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Plancherel Measure for the Quantum Matrix Ball-1

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The Plancherel formula is one of the celebrated results of harmonic analysis on semisimple Lie groups and their homogeneous spaces. The main goal of this work is to find a q-analogue of the Plancherel formula for spherical transform on the unit matrix ball. Here we present an explicit formula for the radial part of the Plancherel measure. The q-Jacobi polynomials as spherical functions naturally arise on the way.

Key words: quantum matrix ball, Plancherel formula, spherical functions, difference operators.

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1. Introduction

Let us recall one of the most common problems of harmonic analysis on homogenous spaces. Let G be a real Lie group, K be a closed subgroup and $d\nu$ be a G-invariant Haar measure on $X = K\backslash G$. The representation of G by right shifts in $L^2(X, d\nu)$

$$R(g) : f(x) \mapsto f(xg), \qquad x \in X, g \in G,$$

is strongly continuous and unitary. It is called a quasiregular representation. The problem is to find a decomposition of R into irreducible representations.

A special case of the Riemannian symmetric space $X = K\backslash G$ and its isometry group was studied in detail ([9, p. 192], [10, p. 506]). Harmonic analysis for these spaces was developed by E. Cartan, I. Gelfand, F. Berezin, Harish-Chandra, S. Gindikin and F. Karpelevich.

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The problem of harmonic analysis is closely connected with the following. Consider the algebra $D_G(X)$ of all G-invariant differential operators on X. An important result of the representation theory is that the decomposition of R can be obtained by using common eigenfunctions of operators from $D_G(X)$. Namely, the shifts of a common eigenfunction generate an irreducible subrepresentation of R.

In the case of a Riemannian symmetric space the algebra $D_G(X)$ is finitely generated and commutative ([9, p. 431]). Consider a set of generators of $D_G(X)$ and their restrictions $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_r$ onto the subspace of smooth K-invariant functions on X with compact support. The Plancherel measure is a Borel measure in r-dimensional space, and the problem is to find this measure.

Let us describe briefly how to find common eigenfunctions. Recall that an irreducible strongly continuous unitary representation T is called a representation of type I if it contains a nonzero K-invariant vector v. We can assume that $(v, v) = 1$. The function $f(g) = (T(g)v, v)$ is called a spherical function. It is constant on double cosets $K\backslash G/K$, so it corresponds to a K-invariant function on X. This function is a common eigenfunction of the operators $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_r$. The problem of the decomposition of K-biinvariant functions on G in terms of the spherical functions naturally arises while solving the general decomposition problem of $L^2(X, d\nu)$.

Consider a more special case. The homogeneous space $SU_{n,n}/S(U_n \times U_n)$ is a Hermitian symmetric space of noncompact type. It has the standard Harish-Chandra realization as the unit ball

$$\mathbb{D} = \{z \in \mathrm{Mat}_n \mid ||z|| < I\}$$

(in the space of complex $n \times n$-matrices with respect to the operator norm). It worth to be mentioned that standard generators of $D_G(X)$ are well known and their common eigenfunctions are Jacobi polynomials [7, 11].

Quantum bounded symmetric domains were introduced in 1998 by L. Vaksman and S. Sinel'shchikov [24]. L. Vaksman with his collaborators managed to develop the noncommutative complex analysis and representation theory on quantum domains. A series of works were dedicated to quantum matrix balls that are the simplest examples of quantum bounded symmetric domains [3, 21, 22, 29, 30].

For the case of quantum disk some problems of noncommutative harmonic analysis are solved [15, 16]. In particular, explicit formulas for the invariant integral, spherical functions, and the Plansherel measure are obtained.

In this paper we generalize the results mentioned above for the quantum matrix ball case. Imitating the classical approach, we construct a family of commuting 'q-differential' operators and find the exact formula for their common eigenfunctions. We use spherical functions which appeare to be q-Jacobi polynomials. We obtain the decomposition of the biinvariant functions in terms of the spherical functions and the exact formula for the 'radial part' of the Plancherel measure.

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2. The Radial Part of the $U_q\mathfrak{sl}_{2n}$-invariant Integral

2.1. Preliminaries on the quantum matrix ball

All results from the next two subsections form the basic notions in the theory of quantum bounded symmetric domains. We refer to [21, 24] for the first appearance and full consideration of these notions.

Let $q \in (0,1)$. All algebras are assumed to be associative and unital, and $\mathbb{C}$ is the ground field.

Consider the well-known quantum universal enveloping algebra (see e.g. [13]) $U_q\mathfrak{sl}_{2n}$ corresponding to the Lie algebra $\mathfrak{sl}_{2n}$. Recall that $U_q\mathfrak{sl}_{2n}$ is a Hopf algebra with the generators $\{E_i,\ F_i,\ K_i,\ K_i^{-1}\}_{i=1}^{2n-1}$ and the relations

$$\begin{gathered}
K_iK_j \;=\; K_jK_i, \quad K_iK_i^{-1} \;=\; K_i^{-1}K_i \;=\; 1;\\
K_iE_i \;=\; q^2E_iK_i, \quad K_iF_i \;=\; q^{-2}F_iK_i;\\
K_iE_j \;=\; q^{-1}E_jK_i, \quad K_iF_j \;=\; qF_jK_i, \quad |i-j|=1;\\
K_iE_j \;=\; E_jK_i, \quad K_iF_j \;=\; F_jK_i, \quad |i-j|\;>\;1;\\
E_iF_j \;-\; F_jE_i \;=\; \delta_{ij}\frac{K_i-K_i^{-1}}{q-q^{-1}};\\
E_i^2E_j \;-\; (q+q^{-1})E_iE_jE_i \;+\; E_jE_i^2 \;=\; 0, \quad |i-j| \;=\; 1;\\
F_i^2F_j \;-\; (q+q^{-1})F_iF_jF_i \;+\; F_jF_i^2 \;=\; 0, \quad |i-j| \;=\; 1;\\
E_iE_j - E_jE_i \;=\; F_iF_j - F_jF_i \;=\; 0, \quad |i-j| \;>\; 1.
\end{gathered}$$

The coproduct, the counit, and the antipode are defined as follows:

$$\begin{aligned}
&\triangle E_j = E_j\otimes 1 + K_j\otimes E_j, && \varepsilon(E_j)=0, && S(E_j) = -K_j^{-1}E_j,\\
&\triangle F_j = F_j\otimes K_j^{-1} + 1\otimes F_j, && \varepsilon(F_j)=0, && S(F_j) = -F_jK_j,\\
&\triangle K_j = K_j\otimes K_j, && \varepsilon(K_j)=1, && S(K_j) = K_j^{-1}, \qquad j=1,\dots,2n-1.
\end{aligned}$$

Equip the Hopf algebra $U_q\mathfrak{sl}_{2n}$ with the involution $*$:

$$(K_j^{\pm1})^* = K_j^{\pm1}, \quad E_j^* = \begin{cases} K_jF_j, & j\neq n,\\ -K_jF_j, & j=n,\end{cases} \qquad F_j^* = \begin{cases} E_jK_j^{-1}, & j\neq n,\\ -E_jK_j^{-1}, & j=n.\end{cases}$$

Then $U_q\mathfrak{su}_{n,n} \stackrel{\mathrm{def}}{=} (U_q\mathfrak{sl}_{2n}, *)$ is a $*$-Hopf algebra. It is a quantum analogue of the algebra $U\mathfrak{su}_{n,n} \otimes_{\mathbb{R}} \mathbb{C}$, where $\mathfrak{su}_{n,n}$ stands for the Lie algebra of the noncompact real Lie group $SU_{n,n}$.

Let $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n) \subset U_q\mathfrak{sl}_{2n}$ denote the Hopf subalgebra generated by E_j, F_j, $j \neq n$, and $K_i, K_i^{-1}, i = 1, \dots, 2n-1$. The corresponding $*$-Hopf subalgebra in $U_q\mathfrak{su}_{n,n}$ is denoted by $U_q\mathfrak{s}(\mathfrak{u}_n \times \mathfrak{u}_n)$.

Recall an important definition of the weight module. A $U_q\mathfrak{sl}_{2n}$-module V is called a weight one if

$$V = \bigoplus_{\lambda \in P} V_\lambda, \qquad V_\lambda = \left\{ v \in V \;\middle|\; K_i v = q^{\lambda_i} v, \quad i = 1, 2, \dots, 2n-1 \right\},$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{2n-1})$ and P is the weight lattice of the Lie algebra $\mathfrak{sl}_{2n}$. Nonzero summand V_λ is called a weight subspace of weight λ.

Further, all $U_q\mathfrak{sl}_{2n}$-modules are assumed to be the weight ones what allows us to introduce the linear operators H_j, $j = 1, \dots, 2n-1$, in V such that

$$H_j v = \eta_j v, \qquad v \in V_\eta.$$

Therefore, one can formally consider

$$K_i^{\pm 1} = q^{\pm H_i}.$$

We recall a definition of the $*$-algebra $\mathrm{Pol}(\mathrm{Mat}_n)_q$ from [21]. First, let $\mathbb{C}[\mathrm{Mat}_n]_q$ denote the well-known algebra with the generators z_a^α, a, $\alpha = 1, \dots, n$, and the relations

$$z_a^\alpha z_b^\beta - q z_b^\beta z_a^\alpha = 0, \qquad a = b \;\;\&\;\; \alpha < \beta, \quad \text{or} \quad a < b \;\;\&\;\; \alpha = \beta, \tag{1}$$

$$z_a^\alpha z_b^\beta - z_b^\beta z_a^\alpha = 0, \qquad \alpha < \beta \;\;\&\;\; a > b, \tag{2}$$

$$z_a^\alpha z_b^\beta - z_b^\beta z_a^\alpha - (q - q^{-1}) z_a^\beta z_b^\alpha = 0, \quad \alpha < \beta \;\;\&\;\; a < b. \tag{3}$$

The algebra $\mathbb{C}[\mathrm{Mat}_n]_q$ is called the algebra of holomorphic polynomials on the quantum n-matrices space (see [13]).

