Phi(\lambda_0)^{-1}$) is a bounded everywhere defined operator. Then the family $R(\lambda)$ is holomorphic in the point λ_0 if and only if the family $(\theta(\lambda) - \Phi(\lambda))^{-1}$ is holomorphic in λ_0.*

R e m a r k 7. If the relation $T(\lambda_0)$ is a bounded everywhere defined operator and the family of relations $T(\lambda)$ is holomorphic in the point λ_0, then the relations $T(\lambda)$ are bounded everywhere defined operators in some neighborhood of λ_0 (see [23, Ch. 7; 24]).

6. Maximal and Minimal Relations in $L_2(H, A(t); 0, b)$. Description of Generalized Resolvents

In this section, we prove that the minimal relation L_0 is symmetric in the space $L_2(H, A(t); 0, b)$ and describe the generalized resolvents of the relation L_0.

Further, we will consider the case of $\mathrm{B} = L_2(H, A(t); 0, b)$, i.e., $p = 2$. Notice that the norm in B is generated by the scalar product

$$(\tilde{f}, \tilde{g})_{\mathrm{B}} = \int_0^b (\tilde{A}(t)f(t), g(t))dt.$$

The space Q_- is a Hilbert space with the scalar product

$$(x_1, x_2)_- = (\tilde{U}(\cdot, 0)x_1, \tilde{U}(\cdot, 0)x_2)_{\mathrm{B}}.$$

This scalar product generates the norm (28) under $r = 2$. The space Q_- can be treated as a space with the negative norm with respect to Q [6, Ch. 2; 9, Ch. 1]. By Q_+, denote the corresponding space with the positive norm. Using the definitions of positive and negative spaces, we get $Q_+ = \tilde{Q}_+$.

Lemma 12. *Let the pairs* $\{\tilde{y}, \tilde{f}\}, \{\tilde{z}, \tilde{g}\} \in L'$. *Then there exist such representatives* $y \in \tilde{y}$, $z \in \tilde{z}$ *that the following equality holds:*

$$(\tilde{f}, \tilde{z})_{\mathrm{B}} - (\tilde{y}, \tilde{g})_{\mathrm{B}} = -(y'(b), z(b)) + (y'(0), z(0)) + (y(b), z'(b)) - (y(0), z'(0)). \quad (35)$$

P r o o f. It follows from Lemma 7 and Remark 2 that there exist such representatives $y \in \tilde{y}$, $z \in \tilde{z}$ that

$$y(t) = U(t,0)v + \int_0^b G^*(s,t)\tilde{A}(s)f(s)ds,$$

$$z(t) = U(t,0)w + \int_0^b G^*(s,t)\tilde{A}(s)g(s)ds,$$

where $v, w \in Q$. Since $\tilde{f}, \tilde{g} \in \mathrm{B}$, we obtain that the functions $\tilde{A}(s)f(s)$, $\tilde{A}(s)g(s)$ belong to $L_1(H; 0, b)$. We chose two sequences $\{f_n\}$ and $\{g_n\}$ of functions such that f_n, g_n are strongly continuous functions in the space $\hat{H}_{+1}$ and the sequences $\{f_n\}$, $\{g_n\}$ converge to the functions $\tilde{A}(s)f(s)$ and $\tilde{A}(s)g(s)$, respectively, in the space $L_1(H; 0, b)$. Moreover, we take two sequences $\{v_n\}$, $\{w_n\}$, where $v_n, w_n \in \hat{H}_{+1}$, such that $\{v_n\}$, $\{w_n\}$ converge to v, w, respectively, in the space H. Then the functions

$$y_n(t) = U(t,0)v_n + \int_0^b G^*(s,t)f_n(s)ds, \; z_n(t) = U(t,0)w_n + \int_0^b G^*(s,t)g_n(s)ds$$

are strong solutions [11] of equation (2) with the right parts f_n, g_n, respectively. Hence, $y_n(t), z_n(t) \in \mathcal{D}(\mathcal{A}_1)$ for each $t \in [0, b]$. Therefore,

$$\int_0^b (l[y_n], z_n)dt - \int_0^b (y_n, l[z_n])dt = \int_0^b (-y_n''(t) + \mathcal{A}_1(t)y_n(t), z_n(t))dt$$
$$- \int_0^b (y_n(t), -z_n''(t) + \mathcal{A}_1(t)z_n(t))dt = - \int_0^b (y_n''(t), z_n(t))dt + \int_0^b (y_n(t), z_n''(t))dt$$
$$= -(y_n'(b), z_n(b)) + (y_n'(0), z_n(0)) + (y_n(b), z_n'(b)) - (y_n(0), z_n'(0)). \quad (36)$$

It follows from (11) that $y_n(0)$, $y_n(b)$, $z_n(0)$, $z_n(b)$ converge to $y(0)$, $y(b)$, $z(0)$, $z(b)$, respectively, in the space H. Since $v_n = \{-y_n'(0), y_n'(b)\}$, $w_n =$

$\{-z_n'(0), z_n'(b)\}$, $v = \{-y'(0), y'(b)\}$, $w = \{-z'(0), z'(b)\}$, we have $y_n'(0)$, $y_n'(b)$, $z_n'(0)$, $z_n'(b)$ converge to $y'(0)$, $y'(b)$, $z'(0)$, $z'(b)$, respectively, in the space H. In (36), we pass to the limit as $n \to \infty$ and obtain (35). The proof of Lemma 12 is complete.

