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Parabolic Foliations on Three-Manifolds

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We prove that every closed orientable three-manifold admits a parabolic foliation.

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1. Introduction

It is well known that every closed orientable three-manifold admits a foliation. When there are additional restrictions on the geometry and topology of the leaves, this statement is not true. For example, the foliations by minimal surfaces do not exist on a three-sphere (relatively to any metric). Analogously, the classes of totally umbilical foliations and totally geodesic foliations do not exist on every three-manifold.

A. Borisenko introduced new classes of foliations on the Riemannian manifolds having the restrictions on the extrinsic geometry of leaves, namely, elliptic, parabolic, and strong saddle (or hyperbolic) foliations.

Definition 1.1 (Borisenko). *A codimension one foliation on a three-manifold is called:*

1) parabolic, if there is a metric such that $K_e = 0$*;*

2) (strong)saddle, if there is a metric such that $(K_e < 0)K_e \leq 0$*;*

3) elliptic, if there is a metric such that $K_e > 0$*,*

where K_e *states for the extrinsic curvature of the leaves.*

The studying of the existence of these foliations on three-manifolds was initiated by D. Bolotov in [2] where he, among other results, defined a metric on the solid torus such that the Reeb component was a parabolic foliation. In [3] he gave the examples of strong saddle foliations on the torus bundles over the circle and on a three-sphere. In particular, a foliation in the Reeb component is not a topological restriction to the existence of strong saddle foliations. In [5], the author showed that in fact every closed orientable three-manifold admits a strong saddle foliation.

It is well known that closed orientable three-manifolds do not admit elliptic foliations. Namely, the existence of them contradicts to a well-known Reeb formula $\int_M H = 0$ (here H stands for the mean curvature) since, if it is elliptic with respect to some metric, then its total mean curvature cannot be zero.

The last open problem was the existence of parabolic foliations on closed orientable three-manifolds. In this paper we give the positive answer to this question.

Theorem 1.2. *Every closed orientable three-manifold admits a parabolic foliation.*

Notice that there are no parabolic foliations on S^3 with respect to a standard metric.

Parabolic foliations of the codimension larger than one were studied in [1].

The paper is organized as follows. In Section 2 we recall some definitions and constructions from the topology of foliations on three-manifolds. In Section 3, several local models of parabolic foliations are constructed. In Section 4 we define a parabolic foliation on the three-sphere which is a turbulization of the Reeb foliation along an arbitrary knot. In Section 5 it is shown how to perform a Dehn surgery on this knot to obtain a parabolic foliation on every closed orientable three-manifold.

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2. Basic Definition

2.1. Foliations on Three-Manifolds

In this section we recall some necessary basic facts about the foliations on three-manifolds.

Let $\mathcal{F}$ be a foliation on a closed three-manifold. It defines a two-dimensional distribution of planes tangent to the leaves. However, not every plane distribution defines a foliation. A distribution is called integrable if it defines a foliation. Classical Frobenius theorem gives necessary and sufficient conditions for the distribution to be integrable. We also recall a three-dimensional version of this theorem.

Theorem 2.1 (Frobenius). *A distribution of planes ξ on a three-manifold is integrable if and only if for every pair of local sections X and Y of ξ its Lie bracket belongs to ξ.*

Notice that a distribution is called transversally orientable if there is a globally defined vector field that is transverse to it. In this case there is a globally defined one-form α such that $Ker(\alpha)_p = T_pL$, where L is a leaf through p. It is easy to rewrite the conditions of Frobenius theorem in the terms of the form α: a distribution is integrable if and only if $\alpha \wedge d\alpha = 0$.

Example 2.2. *Reeb foliation on $D^2 \times S^1$.*
Consider the following C^∞-smooth function on $[0, 1]$:

1) the function f is a smooth increasing function on $[0, 1]$;

2) there is an $\varepsilon > 0$ such that for any $x \in [0, \varepsilon)$ the value of $f(x)$ is equal to zero and $f(x) = 1$.

On the solid torus $D^2 \times S^1$ with cylindrical coordinates $((r, \phi), t)$ define the following one-form:

$$\alpha = f(r)dr + (1 - f(r))dt.$$

By the Frobenius theorem, a distribution of planes defined by the kernel of α is integrable since

$$\alpha \wedge d\alpha = (f(r)dr + (1 - f(r))dt) \wedge (-f'(r)dr \wedge dt) = 0.$$

Therefore α defines a foliation on $D^2 \times S^1$. We denote this foliation by $\mathcal{F}_R$ and call it the Reeb foliation on a solid torus.

Remark 2.2. *The Reeb foliation is usually defined as a foliation of $D^2 \times S^1 = \{((r, \phi), t) : r \in [0, 1],\ \phi,\ t \in [0, 2\pi)\}$ by the levels of function $h(r, \phi, t) = (r^2 - 1)e^t$. It is obvious that the foliation defined above is isotopic to this foliation what justifies the title of Example 2.2.*

2.2. Extrinsic Geometry of Foliations

Assume now that M is a Riemannian manifold with a scalar product g and an associated Levi–Civita connection ∇. Consider a foliation $\mathcal{F}$ on M. For each pair of vector fields X and Y on M that are tangent to $\mathcal{F}$, define the second fundamental form of $\mathcal{F}$ with respect to unit normal n by

$$B(X, Y) = g(\nabla_X Y, n).$$

Using the scalar product in the tangent bundle, we may define the following linear operator A_n:

$$B(X, Y) = g(A_n X, Y),$$

that is called a Weingarten operator. Since A_n is symmetric, it has two real eigenvalues which are principal curvature functions. A product $K_e = k_1 k_2$ is called an extrinsic curvature of $\mathcal{F}$.

Owing to the sign of extrinsic curvature we may define the following classes of foliations on three-manifolds.

Definition 2.3 (Borisenko). *A codimension one foliation on a three-manifold is called:*

1) parabolic, if there is a metric such that $K_e = 0$*;*

2) (strong)saddle, if there is a metric such that $(K_e < 0) K_e \leq 0$*;*

3) elliptic, if there is a metric such that $K_e > 0$*.*

Remark 2.4. *As mentioned in the introduction, there are no elliptic foliations on the closed oriented three-manifolds. Since* $K_e > 0$*, the functions of principal curvatures are nowhere zero and simultaneously they are larger or less than zero. By the Reeb formula*

$$0 = \int\limits_M H = \frac{1}{2} \int\limits_M (k_1 + k_2) \neq 0,$$

what is a contradiction.

Remark 2.5. *Notice that many geometric classes of foliations are subclasses of the introduced classes. Minimal foliations are saddle, totally umbilical foliations have* $K_e \geq 0$*, and totally geodesic foliations are parabolic.*

2.3. Knots and Braids

Recall that a knot in S^3 or $\mathbb{R}^3$ is an image of a circle S^1 under some C^∞-smooth regular embedding. The knots K_0 and K_1 are called isotopic if there is a smooth family of embeddings $K(t) : S^1 \to S^3(\mathbb{R}^3)$ such that $K(0) = K_0$ and $K(1) = K_1$.