Similarly, let $\mathbb{C}[\overline{\mathrm{Mat}}_n]_q$ denote the algebra with the generators $(z_a^\alpha)^*$, a, $\alpha = 1, \dots, n$, and the relations

$$(z_b^\beta)^*(z_a^\alpha)^* - q(z_a^\alpha)^*(z_b^\beta)^* = 0, \; a = b \;\;\&\;\; \alpha < \beta, \quad \text{or} \quad a < b \;\;\&\;\; \alpha = \beta, \tag{4}$$

$$(z_b^\beta)^*(z_a^\alpha)^* - (z_a^\alpha)^*(z_b^\beta)^* = 0, \quad \alpha < \beta \;\;\&\;\; a > b, \tag{5}$$

$$(z_b^\beta)^*(z_a^\alpha)^* - (z_a^\alpha)^*(z_b^\beta)^* - (q - q^{-1})(z_b^\alpha)^*(z_a^\beta)^* = 0, \quad \alpha < \beta \;\;\&\;\; a < b. \tag{6}$$

Moreover, let $\mathbb{C}[\mathrm{Mat}_n \oplus \overline{\mathrm{Mat}}_n]_q$ denote the algebra with the generators z_a^α, $(z_a^\alpha)^*$, $a, \alpha = 1, \ldots, n$, relations (1)–(6), and additional relations

$$(z_b^\beta)^* z_a^\alpha = q^2 \sum_{a',b'=1}^{n} \sum_{\alpha',\beta'=1}^{n} R(b,a,b',a') R(\beta,\alpha,\beta',\alpha') z_{a'}^{\alpha'} \left(z_{b'}^{\beta'}\right)^* + (1-q^2)\delta_{ab}\delta^{\alpha\beta},$$

where δ_{ab}, $\delta^{\alpha\beta}$ are Kronecker symbols and

$$R(j,i,j',i') = \begin{cases} q^{-1}, & i \neq j \ \& \ j = j' \ \& \ i = i', \\ 1, & i = j = i' = j', \\ -(q^{-2}-1), & i = j \ \& \ i' = j' \ \& \ i' > i, \\ 0, & \text{otherwise.} \end{cases}$$

Finally, let $\mathrm{Pol}(\mathrm{Mat}_n)_q \overset{\mathrm{def}}{=} (\mathbb{C}[\mathrm{Mat}_n \oplus \overline{\mathrm{Mat}}_n]_q, *)$ be the $*$-algebra with the natural involution: $* : z_a^\alpha \mapsto (z_a^\alpha)^*$. The algebra $\mathrm{Pol}(\mathrm{Mat}_n)_q$ is called the algebra of polynomials on the quantum n-matrices space (see [13]).

We now recall an irreducible $*$-representation of $\mathrm{Pol}(\mathrm{Mat}_n)_q$ in a pre-Hilbert space. Let $\mathcal{H}$ denote the $\mathrm{Pol}(\mathrm{Mat}_n)_q$-module with one generator v_0 and the defining relations

$$(z_a^\alpha)^* v_0 = 0, \quad a, \alpha = 1, \ldots, n.$$

Let T_F denote the representation of $\mathrm{Pol}(\mathrm{Mat}_n)_q$ which corresponds to $\mathcal{H}$. It is called the Fock representation. All statements of the following proposition are proved in [21].

Proposition 1. *1.* $\mathcal{H} = \mathbb{C}[\mathrm{Mat}_n]_q v_0$.

2. $\mathcal{H}$ *is a simple* $\mathrm{Pol}(\mathrm{Mat}_n)_q$*-module.*

3. There exists a unique sesquilinear form $(\cdot,\cdot)$ *on* $\mathcal{H}$ *with the following properties:*
i) $(v_0, v_0) = 1$*; ii)* $(fv, w) = (v, f^* w)$ *for all* $v, w \in \mathcal{H}$*,* $f \in \mathrm{Pol}(\mathrm{Mat}_n)_q$.

4. The form $(\cdot,\cdot)$ *is positive definite on* $\mathcal{H}$*.*

Also it is proved in [21] that $\mathrm{Pol}(\mathrm{Mat}_n)_q$ is a $U_q\mathfrak{su}_{n,n}$-module algebra*. The action of the generators of $U_q\mathfrak{su}_{n,n}$ is given by the formulae

$$H_n z_a^\alpha = \begin{cases} 2z_a^\alpha, & a = n \ \& \ \alpha = n, \\ z_a^\alpha, & a = n \ \& \ \alpha \neq n \quad \text{or} \quad a \neq n \ \& \ \alpha = n, \\ 0, & \text{otherwise,} \end{cases}$$

*I.e., the multiplication in $\mathrm{Pol}(\mathrm{Mat}_n)_q$ is a morphism of $U_q\mathfrak{su}_{n,n}$-modules, and the involutions in $\mathrm{Pol}(\mathrm{Mat}_n)_q$ and $U_q\mathfrak{su}_{n,n}$ are compatible.

$$F_n z_a^\alpha = q^{1/2} \cdot \begin{cases} 1, & a = n \;\&\; \alpha = n, \\ 0, & \text{otherwise}, \end{cases}$$

$$E_n z_a^\alpha = -q^{1/2} \cdot \begin{cases} q^{-1} z_a^n z_n^\alpha, & a \neq n \;\&\; \alpha \neq n, \\ (z_n^n)^2, & a = n \;\&\; \alpha = n, \\ z_n^n z_a^\alpha, & \text{otherwise}, \end{cases}$$

for all $a = 1, \ldots, n$; $\alpha = 1, \ldots, n$, and with $k \neq n$

$$H_k z_a^\alpha = \begin{cases} z_a^\alpha, & k < n \;\&\; a = k \quad \text{or} \quad k > n \;\&\; \alpha = 2n - k, \\ -z_a^\alpha, & k < n \;\&\; a = k+1 \quad \text{or} \quad k > n \;\&\; \alpha = 2n - k + 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$F_k z_a^\alpha = q^{1/2} \cdot \begin{cases} z_{a+1}^\alpha, & k < n \;\&\; a = k, \\ z_a^{\alpha+1}, & k > n \;\&\; \alpha = 2n - k, \\ 0, & \text{otherwise}, \end{cases}$$

$$E_k z_a^\alpha = q^{-1/2} \cdot \begin{cases} z_{a-1}^\alpha, & k < n \;\&\; a = k+1, \\ z_a^{\alpha-1}, & k > n \;\&\; \alpha = 2n - k + 1, \\ 0, & \text{otherwise}. \end{cases}$$

Let

$$\Lambda_n = \{(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}_+^n \mid \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n\}$$

be the set of partitions of the length not larger than n. Similarly to the classical case, one obtains the decomposition $\mathbb{C}[\mathrm{Mat}_n]_q = \bigoplus_{\lambda \in \Lambda_n} \mathbb{C}[\mathrm{Mat}_n]_{q,\lambda}$ into a sum of $U_q\mathfrak{s}(\mathfrak{u}_n \times \mathfrak{u}_n)$-isotypic components, where $\mathbb{C}[\mathrm{Mat}_n]_{q,\lambda}$ is a simple $U_q\mathfrak{s}(\mathfrak{u}_n \times \mathfrak{u}_n)$-module with highest weight

$$(\lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n, 2\lambda_n, \lambda_{n-1} - \lambda_n, \ldots, \lambda_1 - \lambda_2).$$

This decomposition gives rise to the decomposition

$$\mathcal{H} = \bigoplus_{\lambda \in \Lambda_n} \mathcal{H}_\lambda, \qquad \mathcal{H}_\lambda = \mathbb{C}[\mathrm{Mat}_n]_{q,\lambda} v_0.$$

Recall a quantum analogue of the Harish-Chandra embedding of the Hermitian symmetric space $S(U_n \times U_n)\backslash SU_{n,n} \hookrightarrow \mathrm{Mat}_n$. Let $\mathbb{C}[SL_{2n}]_q$ denote the well-known Hopf algebra with the generators $\{t_{ij}\}_{i,j=1,\ldots,2n}$ and the relations

$$\begin{aligned} & t_{\alpha a} t_{\beta b} - q t_{\beta b} t_{\alpha a} = 0, && a = b \;\&\; \alpha < \beta, \text{ or } a < b \;\&\; \alpha = \beta, \\ & t_{\alpha a} t_{\beta b} - t_{\beta b} t_{\alpha a} = 0, && \alpha < \beta \;\&\; a > b, \\ & t_{\alpha a} t_{\beta b} - t_{\beta b} t_{\alpha a} - (q - q^{-1}) t_{\beta a} t_{\alpha b} = 0, && \alpha < \beta \;\&\; a < b, \\ & \det\nolimits_q \mathbf{t} = 1. \end{aligned}$$

Here $\det_q \mathbf{t}$ is the q-determinant of the matrix $\mathbf{t} = (t_{ij})_{i,j=1,\dots,2n}$ defined by

$$\det_q \mathbf{t} \stackrel{\text{def}}{=} \sum_{s \in S_{2n}} (-q)^{l(s)} t_{1\,s(1)} t_{2\,s(2)} \dots t_{2n\,s(2n)}$$

with $l(s) = \operatorname{card}\{(i,j) |\ i < j \ \&\ s(i) > s(j)\}$. The comultiplication Δ, the counit ε, and the antipode S are defined as follows:

$$\Delta(t_{ij}) = \sum_k t_{ik} \otimes t_{kj}, \qquad \varepsilon(t_{ij}) = \delta_{ij}, \qquad S(t_{ij}) = (-q)^{i-j} \det_q \mathbf{t}_{ji},$$

where $\mathbf{t}_{ji}$ is the matrix derived from $\mathbf{t}$ by discarding its j-th row and its i-th column.

We equip $\mathbb{C}[SL_{2n}]_q$ with the standard $U_q\mathfrak{sl}_{2n}$-module algebra structure as follows (see [21]): for $k = 1, \dots, 2n-1$,

$$E_k \cdot t_{ij} = q^{-1/2} \begin{cases} t_{i\,j-1}, & k = j-1, \\ 0, & \text{otherwise}, \end{cases} \qquad F_k \cdot t_{ij} = q^{1/2} \begin{cases} t_{i\,j+1}, & k = j, \\ 0, & \text{otherwise}, \end{cases} \tag{7}$$

$$K_k \cdot t_{ij} = \begin{cases} q t_{ij}, & k = j, \\ q^{-1} t_{ij}, & k = j-1, \\ t_{ij}, & \text{otherwise}. \end{cases} \tag{8}$$

Denote by $U_q\mathfrak{sl}_{2n}^{\text{op}}$ the Hopf algebra obtained from $U_q\mathfrak{sl}_{2n}$ by changing the multiplication to the opposite one. We can also equip $\mathbb{C}[SL_{2n}]_q$ with a $U_q\mathfrak{sl}_{2n}^{\text{op}}$-module algebra structure as follows: for $k = 1, \dots, 2n-1$,

$$E_k \cdot t_{ij} = q^{-1/2} \begin{cases} t_{i+1\,j}, & k = i, \\ 0, & \text{otherwise}, \end{cases} \qquad F_k \cdot t_{ij} = q^{1/2} \begin{cases} t_{i-1\,j}, & k = i+1, \\ 0, & \text{otherwise}, \end{cases}$$

$$K_k \cdot t_{ij} = \begin{cases} q t_{ij}, & k = i, \\ q^{-1} t_{ij}, & k = i+1, \\ t_{ij}, & \text{otherwise}. \end{cases}$$

So, $\mathbb{C}[SL_{2n}]_q$ is a $U_q\mathfrak{sl}_{2n}^{\text{op}} \otimes U_q\mathfrak{sl}_{2n}$-module algebra (see [21]). The subalgebra

$$\begin{aligned} &\mathbb{C}[SL_{2n}]_q^{(U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n))^{\text{op}} \otimes U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)} = \{ f \in \mathbb{C}[SL_{2n}]_q \quad | \\ &(\xi_1 \otimes \xi_2) f = \varepsilon(\xi_1)\varepsilon(\xi_2) f, \qquad \xi_1 \in U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)^{\text{op}},\ \xi_2 \in U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)\} \end{aligned} \tag{9}$$

will be referred as the subalgebra of $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-biinvariants.