Corollary 5. *The relation L_0 is symmetric.*

P r o o f follows from Remark 2, Lemma 12 and the definition of L_0.

Lemma 13. $L_0^* = L$.

In view of Lemma 7 and Corollary 5, the proof of Lemma 13 is the same as that of the similar assertion in [12].

Theorem 6. *The range $\mathcal{R}(\gamma)$ of the operator γ coincides with $Q_- \times Q_+$, and for any pairs $\{\tilde{y}, \tilde{f}\}, \{\tilde{z}, \tilde{g}\} \in L$ "the Green formula" is valid:*

$$(\tilde{f}, \tilde{z})_{\mathrm{B}} - (\tilde{y}, \tilde{g})_{\mathrm{B}} = (Y_2, Z_1) - (Y_1, Z_2), \tag{37}$$

where $\{Y_1, Y_2\} = \gamma\{\tilde{y}, \tilde{f}\}$, $\{Z_1, Z_2\} = \gamma\{\tilde{z}, \tilde{g}\}$.

P r o o f. The equality $\mathcal{R}(\gamma) = Q_- \times Q_+$ follows from Lemmas 6, 7, Corollary 2 and equalities (33). In view of Lemma 12, formula (37) is proved in the same way as the similar one in [12]. Theorem 6 is proved.

In a particular case of $Q_- = Q_+ = Q$, Theorem 6 implies that the ordered triple (Q, γ_1, γ_2) is a space of boundary values in the sense of papers [20, 21]. Using the argumentation of [20, 21], we obtain the following assertion.

Lemma 14. *For fixed λ, the relations $L_{\theta(\lambda)}$ and $\theta(\lambda)$ are or are not simultaneously accumulative (dissipative, symmetric, maximal accumulative, maximal dissipative, maximal symmetric, selfadjoint).*

When there is no operator weight (i.e., $(A(t) = E)$, the relation L is an operator, and in this case Theorem 6 was proved in [1] for the expression $l[y]$ with a constant operator coefficient $\mathcal{A}_1(t) = \mathcal{A}_1$, and in [2, 3] for $l[y]$ with a variable operator coefficient $\mathcal{A}_1(t)$ satisfying the conditions listed in Sect. 2. The case of $\alpha = 0$ was considered in these papers. In [1], the boundary values did not contain the Green function. In [3], for the variable operator coefficient $\mathcal{A}_1(t)$, the boundary values were constructed so that they did not contain the Green function. Moreover, additional conditions were imposed on the function $\mathcal{A}_1(t)$, and the example proving the necessity of these conditions was given. The boundary values containing the Green function were constructed in [2] as $\alpha = 0$ and in [4, 5] as $\alpha \geqslant 0$, and they differ from the boundary values (33) introduced in the present paper. The papers [1–3, 20, 21] are reviewed in the monograph [6]. Notice that for the first time linear relations were applied to the description of extensions of differential operators in [25] (see also [14]), where the differential expressions with bounded operator coefficients were considered.

We recall a definition of the generalized resolvent. Suppose that $\mathcal{B}$ is a Hilbert space, $\mathcal{L}_0$ is a closed symmetric relation, $\mathcal{L}_0 \subset \mathcal{B} \times \mathcal{B}$. The operator function R_λ, $\mathrm{Im}\lambda \neq 0$, is called a generalized resolvent of the relation $\mathcal{L}_0$ if there exists the Hilbert space $\tilde{\mathcal{B}} \supset \mathcal{B}$ and the selfadjoint relation $\tilde{\mathcal{L}} \supset \mathcal{L}_0$, $\tilde{\mathcal{L}} \subset \tilde{\mathcal{B}} \times \tilde{\mathcal{B}}$ such that the condition $R_\lambda = P(\tilde{\mathcal{L}} - \lambda E)^{-1}|_{\mathcal{B}}$, where P is an orthogonal projection of $\tilde{\mathcal{B}}$ onto $\mathcal{B}$, is satisfied.

Detailed bibliography on generalized resolvents is given in the monograph [14].

In view of Theorems 4, 5 and Lemma 14, the proof of the following theorem is the same as that of the similar assertion in [12], where the case of $\alpha = 0$ was considered.

Theorem 7. *Any generalized resolvent R_λ $(\mathrm{Im}\lambda \neq 0)$ of the relation L_0 is the integral operator* (34)*, where $\theta(\lambda) \subset Q_- \times Q_+$, and $\theta(\lambda)$ is a holomorphic family, the values of which $\theta(\lambda)$ are maximal accumulative relations in the case of* $\mathrm{Im}\lambda > 0$ *and maximal dissipative relations in the case of* $\mathrm{Im}\lambda < 0$*, with $\theta^*(\lambda) = \theta(\bar{\lambda})$. Conversely, if $\theta(\lambda)$ is a family of the linear relations with the mentioned above properties, then the family of operators R_λ of form* (34) *is a generalized resolvent of the relation L_0.*

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