Consider two sets of the points $A = \{(i, 0, 0), i = 1, \dots, n\}$ and $B = \{(i, 0, 1), i = 1, \dots, n\}$ in $\mathbb{R}^3$. A smooth embedded curve $\gamma(t)$ is called descending if its z-coordinate is a strictly decreasing function of the parameter t.

A topological braid K with n strings is a collection of n disjoint descending curves in $\mathbb{R}^3$ which connect the points from the set B with the points of A. We say that two braids are isotopic if there is a smooth family of braids connecting them.

Consider a group with generators $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ and relations $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for all i and also $\sigma_i \sigma_j = \sigma_j \sigma_i$ in the case when $|i - j| \geq 2$. This group is denoted by B_n and called a group of algebraic braids. There is a one-to-one

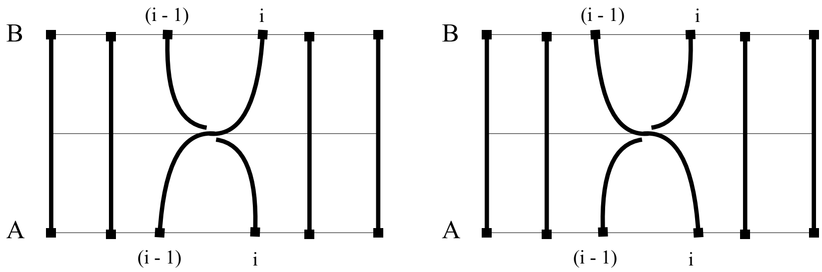

Fig. 1: Possible intersections in the frontal projection.

correspondence between the isotopy classes of topological braids and the elements of B_n. Further we will agree that a topological braid whose frontal projection is in the left part of Figure 1 corresponds to the generator σ_i. The second possible intersection corresponds to the element σ_i^{-1}. Therefore each isotopy class of topological braids can be represented as a product $K = \sigma_1^{\pm 1}\sigma_2^{\pm 1}\dots\sigma_N^{\pm 1}$.

On the set of isotopy classes of topological braids define a new operation — a closure of the braid. A closure of braid K is a link (that is an embedded image of disjoint S^1) which is obtained from K by adding disjoint curves connecting the i-th point of A with the i-th point of B. The following theorem holds (cf. [4]):

Theorem 2.6. *The mapping from the set of isotopy classes of topological braids which maps each braid to its closure is surjective. In particular, each isotopy class of knots contains the closure of some braid.*

2.4. Combinatorial Presentation of Three-Manifolds

In this section we will give a sketch of the proof that every closed orientable three-manifold admits a foliation.

Consider a knot K in S^3. Let N be some tubular neighborhood of K. Denote $X = \overline{S^3 \setminus N}$. Then $\partial X = \partial N = T^2$. Consider some homeomorphism

$$h : \partial X \to \partial(D^2 \times S^1),$$

and let $M = X \cup D^2 \times S^1/(y \sim h(y),\ \text{for all } y \in \partial X)$. It is easy to see that M is a closed manifold.

This construction is called a Dehn surgery on a knot. The importance of this construction follows from the theorem:

Theorem 2.7. *[4] Any closed orientable manifold may be obtained by the Dehn surgery on some knot in S^3.*

Recall a classical construction of the transversally orientable foliation on a closed orientable three-manifold. Consider a solid torus $D^2 \times S^1 = \{((r, \phi), t) : r \in [0, 2], \phi, t \in [0, 2\pi)\}$, and let $\alpha = f(r)dr + (1 - f(r))dt$, where $f(r)$ is some smooth function on the segment $[0, 2]$ satisfying the following conditions:

1) $f(r)$ is a strictly increasing function on $[0, 1]$;

2) there is an ϵ such that for all $r \in (2 - \epsilon, 2]$ the function $f(r) = 0$;

3) $f(r)$ is a strictly decreasing function on $[1, 2 - \epsilon]$;

4) $f(0) = 0$ and $f(1) = 1$.

The form α defines some foliation on $D^2 \times S^1$. Denote it by $\mathcal{F}_T$.

It is obvious that $\mathcal{F}_T$ has a single compact leaf $\{r = 1\}$. $\mathcal{F}_T$, restricted on a solid torus $D^2(1) \times S^1 = \{(r, \phi, t) \in D^2 \times S^1 : r \in [0, 1]\}$, is a Reeb foliation (see Example 2.2).

It is well known that S^3 may be represented as a union of two solid tori that are glued along the boundary torus. Gluing homeomorphism interchanges the generators of the boundary torus. In each solid torus consider the Reeb foliations $\mathcal{F}_R$. Since the gluing homeomorphism maps a leaf of the first Reeb component to the leaf of the second one, we can see that a three-sphere admits the foliation which is the union of two Reeb components. We will also denote this foliation by $\mathcal{F}_R$.

Assume now that K is a knot in S^3. By Theorem 2.4 it is isotopic to the closure of some braid. Further we can isotope this braid to make it everywhere transverse to the foliation of one of solid torus by disks $D^2 \times \{t\}$. Since $\mathcal{F}_R$ is a foliation by disks in a small neighborhood of the core curve $r = 0$, we may assume that K is transverse to $\mathcal{F}_R$. Cut out a small tubular neighborhood of K and glue it back into a solid torus with the foliation $\mathcal{F}_T$ inside. We obtain a new foliation on S^3 which is a turbulization of the initial one along K. Finally, to obtain a foliation on M, we Cut out a tubular neighborhood of K up to the torus leaf and glue it back by diffeomorphism of the boundary. It is easy to verify that, since the boundary of this neighborhood is a leaf, the foliation is correctly defined on M. By Theorem 2.7 any closed orientable three-manifold may be obtained as above, therefore every closed orientable three-manifold admits a foliation.

2.5. Bump Functions on $\mathbb{R}$

Later in the proof we will often meet the situation when in some finite segment $[a, b]$ a smooth function $f(t)$ is defined in such a way that the following conditions are satisfied:

1) $f(a) = f_0$;

2) $f(b) = f_1$;

3) there is an $\epsilon > 0$ such that:
— for all $t \in [a, a+\epsilon)$, $f(t) = f_0$,
— for all $t \in [b-\epsilon, b)$, $f(t) = f_1$;

4) f is monotone on $[a, b]$.

In the paper we refer to these functions as to the bump functions on $[a, b]$.

For example, in the construction of Reeb component $\mathcal{F}_R$, f is an increasing bump function on $[0, 1]$. The function $f(r)$, arising in the construction of $\mathcal{F}_T$, is a union of two bump functions.