Equip $\mathbb{C}[SL_{2n}]_q$ with the involution given by

$$t_{ij}^* = \operatorname{sign}[(i - n - 1/2)(n - j + 1/2)](-q)^{j-i} \det_q \mathbf{t}_{ij}.$$

It can be proved that $\mathbb{C}[w_0SU_{n,n}]_q \stackrel{\mathrm{def}}{=} (\mathbb{C}[SL_{2n}]_q, *)$ is a $U_q\mathfrak{su}_{n,n}$-module $*$-algebra. It is a q-analogue of the algebra of regular functions on the real affine algebraic manifold $w_0SU_{n,n}$, where*

$$w_0 = \begin{pmatrix} 0 & -J \\ J & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 0 & ... & 0 & 1 \\ 0 & 0 & ... & 1 & 0 \\ & & ... & & \\ 0 & 1 & ... & 0 & 0 \\ 1 & 0 & ... & 0 & 0 \end{pmatrix}.$$

For any multiindices $I = \{1 \leq i_1 < i_2 < \ldots < i_k \leq 2n\}$ and $J = \{1 \leq j_1 < j_2 < \ldots < j_k \leq 2n\}$ we use the following standard notation for the corresponding q-minor of the matrix $\mathbf{t}$:

$$t^{\wedge k}_{IJ} \stackrel{\mathrm{def}}{=} \sum_{s \in S_k} (-q)^{l(s)} t_{i_1 j_{s(1)}} t_{i_2 j_{s(2)}} \ldots t_{i_k j_{s(k)}}.$$

We now introduce a short notation for the elements

$$t = t^{\wedge n}_{\{1,2,\ldots,n\}\{n+1,n+2,\ldots,2n\}}, \qquad x = tt^*. \tag{10}$$

Note that t, t^*, and x quasicommute with all generators t_{ij} of $\mathbb{C}[SL_{2n}]_q$, and that $\mathbb{C}[w_0SU_{n,n}]_q$ is an integral domain (see [12]). Let $\mathbb{C}[w_0SU_{n,n}]_{q,x}$ be the localization of $\mathbb{C}[w_0SU_{n,n}]_q$ with respect to the multiplicative set $x^{\mathbb{Z}_+}$ (see [6]). The following statements are proved in [21].

Proposition 2. *There exists a unique extension of the $U_q\mathfrak{su}_{n,n}$-module $*$-algebra structure from $\mathbb{C}[w_0SU_{n,n}]_q$ to $\mathbb{C}[w_0SU_{n,n}]_{q,x}$.*

Proposition 3. *There exists a unique embedding of the $U_q\mathfrak{su}_{n,n}$-module $*$-algebras*

$$i : \mathrm{Pol}(\mathrm{Mat}_n)_q \hookrightarrow \mathbb{C}[w_0SU_{n,n}]_{q,x}$$

such that

$$i(z^\alpha_a) = t^{-1} t^{\wedge n}_{\{1,2,\ldots,n\}J_{a\alpha}},$$

where $J_{a\alpha} = \{n+1, n+2, \ldots, 2n\} \setminus \{2n+1-\alpha\} \cup \{a\}$.

The last proposition gives us a q-analogue of the Harish-Chandra embedding. It allows us to identify $\mathrm{Pol}(\mathrm{Mat}_n)_q$ with its image in $\mathbb{C}[w_0SU_{n,n}]_{q,x}$.

*The matrix w_0 corresponds to the longest element of the Weyl group of the Lie algebra $\mathfrak{sl}_{2n}$.

2.2. The algebra of finite functions and the invariant integral

It is well known that in the classical case $q = 1$ a positive definite $\mathrm{SU}_{n,n}$-invariant integral can not be defined on the polynomial algebra in the unit ball $\mathbb{D} \hookrightarrow \mathrm{Mat}_n$. However, it is well defined on the space of smooth functions with compact support on $\mathbb{D}$. These observations are still applicable for the quantum case. Here we provide the definition and some basic properties of a q-analogue of the algebra of finite functions following [19].

Let us consider a $U_q\mathfrak{su}_{n,n}$-module $*$-algebra $\mathrm{Fun}(\mathbb{D})_q$ obtained from $\mathrm{Pol}(\mathrm{Mat}_n)_q$ by adding a generator f_0 and the relations

$$f_0 = f_0^2 = f_0^*,$$

$$(z_a^\alpha)^* f_0 = 0, \quad f_0 z_a^\alpha = 0, \qquad a, \alpha = 1, 2, \ldots, n.$$

The $U_q\mathfrak{su}_{n,n}$-module algebra structure can be extended from $\mathrm{Pol}(\mathrm{Mat}_n)_q$ to $\mathrm{Fun}(\mathbb{D})_q$ as follows:

$$H_n f_0 = 0, \qquad F_n f_0 = -\frac{q^{1/2}}{q^{-2}-1} f_0 (z_n^n)^*, \qquad E_n f_0 = -\frac{q^{1/2}}{1-q^2} z_n^n f_0,$$

$$H_k f_0 = F_k f_0 = E_k f_0 = 0, \qquad k \neq n.$$

The two-sided ideal $\mathcal{D}(\mathbb{D})_q = \mathrm{Pol}(\mathrm{Mat}_n)_q f_0 \mathrm{Pol}(\mathrm{Mat}_n)_q$ is a $U_q\mathfrak{su}_{n,n}$-module $*$-subalgebra (see [19]). The elements of the two-sided ideal $\mathcal{D}(\mathbb{D})_q$ will be called finite functions on the quantum matrix ball $\mathbb{D}$.

The Fock representation T_F of $\mathrm{Pol}(\mathrm{Mat}_n)_q$ can be extended up to the representation of $\mathrm{Fun}(\mathbb{D})_q$, and so for every finite function $f \in \mathcal{D}(\mathbb{D})_q$ there exists an operator $T_F(f)$, and

$$T_F(\mathcal{D}(\mathbb{D})_q) = \{A \in \mathrm{End}(\mathcal{H}) \mid A|_{\mathcal{H}_\lambda} \neq 0 \text{ for a finite set of indices } \lambda \in \Lambda_n\}.$$

Consider the gradings

$$\mathbb{C}[\mathrm{Mat}_n]_{q,k} = \bigoplus_{|\lambda|=k} \mathbb{C}[\mathrm{Mat}_n]_{q,\lambda}, \qquad k \in \mathbb{Z}_+,$$

and

$$\mathbb{C}[\overline{\mathrm{Mat}_n}]_{q,-k} = \bigoplus_{|\lambda|=k} \mathbb{C}[\overline{\mathrm{Mat}_n}]_{q,\lambda}, \qquad k \in \mathbb{Z}_+,$$

where $|\lambda| = \lambda_1 + \lambda_2 + \ldots + \lambda_n$.

It is evident that

Lemma 1. *The Fock representation T_F has a unique extension to a representation of the $*$-algebra* $\mathrm{Fun}(\mathbb{D})_q$ *such that the element f_0 maps to the orthogonal projection onto the vacuum subspace.*

Let us keep the same notation T_F for this extension.

Proposition 4. *The representation T_F provides the isomorphism of the $*$-algebra $\mathcal{D}(\mathbb{D})_q$ and the $*$-algebra of all finite* linear operators in $\mathcal{H}$.*

P r o o f. T_F is a $*$-representation. So, we have to prove that the restriction of T_F on $\mathcal{D}(\mathbb{D})_q$ is a bijective mapping from $\mathcal{D}(\mathbb{D})_q$ to the algebra of all finite linear operators in $\mathcal{H}$.

Let $\mathcal{D}(\mathbb{D})_{q,i,j} = \mathbb{C}[\mathrm{Mat}_n]_{q,i} \cdot f_0 \cdot \mathbb{C}[\overline{\mathrm{Mat}_n}]_{q,-j}$. If $f \in \mathcal{D}(\mathbb{D})_{q,i,j}$, then the linear operator $T_F(f)$ maps $\mathcal{H}_j$ to $\mathcal{H}_i$ and it is equal to zero on $\bigoplus\limits_{k\neq j} \mathcal{H}_k$. We obtain a linear mapping from $\mathcal{D}(\mathbb{D})_{q,i,j}$ to $\mathrm{Hom}(\mathcal{H}_j, \mathcal{H}_i)$. It is surjective by Proposition 1, and

$$\dim \mathcal{D}(\mathbb{D})_{q,i,j} = \dim \mathrm{Hom}(\mathcal{H}_j, \mathcal{H}_i).$$

Thus the representation T_F provides the isomorphism

$$\mathcal{D}(\mathbb{D})_{q,i,j} = \mathbb{C}[\mathrm{Mat}_n]_{q,i}\, f_0\, \mathbb{C}[\overline{\mathrm{Mat}_n}]_{q,-j} \cong \mathrm{Hom}(\mathcal{H}_j, \mathcal{H}_i).$$

But $\mathcal{D}(\mathbb{D})_q = \bigoplus\limits_{i,j=0}^{\infty} \mathcal{D}(\mathbb{D})_{q,i,j}$, and $\bigoplus\limits_{i,j=0}^{\infty} \mathrm{Hom}(\mathcal{H}_j, \mathcal{H}_i)$ in $\mathrm{End}\,\mathcal{H}$ is the vector space of finite linear operators. ■

Proposition 5. *The representation T_F provides the bijection of the space of $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-invariants in $\mathcal{D}(\mathbb{D})_q$ and the space of finite linear operators in $\mathcal{H}$ that are scalars on every $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-isotypic component $\mathcal{H}_\lambda$, $\lambda \in \Lambda_n$.*

P r o o f.

i) If f is a $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-invariant vector, then $T_F(f)$ maps a highest vector of $\mathcal{H}_\lambda$ to a highest vector of a $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-isotypic component with the same weight.

ii) The action of $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$ in $\mathcal{H}$ is multiplicity free.

iii) Now i) and ii) imply that if f is a $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-invariant vector, then $T_F(f)|_{\mathcal{H}_\lambda}$ is an endomorphism of the simple $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-module $\mathcal{H}_\lambda$. So $T_F(f)$ is scalar on $\mathcal{H}_\lambda$, $\lambda \in \Lambda_n$. ■

*A linear operator A in $\mathcal{H}$ is called finite if $A\mathcal{H}_j = 0$ for all $j \in \mathbb{Z}_+$ except a finite set.

Denote the space of $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-invariants in $\mathcal{D}(\mathbb{D})_q$ by

$$(\mathcal{D}(\mathbb{D})_q)^{U_q\mathfrak{s}(\mathfrak{gl}_n\times\mathfrak{gl}_n)} = \{f \in \mathcal{D}(\mathbb{D})_q \quad | \quad \xi f = \varepsilon(\xi) f, \quad \xi \in U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)\}.$$

Denote

$$\check{\rho} = \frac{1}{2}\sum_{j=1}^{2n-1} j(2n-j)H_j.$$

The following proposition is also stated in [19].

Proposition 6. *The linear functional*

$$\int f d\nu = (1-q^2)^{n^2} \mathrm{tr}(T_F(f) q^{-2\check{\rho}}), \qquad f \in \mathcal{D}(\mathbb{D})_q, \tag{11}$$

is a positive definite $U_q\mathfrak{sl}_{2n}$-invariant integral on $\mathcal{D}(\mathbb{D})_q$, i.e.,

$$\int \xi f d\nu = \varepsilon(\xi) \int f d\nu, \qquad \xi \in U_q\mathfrak{sl}_{2n}$$

and

$$\int f^* f d\nu > 0, \qquad \text{for } f \neq 0.$$

For the sketch of the proof refer to [23, §5].