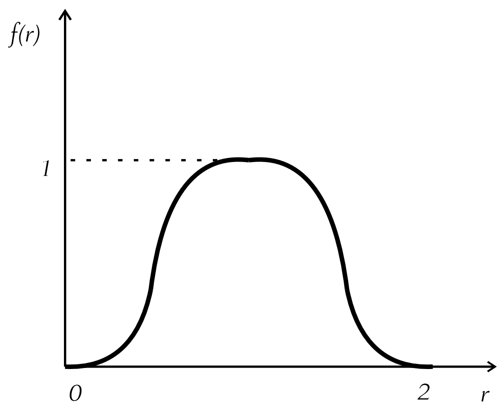

Fig. 2: A graph of $f(r)$ in the construction of turbulization.

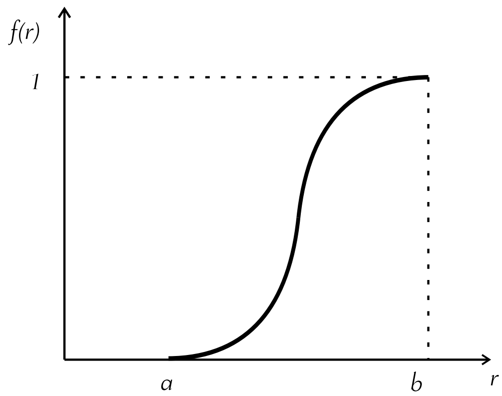

Fig. 3: A typical bump function on $[a, b]$.

3. Local Models of Parabolic Foliations

In this section we describe several local models of parabolic foliations on three-manifolds.

3.1. Parabolic Foliation on $\Sigma^2 \times [0,1]$

Lemma 3.1. *Let Σ^2 be a compact parallelizable surface (possibly with the boundary). Let two Riemannian metrics G and H on Σ^2 coincide in some neighborhood of the boundary $\partial\Sigma^2$. Assume that $\mathcal{F}$ is a foliation of $M = \Sigma^2 \times [0,1]$ by the surfaces $\Sigma^2 \times \{t\}$. Then, there is such Riemannian metric g on M that:*

1) in some tubular neighborhood of $\Sigma^2 \times \{0\}$, $g = dt^2 + G(p)$, for all $p \in \Sigma^2$;

2) in some tubular neighborhood of $\Sigma^2 \times \{1\}$, $g = dt^2 + H(p)$, for all $p \in \Sigma^2$;

3) $\mathcal{F}$ is parabolic on $\Sigma^2 \times [0,1]$ with respect to g;

4) there is a neighborhood U of the boundary $\partial\Sigma^2$ such that for all $t \in [0,1]$, $g(p,t)|_{U\times\{t\}} = G(p)$.

P r o o f. Let $\{X_0, Y_0\}$ be an orthonormal frame on Σ^2 with respect to H. The matrix of G in this frame may be written as

$$G = G(p) = \begin{pmatrix} a(p) & b(p) \\ b(p) & c(p) \end{pmatrix}$$

for all $p \in \Sigma^2$. Since Σ^2 is compact, the functions a, b and c are bounded on Σ^2.

Consider a frame (X, Y, n) on M, where $X = (X_0, 0)$, $Y = (Y_0, 0)$, and $n = \frac{\partial}{\partial t}$. Let N be such a neighborhood of the boundary $\partial\Sigma^2$ that $G|_N = H|_N$. Denote $L = \overline{M\backslash N}$.

We are going to interpolate from G to H on $\Sigma^2 \times [0,1]$ by using the following Riemannian metric:

$$g = g(p,t) = \begin{pmatrix} a(p,t) & b(p,t) & 0 \\ b(p,t) & c(p,t) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $(p,t) \in \Sigma^2 \times [0,1]$, $a(p,t)$, $b(p,t)$ and $c(p,t)$ are functions on $\Sigma^2 \times [0,1]$. The matrix of g is written with respect to the frame (X, Y, n).

By the definition of g, n is a unit normal vector field to $\mathcal{F}$. Calculate the matrix of the second fundamental form of the leaves relative to the normal n in the basis $\{X, Y\}$. By the Koszul formula,

$$2g(\nabla_X X, n) = 2X(g(X,n)) - n(g(X,X)) + g([X,X],n) - 2g([X,n],X),$$

where ∇ is a Levi–Civita connection of g. Since X is independent of t, then $g([X,n],X) = 0$. Therefore $g(\nabla_X X, n) = -\frac{1}{2}\frac{\partial a}{\partial t}$.

Similarly,

$$2g(\nabla_Y Y, n) = 2Y(g(Y,n)) - n(g(Y,Y)) + g([Y,Y],n) - 2g([Y,n],Y) = -ng(Y,Y).$$

Finally, we have

$$2g(\nabla_X Y, n) = X(g(Y,n)) + Y(g(X,n)) - n(g(X,Y)) + g([X,Y],n) - g([Y,n],X)$$
$$-g([X,n],Y) = -n(g(X,Y)) + g([X,Y],n) - g([Y,n],X) - g([X,n],Y).$$

Since $\mathcal{F}$ is a foliation, we have $g([X,Y],n) = 0$. As X and Y do not depend on t, we see that $g([Y,n],X) = 0$ and $g([X,n],Y) = 0$. Therefore $g(\nabla_X Y, n) = -\frac{1}{2}n(g(X,Y))$. Consequently, the second fundamental form of the leaves is given by the matrix

$$B = -\frac{1}{2}\begin{pmatrix} \frac{\partial a}{\partial t} & \frac{\partial b}{\partial t} \\ \frac{\partial b}{\partial t} & \frac{\partial c}{\partial t} \end{pmatrix}.$$

An extrinsic curvature of $\mathcal{F}$ with respect to g equals

$$K_e = \frac{1}{4}\frac{\frac{\partial a}{\partial t}\frac{\partial c}{\partial t} - \frac{\partial b}{\partial t}^2}{ac - b^2}.$$

Let $0 \le t_1 < t_2 < t_3 < t_4 < t_5 \le 1$ be a subdivision of the segment $[0,1]$. Assume that D is a positive real number larger than $\max_{p\in\Sigma^2}\{a(p), c(p)\}$. We will choose the exact value of D later in the proof.

Consider the following function h on Σ^2:

1) $h(p) = 1$, for all $p \in L$;

2) $h(p) = 0$ in some neighborhood of $\partial\Sigma^2$;

3) h is a smooth nonnegative function on Σ^2.

Consider the function $\tilde{a}(p,t) = Df(t) + (1-f(t))a(p)$, where $f(t)$ is an increasing bump function on $[0,t_1]$ with $f(0) = 0$ and $f(t_1) = 1$. Finally, let

$$a(p,t) = h(p)\tilde{a}(p,t) + (1-h(p))a(p).$$

On $[0,t_1]$ define the following matrix (with respect to the frame (X,Y,n)):

$$g_D = g_D(p,t) = \begin{pmatrix} a(p,t) & b(p) & 0 \\ b(p) & c(p) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This matrix clearly defines a metric on $\Sigma^2 \times [0, t_1]$ since, when $t = 0$, it has positive diagonal entries, and $a(p,t)$ is nondecreasing on the segment $[0, t_1]$. By the definition of bump functions, g is equal to $dt^2 + G(p)$ in some tubular neighborhood of $\Sigma^2 \times \{0\}$. A foliation by surfaces $\Sigma \times \{t\}$ is parabolic with respect to the introduced metric.

On the segment $[t_1, t_2]$ we may change $c(p)$ in the same way. Consequently, on the segment $[0, t_2]$ we have:

1) $g_D(p,0) = \begin{pmatrix} G(p) & 0 \\ 0 & 1 \end{pmatrix}$;

2) $g_D(p,t_2) = \begin{pmatrix} a(p,t_2) & b(p) & 0 \\ b(p) & c(p,t_2) & 0 \\ 0 & 0 & 1 \end{pmatrix}$;

3) $\mathcal{F}$ is a parabolic foliation on $\Sigma^2 \times [0, t_2]$ with respect to g_D.

Consider an increasing bump function $f(t)$ on the segment $[t_2, t_3]$ with $f(t_2) = 0$ and $f(t_3) = 1$. On $[t_2, t_3]$ define g_D by the matrix

$$g_D = \begin{pmatrix} a(p,t_2) + f(t)b(p) & b(p)(1-f(t)) & 0 \\ b(p)(1-f(t)) & c(p,t_2) + f(t)b(p) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $b(p)$ is bounded on Σ^2 and equal to zero on N, the matrix of g_D is positively definite for some choice of D (see the definition of D above) and therefore defines a metric.

The foliation is parabolic with respect to the introduced metric since an extrinsic curvature is given by $K_e = \frac{1}{4det(g_D)}(f'(t)^2 b(p)^2 - (-f'(t))^2 b(p)^2) = 0$.

On $\Sigma^2 \times \{t_3\}$ the matrix of g_D is diagonal. All non-diagonal elements of g_D being eliminated, we may freely decrease the diagonal elements of g_D. To do this, consider the decreasing bump function $k(t)$ on $[t_3, t_4]$ with $k(t_3) = 1$ and $k(t_4) = 0$. On $[t_3, t_4]$ define

$$g_D = \begin{pmatrix} k(t)a(p,t_3) + (1-k(t)) & 0 & 0 \\ 0 & c(p,t_3) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For all $t \in [t_3, t_4]$ the matrix g_D is positively definite and therefore defines a metric. At $t = t_4$ the matrix of g_D is given by

$$g_D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c(p,t_3) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Analogously, on $[t_4, t_5]$ we may decrease a diagonal element $c(p, t_3)$. We showed above how to deform g_D into the metric $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ preserving the parabolicity defined of $\mathcal{F}$. This metric is nothing else but a matrix of $dt^2 + H(p)$ written with respect to a frame (X, Y, n).

Let U be an open subset of $\{p \in \Sigma^2, \text{where } h(p) = 0\}$ containing the boundary of Σ^2. On $[0, t_1]$ the function $a(p,t)|_U = a(p)|_U = 1$. The same holds for the function $c(p,t)|_U$ on the segment $[t_1, t_2]$. On $[t_2, t_3]$, since $b(p)|_U = 0$, the matrix of $g|_U$ is an identity matrix. On the segments $[t_3, t_4]$ and $[t_4, t_5]$, when $a(p,t)|_U = c(p,t)|_U = 1$, the matrix $g|_U$ is also an identity matrix. This finishes the proof of the lemma.

Corollary 3.2. *Consider the manifold $M = T^2 \times [0,1]$. Let $G(x,y)$ and $H(x,y)$ be some metrics on T^2. Then there is a metric g on M such that:*

1) *foliation $\mathcal{F}$ by the tori $T^2 \times \{pt\}$ is parabolic;*

2) *the matrix of $g|_{T^2\times\{0\}} = \begin{pmatrix} G(x,y) & 0 \\ 0 & 1 \end{pmatrix}$ and the matrix of $g|_{T^2\times\{1\}} = \begin{pmatrix} H(x,y) & 0 \\ 0 & 1 \end{pmatrix}$;*

3) *g is a direct product metric in some one-sided neighborhood of the boundary.*

3.2. Parabolic Foliation on a Solid Torus

The following proposition is due to D. Bolotov.

Lemma 3.3 (D. Bolotov, [2]). *There is a foliation $\mathcal{F}$ and a metric g on $D^2 \times S^1$ such that:*

1) *$\mathcal{F}$ is parabolic with respect to g;*

2) *the foliation $\mathcal{F}|_{D^2(\frac{1}{3})\times S^1}$ is a foliation by the totally geodesic disks $D^2(\frac{1}{3})\times\{t\}$ and the foliation $\mathcal{F}|_{([\frac{2}{3},1]\times S^1)\times S^1}$ is a foliation by the totally geodesic tori $\{r\} \times S^1 \times S^1$.*

P r o o f. Consider the solid torus $D^2 \times S^1$ with the following coordinates on it:

$$D^2 \times S^1 = \{((r,\phi),t) : r \in [0,1], \phi, t \in [0, 2\pi)\}.$$

Define the one-from α on $D^2 \times S^1$ as

$$\alpha = f(r)dr + (1 - f(r))dt,$$

where $f(r)$ is such a smooth function on $[0,1]$ that

$$f(r) = \begin{cases} 0, \ r \in [0, \frac{1}{3}] \\ \text{is a strictly increasing function when } r \in (\frac{1}{3}, \frac{2}{3}] \\ 1, \ r \in (\frac{2}{3}, 1] \end{cases} .$$

This form defines a "thick" Reeb foliation $\mathcal{F}$ on $D^2 \times S^1$ (that is, there is a subset N such that $\mathcal{F}|_N$ is a Reeb foliation and $\mathcal{F}|_{D^2 \times S^1 \setminus N}$ is diffeomorphic to a product foliation by tori).

Assume that in the coordinates (r, ϕ, t) the matrix of g has a form

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & G(r) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In order to calculate the second fundamental form of $\mathcal{F}$ consider the following sections: $X = \frac{\partial}{\partial \phi}, Y = (1 - f(r))\frac{\partial}{\partial r} - f(r)\frac{\partial}{\partial t}$ of the tangent bundle $T\mathcal{F}$. Let $n = f(r)\frac{\partial}{\partial r} + (1 - f(r))\frac{\partial}{\partial t}$ be a normal vector field.