Further we consider a restriction of the invariant integral (11) to the space of $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-invariants in $\mathcal{D}(\mathbb{D})_q$. We will call this restriction the radial part.

2.3. The radial part of the invariant integral

In this subsection we will describe the support of radial part of the invariant measure $d\nu$ and find an exact formula for the radial part of the invariant integral.

Consider the elements of $\mathbb{C}[w_0 SU_{n,n}]_q$:

$$x_k = q^{k(k-1)} \sum_{\substack{I\subset\{1,2,\ldots,n\},\, J\subset\{n+1,n+2,\ldots,2n\} \\ \mathrm{card}(I)=\mathrm{card}(J)=k}} q^{-2\sum\limits_{m=1}^{k}(n-i_m)} (-q)^{\sum\limits_{m=1}^{k}(j_m-i_m-n)} t^{\wedge k}_{I\,J}\, t^{\wedge(2n-k)}_{I^c\,J^c}.$$

It follows from the results of [4] that x_k, $k = 1, 2, \ldots, n$ are pairwise commuting self-adjoint $U_q\mathfrak{s}(\mathfrak{u}_n \times \mathfrak{u}_n)$-biinvariants. These elements generate the subalgebra of all $U_q\mathfrak{s}(\mathfrak{u}_n \times \mathfrak{u}_n)$-biinvariant elements in $\mathbb{C}[w_0 SU_{n,n}]$, as follows from the results of [2] and [4]. So,

$$\mathbb{C}[w_0 SU_{n,n}]_q^{(U_q\mathfrak{s}(\mathfrak{u}_n\times\mathfrak{u}_n))^{\mathrm{op}}\otimes U_q\mathfrak{s}(\mathfrak{u}_n\times\mathfrak{u}_n)} \cong \mathbb{C}[x_1, x_2, \ldots, x_n].$$

Denote by T the $*$-representation of the $*$-algebra $\mathbb{C}[w_0SU_{n,n}]_q$ corresponding to the permutation

$$\begin{pmatrix} 1 & 2 & \dots & \text{n-1} & \text{n} & \text{n+1} & \text{n+2} & \dots & \text{2n-1} & \text{2n} \\ \text{n+1} & \text{n+2} & \dots & \text{2n-1} & \text{2n} & 1 & 2 & \dots & \text{n-1} & \text{n} \end{pmatrix}$$

(see [21, §4]). This representation admits a unique extension to the representation of $\mathbb{C}[w_0SU_{n,n}]_{q,x}$, where x is defined in (10). It is proved in [21, §4, 5] that the representation T_F is unitary equivalent to the restriction of the representation T to $\mathrm{Pol}(\mathrm{Mat}_n)_q$. Consider the short notation

$$q^\mu = (q^{\mu_1}, q^{\mu_2}, \dots, q^{\mu_n}) \in \mathbb{C}^n, \qquad \mu \in \mathbb{C}^n.$$

It is also proved in [4] that

$$T(x_k)|_{\mathcal{H}_\lambda} = q^{k(k-1)} e_k(q^{-2(\lambda+\delta)}), \qquad k = 1, 2, \dots, n, \quad \lambda \in \Lambda_n,$$

where $\delta = (n-1, n-2, \dots, 1, 0) \in \Lambda_n$, e_k is the elementary symmetric polynomial in n variables of degree k. So, the set of common eigenvalues of the operators $T(x_1)$, $T(x_2), \dots, T(x_n)$ is

$$\Sigma_{\mathbb{D}} = \{(e_1(q^{-2(\lambda+\delta)}), q^2 e_2(q^{-2(\lambda+\delta)}), \dots, q^{n(n-1)} e_n(q^{-2(\lambda+\delta)})) \mid \lambda \in \Lambda_n\}.$$

Thus the algebra $\mathbb{C}[w_0SU_{n,n}]_q^{(U_q\mathfrak{s}(\mathfrak{u}_n\times\mathfrak{u}_n))^{\mathrm{op}}\otimes U_q\mathfrak{s}(\mathfrak{u}_n\times\mathfrak{u}_n)}$ can be identified with the algebra of polynomial functions on $\Sigma_{\mathbb{D}}$. Following Propositions 4, 5 the algebra $\mathcal{D}(\mathbb{D})_q^{U_q\mathfrak{s}(\mathfrak{gl}_n\times\mathfrak{gl}_n)}$ can be identified with the algebra $\mathcal{D}(\Sigma_{\mathbb{D}})$ of functions $f(x_1, x_2, \dots, x_n)$ with finite support on $\Sigma_{\mathbb{D}}$.

Lemma 2. *The mapping*

$$\Lambda_n \to \Sigma_{\mathbb{D}},$$
$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \mapsto (e_1(q^{-2(\lambda+\delta)}), q^2 e_2(q^{-2(\lambda+\delta)}), \dots, q^{n(n-1)} e_n(q^{-2(\lambda+\delta)}))$$

is a bijection.

P r o o f. The surjectivity follows from the definition of $\Sigma_{\mathbb{D}}$. Let us prove the injectivity. The function q^{-l} is strictly increasing as $l \in [0, +\infty)$, so the mapping

$$\Lambda_n \to \mathbb{R}^n, \qquad \lambda \mapsto q^{-2(\lambda+\delta)}$$

is an injection. Due to the Viet theorem, the mapping

$$q^{-2(\lambda+\delta)} \mapsto (e_1(q^{-2(\lambda+\delta)}), q^2 e_2(q^{-2(\lambda+\delta)}), \dots, q^{n(n-1)} e_n(q^{-2(\lambda+\delta)}))$$

is also an injection since $q^{-2(\lambda_1+n-1)} > q^{-2(\lambda_2+n-2)} > \dots > q^{-2\lambda_n}$ for any $\lambda \in \Lambda_n$. Now we have the injectivity of the composition. ■

Consider the algebra $\mathbb{C}[u_1, u_2, \dots, u_n]$ and the injection

$$\mathbb{C}[x_1, x_2, \dots, x_n] \hookrightarrow \mathbb{C}[u_1, u_2, \dots, u_n], \qquad x_k \mapsto q^{k(k-1)} e_k(u_1, \dots, u_n), \tag{12}$$

where e_k are the elementary symmetric polynomials in n variables. This injection allows one to identify the subalgebra $\mathbb{C}[w_0 SU_{n,n}]_q^{(U_q\mathfrak{s}(\mathfrak{u}_n\times\mathfrak{u}_n))^{\mathrm{op}}\otimes U_q\mathfrak{s}(\mathfrak{u}_n\times\mathfrak{u}_n)}$ with the algebra of all symmetric polynomials in variables $u_1, u_2, \dots, u_n$.

Specify

$$\Delta_{\mathbb{D}} = \{q^{-2(\lambda+\delta)} \mid \lambda \in \Lambda_n\}.$$

Let also $\mathcal{D}(\Delta_{\mathbb{D}})$ be the algebra of functions $f(u_1, u_2, \dots, u_n)$ with finite support on the set $\Delta_{\mathbb{D}}$. Then

$$\mathcal{D}(\Delta_{\mathbb{D}}) \cong \mathcal{D}(\Sigma_{\mathbb{D}}).$$

More exactly, the bijection is as follows:

$$\mathcal{D}(\Sigma_{\mathbb{D}}) \to \mathcal{D}(\Delta_{\mathbb{D}}): \ f(x_1, x_2, \dots, x_n) \mapsto f(e_1(u), q^2 e_2(u), \dots, q^{n(n-1)} e_n(u)).$$

Thereby,

$$\mathcal{D}(\Delta_{\mathbb{D}}) \cong \mathcal{D}(\Sigma_{\mathbb{D}}) \cong \mathcal{D}(\mathbb{D})_q^{U_q\mathfrak{s}(\mathfrak{gl}_n\times\mathfrak{gl}_n)}. \tag{13}$$

In the sequel we do not distinguish between $\mathcal{D}(\Delta_{\mathbb{D}})$ and $\mathcal{D}(\mathbb{D})_q^{U_q\mathfrak{s}(\mathfrak{gl}_n\times\mathfrak{gl}_n)}$.

Recall the definition of a multiple Jackson integral with 'base' q^{-2} (see [27]):

$$\int\limits_{q^{-2(n-1)}}^{\infty} \int\limits_{q^{-2(n-2)}}^{q^2 u_n} \dots \int\limits_{1}^{q^2 u_2} \phi(u) d_{q^{-2}} u_1 \dots d_{q^{-2}} u_n \stackrel{\mathrm{def}}{=} (1-q^2)^n \sum_{\lambda\in\Lambda_n} \phi(q^{-2(\lambda+\delta)}) q^{-2|\lambda+\delta|}. \tag{14}$$

Proposition 7. *The restriction of the invariant integral (11) to the space* $\mathcal{D}(\mathbb{D})_q^{U_q\mathfrak{s}(\mathfrak{gl}_n\times\mathfrak{gl}_n)}$ *is*

$$\int f(x_1, x_2, \dots, x_n) \, d\nu$$

$$= \mathcal{N} \int\limits_{q^{-2(n-1)}}^{\infty} \int\limits_{q^{-2(n-2)}}^{q^2 u_n} \dots \int\limits_{1}^{q^2 u_2} f(e_1(u), q^2 e_2(u), \dots, q^{n(n-1)} e_n(u))$$
$$\times \Delta(u)^2 \, d_{q^{-2}} u_1 d_{q^{-2}} u_2 \dots d_{q^{-2}} u_n,$$

where $\Delta(u) = \prod\limits_{1\le i<j\le n} (u_i - u_j)$, $\mathcal{N} = (1-q^2)^{n(n-1)} q^{n(n-1)} \Delta(q^{-2\delta})^{-2}$ *is a positive constant.*

The constant $\mathcal{N}$ can be found easily by calculating the integral for the element f_0:

$$\int f_0 \, d\nu = (1-q^2)^{n^2} = \mathcal{N}(1-q^2)^n \Delta(q^{-2\delta})^2 q^{-2|\delta|}.$$

So,

$$\mathcal{N} = (1-q^2)^{n(n-1)} q^{n(n-1)} \Delta(q^{-2\delta})^{-2}. \tag{15}$$

P r o o f. Consider the integral

$$\widetilde{\eta} : f \mapsto \int\limits_{q^{-2(n-1)}}^{\infty} \int\limits_{q^{-2(n-2)}}^{q^2 u_n} \dots \int\limits_{1}^{q^2 u_2} f(e_1(u), q^2 e_2(u), \dots, q^{n(n-1)} e_n(u))$$
$$\times \Delta(u)^2 \, d_{q^{-2}} u_1 d_{q^{-2}} u_2 \dots d_{q^{-2}} u_n.$$

Let us show that the integrals η and $\widetilde{\eta}$ are equal up to a multiplicative constant on the space $\mathcal{D}(\mathbb{D})_q^{U_q \mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)}$ (the normalizing constant is calculated in (15)).