By a direct computation we obtain a matrix of the second fundamental form that is equal to

$$\frac{1}{2f(r)^2 - 2f(r) + 1} \begin{pmatrix} -f\frac{\partial G}{\partial r} & 0 \\ 0 & -(1-f)\frac{\partial f}{\partial r} \end{pmatrix}.$$

It is obvious, since $f = 0$ on $[0, \frac{1}{3})$, the foliation by disks is totally geodesic for every choice of $G = G(r)$. Define $G = G(r)$ in the following way:

$$G = \begin{cases} r^2 \text{when } r \in [0, \frac{1}{4}) \\ \text{strictly increasing when } r \in [\frac{1}{4}, \frac{1}{3}) \\ 1 \text{ when } r \in [\frac{1}{3}, 1] \end{cases} .$$

For this choice of G, the metric g is regular in the neighborhood of the core curve $r = 0$ and satisfies the conditions of the lemma.

3.3. Parabolic Foliation on $T^2 \times [0,1]$

Using the similar arguments, we may obtain the following result:

Lemma 3.4. *There exists a foliation $\mathcal{F}$ and a metric g on $T^2 \times [0,1]$ such that:*

1) $\mathcal{F}$ is parabolic with respect to g;

2) the foliation $\mathcal{F}|_{T^2\times[0,\frac{1}{3}]}$ *is a foliation by totally geodesic tori* $T^2\times\{r\}$ *and the foliation* $\mathcal{F}|_{S^1\times S^1\times[\frac{2}{3},1]}$ *is a foliation by totally geodesic annuli* $\{t\}\times S^1\times[\frac{2}{3},1]$.

P r o o f. On $T^2\times[0,1]$ define the following coordinates:

$$T^2\times[0,1]=\{((\phi,t),r):\ r\in[1,2],\phi,t\in[0,2\pi)\}$$

and consider the one-from α

$$\alpha=f(r)dr+(1-f(r))dt,$$

where $f(r)$ is such a smooth function on $[0,1]$ that

$$f(r)=\begin{cases}1,\ r\in[0,\frac{1}{3}]\\ \text{strictly decreasing, when } r\in(\frac{1}{3},\frac{2}{3}]\\ 0,\ r\in(\frac{2}{3},1]\end{cases}.$$

This form defines a foliation $\mathcal{F}$ on $T^2\times[0,1]$. Analogously as in the proof of Lemma 3.3 we may define a matrix of g (with respect to the coordinates $((\phi,t),r)$) in the form

$$g=\begin{pmatrix}G(r)&0&0\\0&1&0\\0&0&1\end{pmatrix}.$$

To calculate the second fundamental form of the leaves consider the following sections of $T\mathcal{F}$: $X=\frac{\partial}{\partial\phi}, Y=(1-f(r))\frac{\partial}{\partial r}-f(r)\frac{\partial}{\partial t}$. The field $n=f(r)\frac{\partial}{\partial r}+(1-f(r))\frac{\partial}{\partial t}$ is a normal vector field.

The second fundamental form of $\mathcal{F}$ with respect to the unit normal $\frac{n}{|n|}$ is given by

$$\frac{1}{2f(r)^2-2f(r)+1}\begin{pmatrix}-f\frac{\partial G}{\partial r}&0\\0&-(1-f)\frac{\partial f}{\partial r}\end{pmatrix}.$$

Since $f=0$ on $(\frac{2}{3},1]$,it is obvious that the foliation $\mathcal{F}_{T^2\times(\frac{2}{3},1]}$ by horizontal annuli is totally geodesic for an arbitrary choice of $G(r)$. Define $G=G(r)$ by the following formula:

$$G=\begin{cases}1,\text{when } r\in[0,\frac{2}{3})\\ \text{strictly decreasing, when } r\in[\frac{2}{3},\frac{3}{4})\\ \text{strictly increasing, when } r\in[\frac{3}{4},\frac{4}{5})\\ r^2,\text{ when } r\in[\frac{4}{5},1].\end{cases}$$

For this choice of G, a metric g satisfies all conditions of the lemma.

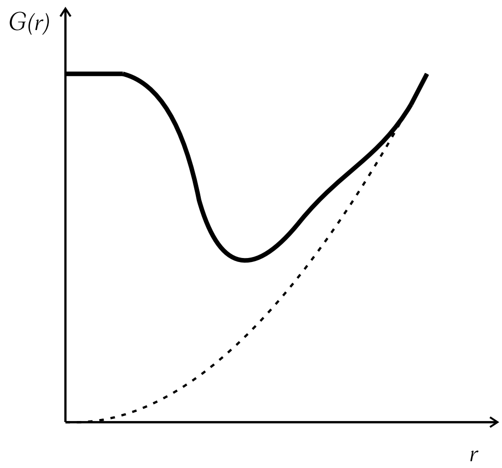

Fig. 4: The function $G(r)$ in the construction of a metric on $T^2 \times [0,1]$.

4. Parabolic Foliations on Three-Sphere

The aim of this section is to define a parabolic turbulization of $\mathcal{F}$ on S^3 along a knot K. We define this foliation in several steps. First, we define a parabolic Reeb foliation on S^3. Then we construct a special parabolic turbulization of $\mathcal{F}_R$ along the trivial link consisting of n components. For the knot K we consider its special presentation which coincides with the trivial link everywhere except double points in the frontal projection. Thus we define a parabolic foliation everywhere except some balls in S^3 containing these double points. To define a parabolic foliation inside these balls we "twist" the turbulization along a trivial link with two components. Further we will show how to glue these foliations back into the sphere to get the desired foliation.

4.1. Parabolic Reeb Foliation on S^3

Proposition 4.1 [2]. *A three-sphere admits a parabolic foliation.*

P r o o f. Take some presentation of the three-sphere $S^3 = D_1^2 \times S^1 \cup_h D_2^2 \times S^1$ as a union of two solid tori. Define the parabolic foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ inside these solid tori as in Lemma 3.3. In the coordinates (t, ϕ) on $\partial D^2 \times S^1$ the gluing diffeomorphism h is given by the matrix $h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. It is obvious that h is an isometry of the boundary torus $\partial(D_1^2 \times S^1)$. Since the metrics on the solid tori are direct product metrics and the foliations are direct product foliations in the (one-sided) neighborhoods of boundary torus, then there is a well-defined glued foliation $\mathcal{F}_R$ and the metric on S^3. This foliation is parabolic with respect to the glued metric.

4.2. Parabolic Turbulization along the Trivial Link

Proposition 4.2. *For every $n \in \mathbb{N}$ there is a parabolic foliation on S^3 with n "thick" Reeb components inside the solid torus $D^2(\frac{1}{3}) \times S^1 \subset D_1^2 \times S^1 \subset S^3$.*

P r o o f. Consider a foliation $\mathcal{F}_R$ on S^3 defined as in Proposition 4.1. Notice that the metric inside disk $D^2(\frac{1}{3}) \times \{0\}$ is a standard euclidian metric. Let $\{x_1, x_2, \ldots, x_n\}$ be a set of vertices of the regular polygon lying inside $D^2(\frac{1}{3}) \times \{0\}$ with the center at $r = 0$ and the radius of the circumscribed circle equal to $\frac{1}{8}$ with respect to the metric induced on the disk $D^2(\frac{1}{3}) \times \{0\}$. Instead of the radius $\frac{1}{8}$ we may choose any other one such that the circle with the center at the median of the side $\overline{x_i x_{i+1}}$ and the radius equal to its length would entirely lie inside the disk $D^2(\frac{1}{3})$. Take such ε that any two circles with the centers at vertices of polygon and of radius ε are disjoint. Consider a set of vertical circles $\{x_i \times S^1\}$ passing through the vertices x_i.

On the solid torus $D^2 \times S^1$ take the following coordinates:

$$D^2 \times S^1 = \{((r, \phi), t) : \ r \in [0, \varepsilon], \phi, t \in [0, 2\pi)\}$$

and consider the function f given by the formula

$$f(r) = \begin{cases} 0, \text{when } r \in [0, \frac{\varepsilon}{6}], \\ \text{strictly increasing, when } r \in (\frac{\varepsilon}{6}, \frac{\varepsilon}{3}], \\ 1, \text{when } r \in (\frac{\varepsilon}{3}, \frac{2\varepsilon}{3}], \\ \text{strictly decreasing, when } r \in (\frac{2\varepsilon}{3}, \frac{5\varepsilon}{6}], \\ 0, \text{when } r \in (\frac{5\varepsilon}{6}, \varepsilon]. \end{cases}$$

The one-form $\alpha = f(r)dr + (1 - f(r))dt$ defines a foliation $\mathcal{F}'$ on $D^2 \times S^1$.

To define a metric on $D^2 \times S^1$ consider the following function $G = G(r)$ on it:

$$G = \begin{cases} r^2, \text{when } r \in [0, \frac{\varepsilon}{8}), \\ \text{strictly increasing, when } r \in [\frac{\varepsilon}{8}, \frac{\varepsilon}{6}), \\ \epsilon, \ r \in [\frac{\varepsilon}{6}, \frac{5\varepsilon}{6}), \\ \text{strictly increasing, when } r \in [\frac{5\varepsilon}{6}, \frac{6\varepsilon}{7}], \\ r^2, \text{ when } r \in [\frac{6\varepsilon}{7}, \varepsilon]. \end{cases}$$

Define a Riemannian metric g on $D^2 \times S^1$ by the following matrix:

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & G(r) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easy to verify that $\mathcal{F}'$ is parabolic with respect to this metric.

Cut out the ϵ-tubular neighborhoods of the circles $\{x_i\} \times S^1$ and glue the solid tori $(D^2 \times S^1, \mathcal{F}')$ instead of these neighborhoods by identity map. Since

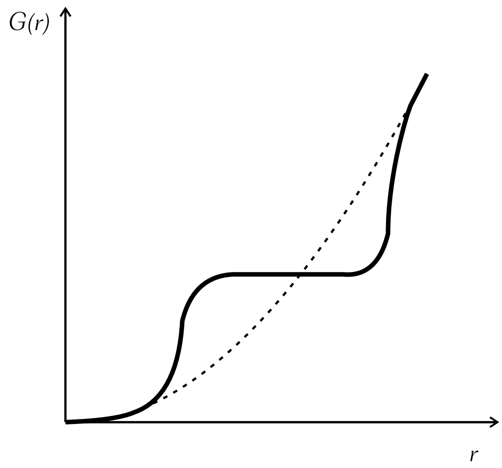

Fig. 5: Construction of metric on $D^2 \times S^1$.

$\mathcal{F}'|_{\{T^2\times[\frac{5\epsilon}{3},2\epsilon]\}}$ is a foliation by totally geodesic annuli, it glues correctly to the foliation of $D^2(\frac{1}{3}) \times S^1$ by horizontal disks.

By the construction, the metric on each $D^2 \times S^1$ may be extended by the euclidian metric on $D^2(\frac{1}{3}) \times S^1$ defined in Lemma 4.1. Denote the obtained foliation by $\mathcal{F}_n$.

Definition 4.3. *We call thus defined foliation $\mathcal{F}_n$ a trivial turbulization with n strings.*

4.3. Standard Presentation of a Knot

Assume that K is a knot in S^3. We may isotope it in such a way that K is a closure of some braid lying inside the solid torus $D^2(\frac{1}{3}) \times S^1$ and it is transverse to a foliation of $D^2(\frac{1}{3}) \times S^1$ by totally geodesic disks. Write a presentation of K as a product of transpositions $K = \sigma_1^{\pm 1}\sigma_2^{\pm 1} \dots \sigma_N^{\pm 1}$.

Without loss of generality, we may assume that in the frontal projection

$$f : D^2 \times S^1 \to [-\frac{1}{3}, \frac{1}{3}] \times S^1 \quad f(x, y, t) = (x, t)$$

there is a finite number of levels $t_1, t_2, \dots, t_N \in [-\frac{1}{3}, \frac{1}{3}]$ such that at these points K has transverse double points (each point corresponds to transposition). Now we can isotope K so that it becomes a subset of $\bigcup_{k=1}^{n}\{x_k \times S^1\}$ for some n, maybe except the neighborhoods of inverse images of double points $f^{-1}(t_i)$, $i = 1, 2, \dots, N$.

Definition 4.4. *We call such a presentation of K a standard presentation of a knot with n strings.*

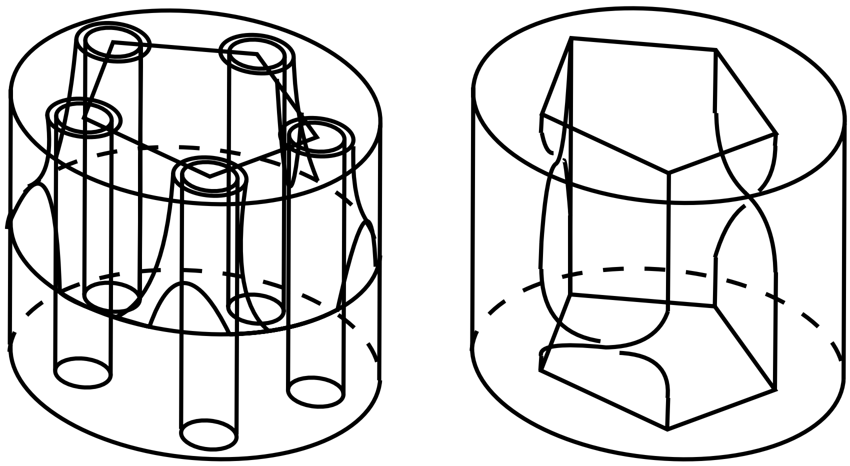

Fig. 6: Trivial turbulization with 5 strings (left). Standard presentation of the knot (right).