Let us compute $\widetilde{\eta}(f)$:

$$\widetilde{\eta}(f) = \text{const} \int\limits_{q^{-2(n-1)}}^{\infty} \dots \int\limits_{1}^{q^2 u_2} f(e_1(u), q^2 e_2(u), \dots, q^{n(n-1)} e_n(u)) \Delta(u)^2 \, d_{q^{-2}} u_1 \dots d_{q^{-2}} u_n$$

$$= \text{const} \sum_{\lambda \in \Lambda_n} f(e_1(q^{-2(\lambda+\delta)}), q^2 e_2(q^{-2(\lambda+\delta)}), \dots, q^{n(n-1)} e_n(q^{-2(\lambda+\delta)}))$$
$$\times \Delta(q^{-2(\lambda+\delta)})^2 \, q^{-2|\lambda+\delta|}.$$

Let us also compute $\eta(f)$:

$$\eta(f) = \text{const} \operatorname{tr}(T_F(f) q^{-2\check{\rho}}) = \text{const} \sum_{\lambda \in \Lambda_n} \operatorname{tr} \left(\, T_F(f)|_{\mathcal{H}_\lambda} \, q^{-2\check{\rho}}|_{\mathcal{H}_\lambda} \, \right)$$

$$= \text{const} \sum_{\lambda \in \Lambda_n} d_\lambda \; f(e_1(q^{-2(\lambda+\delta)}), q^2 e_2(q^{-2(\lambda+\delta)}), \dots, q^{n(n-1)} e_n(q^{-2(\lambda+\delta)})),$$

where $d_\lambda = \operatorname{tr} \left(\, q^{-2\check{\rho}}|_{\mathcal{H}_\lambda} \, \right)$. In the last computation we essentially use the fact that the operators $T_F(f)$, $f \in \mathcal{D}(\mathbb{D})_q^{U_q \mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)}$ are scalar on each $\mathcal{H}_\lambda$.

Introduce the notation

$$H_0 = nH_n + \sum_{j=1}^{n-1} jH_j + \sum_{j=1}^{n-1} jH_{2n-j},$$

then

$$-2\check{\rho} = -nH_0 - \sum_{j=1}^{n-1} j(n-j)H_j - \sum_{j=1}^{n-1} j(n-j)H_{2n-j}.$$

Consider the subalgebra in $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$ generated by $\{E_j, F_j, K_j^{\pm 1}\}_{j\neq n}$. It is isomorphic to $U_q\mathfrak{sl}_n \otimes U_q\mathfrak{sl}_n$. The restriction of the representation of $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$ in $\mathcal{H}_\lambda$ to the subalgebra $U_q\mathfrak{sl}_n \otimes U_q\mathfrak{sl}_n$ is equivalent to the representation $\pi \boxtimes \pi$, where π is the irreducible representation of $U_q\mathfrak{sl}_n$ with highest weight $(\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{n-1} - \lambda_n)$. Consequently (see [13, §7.1.4]),

$$d_\lambda = \mathrm{tr}\ (\ q^{-nH_0}|_{\mathcal{H}_\lambda})\ (\ \mathrm{tr}(\pi(q^{-2\check{\rho}^{(n)}})|_{\mathcal{H}_\lambda^{(n)}})\ )^2 = q^{-2|\lambda|}S_\lambda(q^{-2\delta})^2,$$

where $\check{\rho}^{(n)} = \sum\limits_{j=1}^{n-1} j(n-j)H_j$, and

$$S_\lambda(z_1, z_2, \ldots, z_n) = \frac{\det(z_i^{\lambda_j+j-1})_{i,j=1,2,\ldots,n}}{\det(z_i^{j-1})_{i,j=1,2,\ldots,n}}$$

is the Schur polynomial [17, §1.3]. So $S_\lambda(q^{-2\delta}) = \frac{\Delta(q^{-2(\lambda+\delta)})}{\Delta(q^{-2\delta})}$, and

$$\eta(f) = \mathrm{const} \sum_{\lambda\in\Lambda_n} d_\lambda\ f(e_1(q^{-2(\lambda+\delta)}), q^2e_2(q^{-2(\lambda+\delta)}), \ldots, q^{n(n-1)}e_n(q^{-2(\lambda+\delta)}))$$

$$= \mathrm{const} \sum_{\lambda\in\Lambda_n} q^{-2|\lambda|}\ \Delta(q^{-2(\lambda+\delta)})^2$$

$$\times f(e_1(q^{-2(\lambda+\delta)}), q^2e_2(q^{-2(\lambda+\delta)}), \ldots, q^{n(n-1)}e_n(q^{-2(\lambda+\delta)})).$$

Now it is obvious that the integrals η and $\widetilde{\eta}$ are equal up to a multiplier. ■

3. Spherical Functions on Quantum Grassmanian

Consider the involution $\star$ in $U_q\mathfrak{sl}_{2n}$ determined by

$$(K_j^{\pm 1})^\star = K_j^{\pm 1}, \qquad E_j^\star = K_jF_j, \qquad F_j^\star = E_jK_j^{-1}.$$

Then $U_q\mathfrak{su}_{2n} = (U_q\mathfrak{sl}_{2n}, \star)$ is a $*$-Hopf algebra. It is a quantum analogue of $U\mathfrak{su}_{2n} \otimes_\mathbb{R} \mathbb{C}$.

Consider also the involution $\star$ in $\mathbb{C}[SL_{2n}]_q$ determined by

$$t_{ij}^\star = (-q)^{j-i} t^{\wedge 2n-1}_{\{1,2,\ldots,2n\}\setminus\{i\},\{1,2,\ldots,2n\}\setminus\{j\}}.$$

The $*$-Hopf algebra $\mathbb{C}[SU_{2n}]_q \stackrel{\text{def}}{=} (\mathbb{C}[SL_{2n}]_q, \star)$ is a $U_q\mathfrak{su}_{2n}$-module $*$-Hopf algebra. It is a well-known quantum analogue of the algebra of regular functions on the Lie group SU_{2n} (see [31, 32]).

It is well known that in the classical case the Cartan duality between compact and noncompact Hermitian symmetric spaces allows one to predict some results of harmonic analysis in the noncompact case using the easier compact case. In this subsection we explore this observation. We construct a family of difference operators for the quantum Grassmanians. These operators are obtained using the action of the center of $U_q\mathfrak{sl}_{2n}$. Afterwards, our construction allows us to introduce difference operators in the case of quantum matrix ball.

3.1. Spherical functions

It is well known that for any finite-dimensional irreducible $U_q\mathfrak{sl}_{2n}$-module V

$$\dim V^{U_q\mathfrak{s}(\mathfrak{gl}_n\times\mathfrak{gl}_n)} \le 1.$$

Hence $(U_q\mathfrak{sl}_{2n}, U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n))$ is a "quantum Gelfand pair". As in the classical case, let us define a simple finite-dimensional weight $U_q\mathfrak{sl}_{2n}$-module to be spherical, if $\dim V^{U_q\mathfrak{s}(\mathfrak{gl}_n\times\mathfrak{gl}_n)} = 1$.

Remark 1. *It is well known ([25, Th. 4.4.1]; [26]) that a simple finite-dimensional weight $U_q\mathfrak{sl}_{2n}$-module is $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-spherical if and only if its highest weight has the following form:*

$$\widehat{\lambda} = (\lambda_1-\lambda_2, \lambda_2-\lambda_3, \ldots, \lambda_{n-1}-\lambda_n, 2\lambda_n, \lambda_{n-1}-\lambda_n, \ldots, \lambda_2-\lambda_3, \lambda_1-\lambda_2), \qquad \lambda \in \Lambda_n.$$

We will denote by L_λ the $U_q\mathfrak{sl}_{2n}$-module with highest weight $\widehat{\lambda}$.

A scalar product* $(\cdot,\cdot)$ in V is called $U_q\mathfrak{su}_{2n}$-invariant if for any $\xi \in U_q\mathfrak{sl}_{2n}$ and for any $v_1, v_2 \in V$

$$(\xi v_1, v_2) = (v_1, \xi^\star v_2).$$

Any spherical $U_q\mathfrak{sl}_{2n}$-module V can be equipped with a $U_q\mathfrak{su}_{2n}$-invariant scalar product. Fix $v \in V^{U_q\mathfrak{s}(\mathfrak{gl}_n\times\mathfrak{gl}_n)}$ by the requirement $(v,v) = 1$. Recall (see [13, §11.6.4]) that the matrix element $\varphi_V(\xi) = (\xi v, v)$ corresponding to the $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-invariant vector is called the spherical function on the quantum group SU_{2n} corresponding to V.

Thus φ_V is a $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-biinvariant element of $\mathbb{C}[SU_{2n}]_q$ such that

$$\varphi_V(1) = 1.$$

The lemma below follows from the results of [14].

*A sesquilinear positive definite Hermitian symmetric form.

Lemma 3. $(\varphi_V)^\star = \varphi_V$.

It follows from Proposition 7 of [4] and Lemma 1 of [2] that the subalgebra of $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-biinvariant functions in $\mathbb{C}[SU_{2n}]_q$ is generated by the pairwise commuting elements $x_1, x_2, \ldots, x_n$. In particular, every spherical function φ_V is a polynomial in $x_1, x_2, \ldots, x_n$. Denote by $\varphi_\lambda(x_1, x_2, \ldots, x_n)$ the spherical function corresponding to the module L_λ. In this section we will find an exact formula for $\varphi_\lambda(x_1, x_2, \ldots, x_n)$.

3.1.1. Little q-Jacobi polynomials

We will use the following partial order on Λ_n

$$\eta \le \lambda \overset{\text{def}}{\Longleftrightarrow} \sum_{j=1}^{k} \eta_j \le \sum_{j=1}^{k} \lambda_j, \qquad k = 1, 2, \ldots, n.$$

As usual, $\eta < \lambda \overset{\text{def}}{\Longleftrightarrow} \eta \le \lambda \quad \& \quad \eta \ne \lambda$.

Introduce the short notation $\mathbf{1}^k = (\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0)$. Let us denote by m_λ the monic symmetric polynomial

$$m_\lambda(z_1, z_2, \ldots, z_n) = \sum_{w \in S_n} z_{w(1)}^{\lambda_1} z_{w(2)}^{\lambda_2} \cdots z_{w(n)}^{\lambda_n}.$$

Let P_λ be a unique symmetric polynomial which satisfies the following two conditions:

$$1) \qquad P_\lambda(z) = m_\lambda(z) + \sum_{\eta<\lambda} d_{\lambda,\eta} m_\eta(z), \quad d_{\lambda,\eta} \in \mathbb{R},$$

$$2) \qquad \int_0^{q^2} \ldots \int_0^{q^2 z_2} P_\lambda(z) m_\eta(z) \Delta(z)^2 d_{q^2} z_1 \ldots d_{q^2} z_n = 0, \quad \eta < \lambda,$$

where the multiple Jackson integral (cf. (14)) is defined as

$$\int_0^{q^2} \ldots \int_0^{q^2 z_2} \phi(z) d_{q^2} z_1 \ldots d_{q^2} z_n = (1-q^2)^n \sum_{\lambda \in \Lambda_n} \phi(q^{2(\lambda+\delta+\mathbf{1}^n)}) q^{2|\lambda+\delta+\mathbf{1}^n|}.$$

Remark 2. *It is easy to see that*

$$P_\lambda(z) = P_\lambda(z; 0, 0; q^2),$$

where $P_\lambda(z; a, b; q)$ are Little q-Jacobi polynomials (see [27]).