The trivial turbulization $\mathcal{F}_n$ with n strings coincides with the turbulization along a standard presentation of K everywhere except some balls around the inverse images of double points.

4.4. Parabolic Foliation in the Neighborhood of Transposition

To define turbulizations along transpositions consider a trivial turbulization $\mathcal{F}_2$ with two strings on $D^2(\frac{1}{3}) \times [0,1]$ (here we slightly abuse the notation and call by trivial turbulization the foliation induced on $D^2(\frac{1}{3}) \times [0,1] = \overline{D^2 \times S^1 \backslash D^2}$). The metric, where $\mathcal{F}_2$ is parabolic, denote by g defined in Proposition 4.2. Let δ be some small enough real number. Define the following bump function $f = f(r)$ on $[0, \frac{1}{3}]$:

$$f(r) = \begin{cases} \pi, \ r \in [0, \frac{1}{4} + \delta_0] \\ \text{strictly decreasing, when } r \in (\frac{1}{4} + \delta_0, \frac{1}{3} - \delta] \\ 0, \ r \in (\frac{1}{3} - \delta, \frac{1}{3}] \end{cases}$$

for some sufficiently small δ_0.

Consider a one-dimensional dynamical system $(D^2(\frac{1}{3}), \psi_t)$ generated by the flow of the vector field $X = f(r)\frac{\partial}{\partial \phi}$ on the disk $D^2(\frac{1}{3}) = \{(r, \phi) : r \in [0, \frac{1}{3}], \phi \in [0, 2\pi)\}$.

Let $h(t)$ be a smooth increasing bump function on $[0,1]$ such that $h(0) = 0$ and $h(1) = 1$. Associate with it the following diffeomorphism Φ of $D^2 \times [0,1]$:

$$\Phi(r, \phi, t) = (\psi_{h(t)}(r, \phi), t).$$

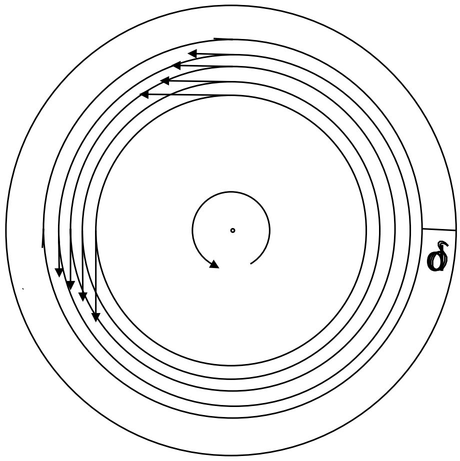

Fig. 7: A dynamical system used to define the left parabolic transposition.

This diffeomorphism defines a foliation $\Phi(\mathcal{F}_2)$ on $D^2\times[0,1]$. It is clearly parabolic with respect to the pull-back metric $(\Phi^{-1})^*g$.

Assume that $G(r,\phi)$ is a metric induced on the disk $D^2(\frac{1}{3})\times\{1\}$ by metric g, and $H(r,\phi)$ is the metric induced on this disk by $(\Phi^{-1})^*g$. We may use Lemma 3.1 to interpolate between these two metrics.

Notice that $G(r,\phi) = H(r,\phi)$ on a disk $D^2(\frac{1}{4})\times\{1\}$ since $\mathcal{F}_2$ is invariant under the rotation by π. It is also clear that $G(r,\phi) = H(r,\phi)$ on $(D^2(\frac{1}{3})\backslash D^2(\frac{1}{3}-\delta))\times\{1\}$.

Let $N = D^2(\frac{1}{3}-\delta+\delta_0)\backslash D^2(\frac{1}{4})\times[0,1]$. By Lemma 3.1 there is such a metric g on N that:

1) in some tubular neighborhood of $(D^2(\frac{1}{3}-\delta+\delta_0)\backslash D^2(\frac{1}{4}))\times\{0\}$ the metric $g(p,t) = G(p)+dt^2$ for all $p\in D^2(\frac{1}{3}-\delta+\delta_0)\backslash D^2(\frac{1}{4})$;

2) in some tubular neighborhood of $(D^2(\frac{1}{3}-\delta+\delta_0)\backslash D^2(\frac{1}{4}))\times\{1\}$ the metric $g(p,t) = H(p)+dt^2$ for all $p\in D^2(\frac{1}{3}-\delta+\delta_0)\backslash D^2(\frac{1}{4})$;

3) $\mathcal{F}$ is parabolic on N with respect to g;

4) There is a neighborhood U of the boundary $\partial(D^2(\frac{1}{3}-\delta+\delta_0)\backslash D^2(\frac{1}{4}))$ such that for all $t\in[0,1]$, $g(p,t) = G(p)+dt^2$.

On $(D^2(\frac{1}{3})\backslash D^2(\frac{1}{3}-\delta))\times[1,2]$ and $D^2(\frac{1}{4})\times[1,2]$ consider the direct product foliations. They are parabolic (even totally geodesic) with respect to the metric $ds^2 = dr^2+r^2d\phi^2+dt^2$. Since in the neighborhood of boundary ∂N the foliation is a direct product foliation and the metric is a direct product metric, then there

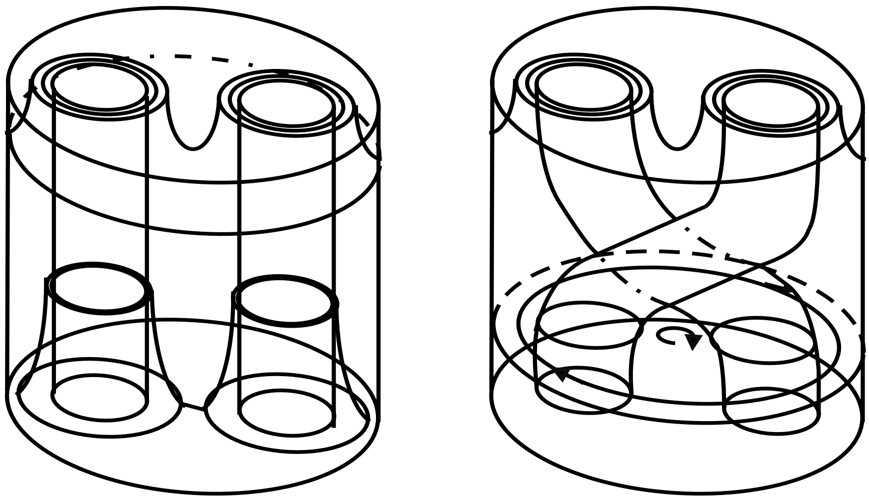

Fig. 8: A trivial parabolic transposition with two strings (left). The left parabolic transposition (right).

is a parabolic foliation correctly defined on the union $L = (D^2(\frac{1}{3}) \backslash D^2(\frac{1}{3} - \delta)) \times [1,2] \cup N \cup D^2(\frac{1}{4}) \times [1,2]$.

Finally, consider the gluing $\Phi(\mathcal{F}_2) \cup L$. It is obvious that the foliations and the metrics on L and $\Phi(\mathcal{F}_2)$ are smoothly glued with each other and they define the structure of parabolic foliation on the union. We are remained to "normalize" this foliation in the t direction. For this consider the map

$$F : D^2 \times [0,1] \to \Phi(\mathcal{F}_2) \cup L$$

defined by the formula $F((r,\phi),t) = ((r,\phi),2t)$. A foliation formed by inverse images of the leaves is parabolic in the pull-back metric F^*g.

We call the foliation obtained (together with the Riemannian metric) on $F(\Phi(\mathcal{F}_2) \cup X)$ a standard left (right) parabolic transposition.

Remark 4.5. *Notice that we cannot use Lemma* 3.1 *directly to the foliation* $\Phi(\mathcal{F}_2)$ *since it is not a foliation by disks.*

4.5. Parabolic Turbulization along K on a Three-Sphere S^3

Lemma 4.6. *For any topological type of the knot K there is a parabolic foliation $\mathcal{F}_K$ on S^3 such that $\mathcal{F}_K$ is a parabolic turbulization along K of the parabolic Reeb foliation on S^3 (see Proposition* 4.1*).*

P r o o f. Consider a standard presentation of K where n is the number of strings in it. Recall that K is a subset of the union of vertical circles $\{x_i \times S^1\}$

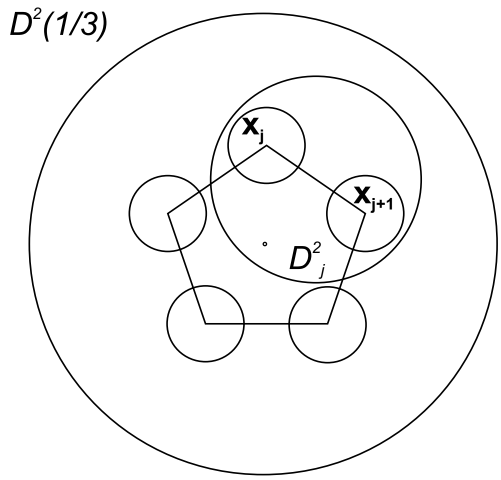

Fig. 9: A disk $D^2(\frac{1}{3}) \times \{0\}$.

everywhere except some neighborhoods of the inverse images of double points of K in the frontal projection.

In the frontal projection, let $t_1, \dots, t_N$ denote a set of t-coordinates of double points of K.

Write the presentation of K as a product $K = \sigma_1^{\pm 1} \sigma_2^{\pm 1} \dots \sigma_N^{\pm 1}$. Recall that with each vertex of the regular polygon we associated the disk with the center at the vertex and with radius ε(see p. 15). For each σ_j consider a disk D_j^2 with the center at the median of edge $\overline{x_{j-1}x_j}$ and radius $d(\overline{x_{j-1}x_j})/2 + 2\varepsilon$. We choose a disk with this radius for the small disks with the centers at vertices and of radius ε to be inside it (see Fig. 9).

Notice that since x_j are the vertices of regular polygon, the points x_{j-1} and x_j are the only points from the set $\{x_1, x_2, \dots, x_n\}$, which are inside the disk D_j^2. Consider a set of disjoint intervals $I(t_j), j = 1 \dots N \subset [0,1]$ such that $t_j \in I(t_j)$.

On S^3 define a standard turbulization with n strings and a metric (see Prop. 4.2) such that $\mathcal{F}_n$ is parabolic with respect to it. For each j let $(D^2(\frac{1}{3}) \times [0,1], \mathcal{F}', g)$ be a left (or right) standard parabolic transposition depending on a degree of corresponding σ_j. Denote the radius of D_j^2 by r_j and the length of the segment $I(t_j)$ by d_j. Consider the map

$$F_j : D^2(\frac{1}{3}) \times [0,1] \to D_j^2 \times I(t_j)$$

which is given by the formula

$$F_j((r,\phi),t) = ((\frac{3r}{r_j}, \phi), \frac{t}{d_j}).$$

This map defines a foliation $F_j(\mathcal{F}')$ inside the ball $D^2_j \times I(t_j)$. An inverse map F_j^{-1} defines on $D^2_j \times I(t_j)$ such a metric that $F_j(\mathcal{F}')$ is parabolic with respect to it. Since in the neighborhood of gluing the foliation is a direct product foliation and the metric is a direct product metric, it glues correctly to $\mathcal{F}_n$. Denote the obtained foliation by $\mathcal{F}_K$. This foliation is parabolic with respect to the glued metric.

5. Gluing the Solid Torus

P r o o f of Theorem 1.1. In order to make the proof of the theorem complete we have to perform a Dehn surgery on a knot K.

Consider the foliation $\mathcal{F}_K$ on S^3. Let N denote such a tubular neighborhood of K that $\partial N = T^2$ is a leaf of $\mathcal{F}_K$. Let $X = \overline{S^3 \backslash N}$ and consider an arbitrary diffeomorphism f

$$f \partial X \to \partial(D^2 \times S^1).$$

This diffeomorphism is defined up to isotopy by the map it induces in first homology

$$f_* : H_1(T^2) \to H_1(T^2) \quad f_* \in SL_2(\mathbb{Z}).$$

In particular, we may think that $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a linear map.

On $D^2 \times S^1$ define a parabolic foliation and a metric as in Lemma 3.3. The metric is euclidian in some neighborhood of boundary torus. Therefore f defines the following metric on ∂X:

$$G = \begin{pmatrix} a^2 + c^2 & ac + bd \\ ac + bd & b^2 + d^2 \end{pmatrix}.$$

In its turn, there is a metric $H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ on ∂X. To interpolate from G to H, consider the union $X \cup T^2 \times [0,1] \cup_f D^2 \times S^1$. By Corollary 3.2 there is a metric on $T^2 \times [0,1]$ which deforms G to H in such a way that the foliation by tori $T^2 \times \{pt\}$ is parabolic. Since in the (one-sided) neighborhoods of ∂X and $\partial(D^2 \times S^1)$ the metrics are direct product metrics, on the union $X \cup T^2 \times [0,1] \cup_f D^2 \times S^1$ we obtain a smooth Riemannian metric. As $X \cup T^2 \times [0,1] \cup_f D^2 \times S^1$ is diffeomorphic to $X \cup_f D^2 \times S^1$, every closed oriantable three-manifold admits a parabolic foliation.

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