Let $\widetilde{P}_\lambda$ be a polynomial such that

$$P_\lambda(z) = \widetilde{P}_\lambda(e_1(z), q^2 e_2(z), \ldots, q^{n(n-1)} e_n(z)).$$

From the results of Subsection 2.3 and [25, Th. 4.7.5], [26], one can deduce the following theorem:

Theorem 1. *The spherical function φ_λ is equal (up to a multiplicative constant) to*

$$\widetilde{P}_\lambda(x_1, x_2, \ldots, x_n).$$

Denote the fundamental spherical weights by

$$\mu_k = \widehat{\mathbf{1}^k}, \qquad k \in \{1, 2, \ldots, n\},$$

and denote by

$$P_+^{\text{spher}} = \bigoplus_{k=1}^{n} \mathbb{Z}_+ \mu_k = \{\widehat{\lambda} \mid \lambda \in \Lambda_n\}$$

the set of positive spherical weights, and by

$$P^{\text{spher}} = \bigoplus_{k=1}^{n} \mathbb{Z} \mu_k = \{\widehat{\lambda} \mid \lambda \in \mathbb{Z}^n\} \tag{16}$$

the set of all spherical weights.

Stokman proved the following formula in [27, Prop. 5.9]:

$$P_\lambda(z_1, z_2, \ldots, z_n) = \Delta(z)^{-1} \sum_{w \in S_n} \operatorname{sign}(w) \prod_{i=1}^{n} P_{(\lambda+\delta)_{w(i)}}(z_i),$$

where $P_m(z)$ are Little q-Jacobi polynomials in one variable.

Recall the 'coordinates' $u_1, u_2, \ldots, u_n$ appeared in (12).

Corollary 1. *Let $\lambda \in \Lambda_n$. Then*

$$\varphi_\lambda(u) = \text{const } P_\lambda(u) = \text{const} \Delta(u)^{-1} \sum_{w \in S_n} \operatorname{sign}(w) \prod_{i=1}^{n} P_{d(\lambda,w,i)}(u_i),$$

where $d(\lambda, w, i) = (\lambda + \delta)_{w(i)} \in \mathbb{Z}$.

3.2. Difference operators and the action of the center of $U_q^{\text{ext}}\mathfrak{sl}_{2n}$

Let a_{ij} be the Cartan matrix of the Lie algebra $\mathfrak{sl}_{2n}$. Denote by α_i, $i = 1, 2, \ldots, 2n-1$, the simple roots such that $\alpha_i(H_j) = a_{ji}$ and by Φ the root system of the Lie algebra $\mathfrak{sl}_{2n}$.

In this subsection we will consider the action of the center of $U_q\mathfrak{sl}_{2n}$ in weight modules. Note that it is more convenient to use the center of the extended quantum universal enveloping algebra $U_q^{\text{ext}}\mathfrak{sl}_{2n}$. Essentially, $U_q^{\text{ext}}\mathfrak{sl}_{2n}$ can be obtained from $U_q\mathfrak{sl}_{2n}$ by adding the elements

$$K_\lambda = K_1^{a_1} K_2^{a_2} \ldots K_{2n-1}^{a_{2n-1}}, \qquad \lambda = \sum_{i=1}^{2n-1} a_i \alpha_i,$$

for all λ in the weight lattice P. In particular, the action of $U_q\mathfrak{sl}_{2n}$ in any weight module admits a unique extension to the action of $U_q^{\mathrm{ext}}\mathfrak{sl}_{2n}$. Denote by $Z(U_q^{\mathrm{ext}}\mathfrak{sl}_{2n})$ the center of the extended universal enveloping algebra.

Recall some definitions, cf. [5]. Consider the real linear span $\mathfrak{h}_{\mathbb{R}}^*$ of all simple roots of the Lie algebra $\mathfrak{sl}_{2n}$. It is well known that there is a positive definite scalar product $(\cdot,\cdot)$ in $\mathfrak{h}_{\mathbb{R}}^*$. Denote by $(\mathfrak{h}_{\mathbb{R}}^*)^- \subset \mathfrak{h}_{\mathbb{R}}^*$ the real subspace spanned by the strictly orthogonal noncompact positive roots

$$\gamma_k = \alpha_k + \alpha_{k+1} + \ldots + \alpha_{2n-k-1} + \alpha_{2n-k}, \qquad k \in \{1, 2, \ldots, n\},$$

and by $(\mathfrak{h}_{\mathbb{R}}^*)^+ \subset \mathfrak{h}_{\mathbb{R}}^*$ its orthogonal complement. It is known that the orthogonal projection of the root system Φ to $(\mathfrak{h}_{\mathbb{R}}^*)^-$ is a root system of type C_n and it is called the system of restricted roots Φ^{res}. The Weyl group W^{res} of the root system Φ^{res} is called the restricted Weyl group.

Let $\mathbb{C}[P^{\mathrm{spher}}]_q$ be an algebra generated by the following functions on P^{spher}:

$$\lambda \mapsto q^{(\eta,\lambda)}, \quad \eta \in P^{\mathrm{spher}}.$$

This algebra is naturally isomorphic to the group algebra of the lattice P^{spher}. Denote by $\mathbb{C}[P^{\mathrm{spher}}]_q^{W^{\mathrm{res}}}$ the subalgebra of W^{res}-invariants in $\mathbb{C}[P^{\mathrm{spher}}]_q$:

$$\mathbb{C}[P^{\mathrm{spher}}]_q^{W^{\mathrm{res}}} = \{f \in \mathbb{C}[P^{\mathrm{spher}}]_q \,|\, f(w\lambda) = f(\lambda) \,\text{for all}\, w \in W^{\mathrm{res}},\, \lambda \in P^{\mathrm{spher}}\}.$$

Here we provide a well-known description of the image of the center $Z(U_q^{\mathrm{ext}}\mathfrak{sl}_{2n})$ under the Harish-Chandra homomorphism $\gamma^{\mathrm{spher}} : Z(U_q^{\mathrm{ext}}\mathfrak{sl}_{2n}) \to \mathbb{C}[P^{\mathrm{spher}}]_q$ (see [1]).

Proposition 8. *The image of $Z(U_q^{\mathrm{ext}}\mathfrak{sl}_{2n})$ under the Harish-Chandra homomorphism is the subalgebra $\mathbb{C}[P^{\mathrm{spher}}]_q^{W^{\mathrm{res}}}$.*

Set for $\lambda \in \mathbb{C}^n$

$$a(\lambda + \delta) \stackrel{\mathrm{def}}{=} (a(\lambda_1 + n - 1), a(\lambda_2 + n - 2), \ldots, a(\lambda_n)),$$

where

$$a(l) = \frac{(1 - q^{-2l})(1 - q^{2l+2})}{(1 - q^2)^2}, \qquad l \in \mathbb{C}. \tag{17}$$

Proposition 9. *There are the elements $C_k \in Z(U_q^{\mathrm{ext}}\mathfrak{sl}_{2n})$, $k = 1, 2, \ldots, n$, such that*

$$C_k \varphi_\lambda = e_k(a(\lambda + \delta))\varphi_\lambda, \qquad \lambda \in \Lambda_n. \tag{18}$$

P r o o f. Consider the mapping

$$\Lambda_n \to \mathbb{R}^n, \qquad \lambda \mapsto \eta(\lambda) = (\lambda_1 - \frac{2n-1}{2}, \lambda_2 - \frac{2n-3}{2}, \ldots, \lambda_{n-1} - \frac{3}{2}, \lambda_n - \frac{1}{2}).$$

Then

$$\widehat{\eta(\lambda)} = \widehat{\lambda} - \rho,$$

where

$$\widehat{\lambda} = (\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{n-1} - \lambda_n, 2\lambda_n, \lambda_{n-1} - \lambda_n, \ldots, \lambda_2 - \lambda_3, \lambda_1 - \lambda_2) \in P.$$

We need the following functions on P^{spher}:

$$\psi_k : \widehat{\lambda} \mapsto e_k(a(\eta(\lambda) + \delta)), \qquad k \in \{1, 2, \ldots, n\}. \tag{19}$$

$\lambda \in \mathbb{Z}^n$ is uniquely defined by the spherical weight $\widehat{\lambda} \in P^{\mathrm{spher}}$, see (16).

Due to Proposition 8 we only need to check the W^{res}-invariance of the functions ψ_k.

It is easy to see that

$$e_k(a(\eta(\lambda) + \delta)) = (1 - q^2)^{-2k}$$

$$\times e_k((1-q^{-2\lambda_1+1})(1-q^{2\lambda_1+1}), (1-q^{-2\lambda_2+1})(1-q^{2\lambda_2+1}), \ldots, (1-q^{-2\lambda_n+1})(1-q^{2\lambda_n+1})).$$

Besides,

$$\widehat{\lambda} = \lambda_1\gamma_1 + \lambda_2\gamma_2 + \ldots + \lambda_n\gamma_n.$$

As the group W^{res} acts on γ_k by permutations and sign changes, the function (19) is W^{res}-invariant. ∎

Let $\mathcal{L}_k$ be the linear operator in $\mathbb{C}[SL_{2n}]_q$ defined by $\mathcal{L}_k f = C_k f$.

The action of $U_q^{\mathrm{ext}}\mathfrak{sl}_{2n}$ in the space of $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-biinvariant functions determines the homomorphism

$$Z(U_q^{\mathrm{ext}}\mathfrak{sl}_{2n}) \to \mathrm{End}(\mathbb{C}[u_1, u_2, \ldots, u_n]^{S_n}),$$

as

$$\mathbb{C}[u_1, u_2, \ldots, u_n]^{S_n} \cong \mathbb{C}[x_1, x_2, \ldots, x_n] \cong \mathbb{C}[SL_{2n}]_q^{(U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n))^{\mathrm{op}} \otimes U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)}, \tag{20}$$

(see Subsect. 2.3). Here we will describe the action of the linear operators $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n$ in the space (20).

Let us define the difference operator $\Box_{u_i}$ in the space $\mathbb{C}[u_1, u_2, \ldots, u_n]$ with

$$\Box_{u_i} f(u_1, \ldots, u_n) = D_{u_i} u_i (1 - q^{-1}u_i) D_{u_i} f(u_1, \ldots, u_n), \tag{21}$$

where $D_{u_i} f(u_1, \ldots, u_n) = \frac{f(u_1, \ldots, u_{i-1}, q^{-1}u_i, u_{i+1}, \ldots, u_n) - f(u_1, \ldots, u_{i-1}, qu_i, u_{i+1}, \ldots, u_n)}{q^{-1}u_i - qu_i}$.

Proposition 10.

$$\mathcal{L}_k|_{\mathbb{C}[u_1,u_2,\ldots,u_n]^{S_n}} = \frac{1}{\Delta(u)}\, e_k(\Box_{u_1},\ldots,\Box_{u_n})\,\Delta(u). \tag{22}$$

P r o o f. In Subsection 4.1.3 it will be shown that in the case of one variable

$$\Box_u\, \varphi_l(u) = a(l)\varphi_l(u).$$

From (17) and the determinant decomposition described in Corollary 1 it follows that

$$\frac{1}{\Delta(u)}\, e_k(\Box_{u_1},\ldots,\Box_{u_n})\,\Delta(u)\,\varphi_\lambda(u) = e_k(a(\lambda+\delta))\varphi_\lambda(u), \qquad \lambda\in\Lambda_n.$$

Equality (22) follows from Proposition 9, as the set $\{\varphi_\lambda\}_{\lambda\in\Lambda_n}$ is a basis of the vector space $\mathbb{C}[SL_{2n}]_q^{(U_q\mathfrak{s}(\mathfrak{gl}_n\times\mathfrak{gl}_n))^{\mathrm{op}}\otimes U_q\mathfrak{s}(\mathfrak{gl}_n\times\mathfrak{gl}_n)}$. ∎

4. Plancherel Measure for the Quantum Matrix Ball

4.1. The Plancherel measure for a family of the operators $\mathcal{L}_1^{\mathrm{radial}}$, $\mathcal{L}_2^{\mathrm{radial}}$, …, $\mathcal{L}_n^{\mathrm{radial}}$

4.1.1. Linear operators $\mathcal{L}_1^{\mathrm{radial}},\mathcal{L}_2^{\mathrm{radial}},\ldots,\mathcal{L}_n^{\mathrm{radial}}$ in the space $L^2(\Delta_{\mathbb{D}},d\nu_q)$

Let us consider the elements

$$C_1,C_2,\ldots,C_n\in Z(U_q^{\mathrm{ext}}\mathfrak{sl}_{2n})$$

defined in (18). Let also $\mathcal{L}_k$ be the linear operator in $\mathcal{D}(\mathbb{D})_q$ defined by

$$\mathcal{L}_k f = C_k f.$$

Now we describe the restriction of the linear operator $\mathcal{L}_k$, $k=1,2,\ldots,n$ to the space $\mathcal{D}(\mathbb{D})_q^{U_q\mathfrak{s}(\mathfrak{gl}_n\times\mathfrak{gl}_n)}$ of $U_q\mathfrak{s}(\mathfrak{gl}_n\times\mathfrak{gl}_n)$-invariants in $\mathcal{D}(\mathbb{D})_q$.

Let us introduce the short notation $\mathcal{L}_k^{\mathrm{radial}}$ for the restriction of $\mathcal{L}_k$ to $\mathcal{D}(\mathbb{D})_q^{U_q\mathfrak{s}(\mathfrak{gl}_n\times\mathfrak{gl}_n)}$.

Proposition 11.

$$\mathcal{L}_k^{\mathrm{radial}} = \frac{1}{\Delta(u)}\, e_k(\Box_{u_1},\ldots,\Box_{u_n})\,\Delta(u), \tag{23}$$

where $\Box_{u_j}$ *are the difference operators in the vector space* (13) *defined by the same formula as in* (21).

P r o o f. Following Subsection 2.3, the vector space of $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-invariants in $\mathcal{D}(\mathbb{D})_q$ can be identified with the space $\mathcal{D}(\Sigma_{\mathbb{D}})$ of functions on $\Sigma_{\mathbb{D}}$ with finite support. Using Lemma 2 one can obtain that the vector space of $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-invariants in $\mathcal{D}(\mathbb{D})_q$ is canonically isomorphic to the space $\mathcal{D}(\Delta_{\mathbb{D}})$ of functions on $\Delta_{\mathbb{D}}$ with finite support. Consider the pointwise convergence topology on $\Delta_{\mathbb{D}}$.

The space of symmetric polynomials in $\Delta_{\mathbb{D}}$ is dense in the topological space $\mathcal{F}(\Delta_{\mathbb{D}})$ of functions on $\Delta_{\mathbb{D}}$, and equation (23) takes place for symmetric polynomials following (22) and (20).

The linear operators in both parts of equation (23) can be extended continuously from the space of symmetric polynomials in $\Delta_{\mathbb{D}}$ to $\mathcal{F}(\Delta_{\mathbb{D}})$, and equation (23) takes place for the whole space $\mathcal{F}(\Delta_{\mathbb{D}})$. ∎

Now we recall the measure on $\Delta_{\mathbb{D}}$:

$$d\nu_q(u) = \mathcal{N}\ \Delta(u)^2 d_{q^{-2}}u_1 d_{q^{-2}}u_2 \dots d_{q^{-2}}u_n, \tag{24}$$

where $\mathcal{N}$ is defined in (15). It is the restriction of the invariant measure to the space $\mathcal{D}(\mathbb{D})_q^{U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)}$, which we already identified with $\mathcal{D}(\Delta_{\mathbb{D}})$ (see Prop. 7).

Let us introduce the Hilbert space $L^2(\Delta_{\mathbb{D}}, d\nu_q)$ of functions on the set $\Delta_{\mathbb{D}}$ which satisfy

$$\int\limits_{\Delta_{\mathbb{D}}} |f(u)|^2 d\nu_q(u) < \infty,$$

where

$$(f, g) = \int\limits_{\Delta_{\mathbb{D}}} \overline{g(u)} f(u) d\nu_q(u).$$

It will be proved in the sequel (Lemma 7) that the linear operators $\mathcal{L}_1^{\text{radial}}$, $\mathcal{L}_2^{\text{radial}}, \dots, \mathcal{L}_n^{\text{radial}}$ can be continuously extended to bounded pairwise commuting selfadjoint operators in $L^2(\Delta_{\mathbb{D}}, d\nu_q)$.

Our goal is to find a Plancherel measure $d\Sigma$ on the joint spectrum of commuting selfadjoint linear operators $\mathcal{L}_1^{\text{radial}}$, $\mathcal{L}_2^{\text{radial}}, \dots, \mathcal{L}_n^{\text{radial}}$ and a unitary operator $\mathcal{F} : L^2(\Delta_{\mathbb{D}}, d\nu_q) \to L^2(d\Sigma)$ which provides a unitary equivalence between the operators $\mathcal{L}_1^{\text{radial}}$, $\mathcal{L}_2^{\text{radial}}, \dots, \mathcal{L}_n^{\text{radial}}$ and the operators of multiplication by independent variable, such as $\mathcal{F} f_0 = 1$.

The element $f_0 \in L^2(\Delta_{\mathbb{D}}, d\nu_q)$ is a cyclic vector under the action of $\mathcal{L}_1^{\text{radial}}$, $\mathcal{L}_2^{\text{radial}}, \dots, \mathcal{L}_n^{\text{radial}}$ (one can prove it explicitly, see Subsect. 4.1.2). However, it follows from the isometry of the operator $\mathcal{F}$ and Remarks 8 and 9.

The considered problems are typical for the theory of commutative operator $*$-algebras with a cyclic vector [18, p. 570, 571], [28, p. 103].

4.1.2. The cyclic vector f_0

Here we discuss the fact that the element $f_0 \in L^2(\Delta_{\mathbb{D}}, d\nu_q)$ is a cyclic vector under the action of the operators $\mathcal{L}_1^{\mathrm{radial}}, \mathcal{L}_2^{\mathrm{radial}}, \ldots, \mathcal{L}_n^{\mathrm{radial}}$.

By direct computation we obtain the following lemma.

Lemma 4. *In the case of quantum disk*

$$\Box_u f_0(q^{2k}u) = c_{k,-1} f_0(q^{2k-2}u) + c_{k,0} f_0(q^{2k}u) + c_{k,1} f_0(q^{2k+2}u), \qquad k \in \mathbb{Z}_+, \quad (25)$$

where $c_{k,-1}$, $c_{k,0}$, $c_{k,1}$ are nonzero constants.

For example,

$$\Box_u f_0(u) = D_u u(1 - q^{-1}u) D_u f_0(u) = \frac{f_0(u)}{1-q^2} - \frac{q^2 f_0(q^2 u)}{1-q^2}.$$

Here $f_0(q^{-2}u) = 0$ for $u \in q^{-2\mathbb{Z}_+}$, so the first term in (25) vanishes.

The lemma below follows from the previous one by induction.

Lemma 5.

$$\mathcal{L}_i^{\mathrm{radial}} f_0(q^{2(\lambda+\delta)}u) = \sum c_{\mathbf{d}} f_0(q^{2(\lambda+\delta+\mathbf{d})}u),$$

where $\mathbf{d} \in \{-1, 0, 1\}^n$, $\mathrm{card}\{j | d_j \neq 0\} \leq i$ *and* $c_{\mathbf{d}} \neq 0$.

Lemma 6. *The linear span of the action of $\mathcal{L}_1^{\mathrm{radial}}, \mathcal{L}_2^{\mathrm{radial}}, \ldots, \mathcal{L}_n^{\mathrm{radial}}$ on f_0 contains the set of finite functions on $\mathcal{D}(\Delta_{\mathbb{D}})$.*

Sketch of the proof. Lemma 5 implies that the linear span of the action of $\mathcal{L}_1^{\mathrm{radial}}, \mathcal{L}_2^{\mathrm{radial}}, \ldots, \mathcal{L}_n^{\mathrm{radial}}$ on f_0 contains the set

$$\mathcal{S}_{\mathbb{D}} = \{f_0(q^{2(\lambda+\delta)}u) \mid \lambda \in \Lambda_n\}$$

of characteristic functions of points of $\Delta_{\mathbb{D}}$.

The last lemma implies that f_0 is cyclic as the set of finite functions $\mathcal{D}(\Delta_{\mathbb{D}})$ is dense in $L^2(\Delta_{\mathbb{D}}, d\nu_q)$.

4.1.3. Example: the quantum disk

In this subsection we recall the Plancherel measure $d\sigma$ for the quantum disk found in [20].

Consider the Hilbert space $L^2(q^{-2\mathbb{Z}_+})$ of functions on the geometric series $q^{-2\mathbb{Z}_+}$ which satisfy the condition

$$\int_1^\infty |f(u)|^2 d_{q^{-2}}u < \infty$$

with the scalar product

$$(f,g)=\int_1^{\infty}\overline{g(u)}f(u)d_{q^{-2}}u.$$

Recall the notation for the difference operator $\Box_u$ which acts in the space of functions on geometric series $q^{-2\mathbb{Z}_+}$ by

$$\Box_u f(u)=D_u u(1-q^{-1}u)D_u f(u),$$

where

$$D_u f(u)\mapsto\frac{f(q^{-1}u)-f(qu)}{q^{-1}u-qu}.$$

Then $\mathcal{L}_1^{\mathrm{radial}}=\Box_u$.

Let us describe the eigenfunctions of the difference operator $\Box_u$. Introduce the notation

$$\Phi_l(x)={}_3\Phi_2\left(\begin{matrix}q^{-2l},q^{2(l+1)},x\\q^2,0\end{matrix};q^2,q^2\right),\qquad l\in\mathbb{C},$$

for the basic hypergeometric function (see [8]).

Proposition 12. *([20, §8]).*

$$\Box_u\Phi_l(u)=a(l)\Phi_l(u),$$

where $a(l)$ is defined in (17)*:*

$$a(l)=\frac{(1-q^{-2l})(1-q^{2l+2})}{(1-q^2)^2}.$$

Remark 3. $\Phi_l(1)=1$.

Remark 4. *$\Phi_l(u)$ is equal up to a multiplicative constant to $\varphi_l(u)$.*

Let

$$c(l)=\frac{\Gamma_{q^2}(2l+1)}{(\Gamma_{q^2}(l+1))^2}$$

be a q-analogue of the Harish-Chandra c-function. Here $\Gamma_{q^2}(x)=\frac{(q^2,q^2)_\infty}{(q^{2x},q^2)_\infty}(1-q^2)^{1-x}$ is a well-known q-analogue of the Gamma function $\Gamma(x)$.

Let us consider the measure

$$d\sigma(\rho)=\frac{1}{2\pi}\cdot\frac{h}{1-q^2}\cdot\frac{d\rho}{c(-\frac12+i\rho)c(-\frac12-i\rho)}$$

on the interval $[0,\pi/h]$, where $h=-2\ln q$.

Consider the operator

$$\mathcal{F}:\ f \mapsto \widehat{f}(\rho) = \int\limits_1^\infty \Phi_{-\frac{1}{2}+i\rho}(u) f(x) d_{q^{-2}} u$$

defined in the space of finite functions on the geometric series $q^{-2\mathbb{Z}_+}$. It is shown in [20, Th. 9.2] that this operator can be extended to a unitary operator $\mathcal{F} : L^2(q^{-2\mathbb{Z}_+}) \to L^2([0, \pi/h], d\sigma)$ such as

$$\mathcal{F}\ \Box_u\ f = a(-\frac{1}{2} + i\rho)\mathcal{F} f, \qquad f \in L^2([0, \pi/h], d\sigma),$$

where $a(l)$ is defined in (17). The inverse operator $\mathcal{F}^{-1}$ has the form

$$\widehat{f}(\rho) \mapsto \int\limits_0^{\pi/h} \widehat{f}(\rho)\Phi_{-\frac{1}{2}+i\rho}(u) d\sigma(\rho).$$

4.1.4. The quantum matrix ball

We will call the eigenfunction of a difference operator a generalized one if it does not belong to L^2. These functions are used in the sequel for the construction of the operator $\mathcal{F}$.

Consider the isometric linear operator*

$$\mathcal{I} : L^2(\Delta_{\mathbb{D}}, d\nu_q) \to L^2(q^{-2\mathbb{Z}_+^n}),$$

$$\mathcal{I} : f(u) \mapsto \Delta(u)\widetilde{f}(u), \tag{26}$$

where $\widetilde{f}$ is defined in the following way: for every $u = (u_1, \ldots, u_n)$ with $u_i \neq u_j$ for $i \neq j$ there exists a unique permutation $w \in S_n$ such as $u_{w_1} > u_{w_2} > \ldots > u_{w_n}$. Then

$$\widetilde{f}(u) = \begin{cases} \frac{1}{\sqrt{n!}} f(u_{w_1}, \ldots, u_{w_n}), & u_i \neq u_j, \quad i \neq j, \\ 0, & \text{otherwise}, \end{cases} \qquad u_1, u_2, \ldots, u_n \in q^{-2\mathbb{Z}_+}.$$

*$L^2(q^{-2\mathbb{Z}_+^n})_q$ is a short notation for $L^2(\underbrace{q^{-2\mathbb{Z}_+} \times \ldots \times q^{-2\mathbb{Z}_+}}_{n})_q$ with the product measure multiplicated by $\mathcal{N}$.

Consider the notation

$$\widetilde{\mathcal{L}_k} = e_k(\Box_{u_1}, \Box_{u_2}, \dots, \Box_{u_n}), \qquad k = 1, 2, \dots, n, \tag{27}$$

for the difference operators in $L^2(q^{-2\mathbb{Z}_+^n})_q$. Then the following diagram is commutative:

$$\begin{array}{ccc} L^2(\Delta_{\mathbb{D}}, d\nu_q) & \xrightarrow{\mathcal{I}} & L^2(q^{-2\mathbb{Z}_+^n}) \\ \Big\downarrow {\scriptstyle \mathcal{L}_k^{\text{radial}}} & & \Big\downarrow {\scriptstyle \widetilde{\mathcal{L}_k}} \\ L^2(\Delta_{\mathbb{D}}, d\nu_q) & \xrightarrow{\mathcal{I}} & L^2(q^{-2\mathbb{Z}_+^n}). \end{array}$$

Lemma 7. *The operators $\mathcal{L}_1^{\text{radial}}, \mathcal{L}_2^{\text{radial}}, \dots, \mathcal{L}_n^{\text{radial}}$ in the Hilbert space $L^2(\Delta_{\mathbb{D}}, d\nu_q)$ are bounded selfadjoint and pairwise commuting.*

P r o o f. The explicit formulas of Subsection 4.1.3 imply that the operators $\Box_{u_i}$ are bounded for all $1 \le i \le n$ (unlike in the classical case), so the same holds for $\widetilde{\mathcal{L}}_1, \widetilde{\mathcal{L}}_2, \dots, \widetilde{\mathcal{L}}_n$. Moreover, it is easy to see that the operators $\Box_{u_i}$ and $\Box_{u_j}$ commute for $1 \le i < j \le n$ (as they act in different variables), so the operators $\widetilde{\mathcal{L}_i}, \widetilde{\mathcal{L}_j}$ commute for $1 \le i < j \le n$, too.

Also, the operators $\Box_{u_i}$, $i = 1, 2, \dots, n$ are symmetric, so they are bounded selfadjoint operators in $L^2(q^{-2\mathbb{Z}_+^n})_q$. Thus, $\widetilde{\mathcal{L}}_i$, $i = 1, 2, \dots, n$ are pairwise commuting bounded selfadjoint linear operators in $L^2(q^{-2\mathbb{Z}_+^n})_q$. As the mapping $\mathcal{I}$ is isometric, the operators $\mathcal{L}_1^{\text{radial}}, \mathcal{L}_2^{\text{radial}}, \dots, \mathcal{L}_n^{\text{radial}}$ are also bounded selfadjoint and pairwise commuting. ∎

Using Proposition 12, one can easily show that the functions $\Phi_{l_1}(u_1)\Phi_{l_2}(u_2)\dots\Phi_{l_n}(u_n)$ on $q^{-2\mathbb{Z}_+^n}$ are common generalized eigenfunctions of the operators (27). We will need the common eigenfunctions which are in the image of the operator $\mathcal{I}$. It is easy to see that

$$\widetilde{\phi}_{l_1, l_2, \dots, l_n}(u_1, u_2, \dots, u_n) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\ \Phi_{l_1}(u_{\sigma_1})\Phi_{l_2}(u_{\sigma_2})\dots\Phi_{l_n}(u_{\sigma_n}) \in \operatorname{Im}\mathcal{I}$$

are common generalized eigenfunctions. Let

$$\mathcal{R} = \{(\rho_1, \rho_2, \dots, \rho_n) \in [0, \pi/h]^n, \quad \rho_1 > \rho_2\ \dots > \rho_n\}.$$

Lemma 8. *The pairwise commuting bounded selfadjoint operators $\widetilde{\mathcal{L}_k}$, $k = 1, 2, \dots, n$ are unitary equivalent to the operators of multiplication by*

$$e_k(a(-\frac{1}{2} + i\rho_1), a(-\frac{1}{2} + i\rho_2), \dots, a(-\frac{1}{2} + i\rho_n)), \qquad k = 1, 2, \dots, n,$$

(respectively) in the Hilbert space $L^2(\mathcal{R},(n!)\mathcal{N}(d\sigma)^n|_{\mathcal{R}})$. *The unitary equivalence is provided by the mapping*

$$\widetilde{\mathcal{U}}:\operatorname{Im}\mathfrak{I}\to L^2(\mathcal{R},(n!)\mathcal{N}(d\sigma)^n|_{\mathcal{R}}),$$

$$\widetilde{\mathcal{U}}:\ f(u_1,u_2,\ldots,u_n)\mapsto\widehat{f}(\rho_1,\rho_2,\ldots,\rho_n)$$

$$=\mathcal{N}\int_1^\infty\ldots\int_1^\infty\widetilde{\phi}_{-\frac12+i\rho_1,-\frac12+i\rho_2,\ldots,-\frac12+i\rho_n}(u)f(u)d_{q^{-2}}u_1\ldots d_{q^{-2}}u_n.$$

The inverse operator is

$$\widetilde{\mathcal{U}}^{-1}:\widehat{f}(\rho_1,\rho_2,\ldots,\rho_n)\mapsto$$

$$\mathcal{N}\underbrace{\int\ldots\int}_{\mathcal{R}}\widehat{f}(\rho_1,\rho_2,\ldots,\rho_n)\widetilde{\phi}_{-\frac12+i\rho_1,-\frac12+i\rho_2,\ldots,-\frac12+i\rho_n}(u)\ (n!)d\sigma(\rho_1)\ldots d\sigma(\rho_n).$$

P r o o f. This lemma follows from the results of subsection 4.1.3 and the explicit formulas for the operators $\widetilde{\mathcal{L}_1},\ldots,\widetilde{\mathcal{L}_n}$.

Remark 5. *The last equalities define* $\widetilde{\mathcal{U}}$ *on a dense linear manifold of the functions with finite support on the set* $q^{-2\mathbb{Z}_+^n}$.

Let us introduce the notation

$$\Phi_{l_1,l_2,\ldots,l_n}(u)=\frac{\sum\limits_{\sigma\in S_n}\operatorname{sign}(\sigma)\ \Phi_{l_1}(u_{\sigma_1})\Phi_{l_2}(u_{\sigma_2})\ldots\Phi_{l_n}(u_{\sigma_n})}{\Delta(u)}. \tag{28}$$

Remark 6. *(See Corollary 1 and Remark 4). The spherical function* $\varphi_\lambda(u)$, $\lambda\in\Lambda_n$ *is equal up to a multiplicative constant to* $\Phi_{l_1,l_2,\ldots,l_n}(u)$, *where* $l_i=(\lambda+\delta)_i\in\mathbb{Z}$.

Using this lemma and the definition (26) of the operator $\mathfrak{I}$, one can easily obtain the following lemma.

Lemma 9. *The pairwise commuting bounded selfadjoint operators* $\mathcal{L}_k^{\mathrm{radial}}$, $k=1,2,\ldots,n$, *are unitary equivalent to the operators of multiplication by*

$$e_k(a(-\frac12+i\rho_1),a(-\frac12+i\rho_2),\ldots,a(-\frac12+i\rho_n)),\quad k=1,2,\ldots,n,$$

(respectively) in the Hilbert space $L^2(\mathcal{R},(n!)\mathcal{N}(d\sigma)^n|_{\mathcal{R}})$. *The unitary equivalence is provided by the mapping*

$$\mathcal{U}:L^2(\Delta_{\mathbb{D}},d\nu_q)\to L^2(\mathcal{R},(n!)\mathcal{N}(d\sigma)^n|_{\mathcal{R}}),$$

$$\mathcal{U}:\ f(u)\mapsto\widehat{f}(\rho_1,\rho_2,\ldots,\rho_n)=\int_{\Delta_{\mathbb{D}}}\Phi_{-\frac12+i\rho_1,-\frac12+i\rho_2,\ldots,-\frac12+i\rho_n}(u)f(u)d\nu_q(u),$$

where the measure $d\nu_q(u)$ *is defined in* (24).

The inverse operator is

$$\mathcal{U}^{-1} : \widehat{f}(\rho_1, \rho_2, \ldots, \rho_n)$$

$$\mapsto \int\limits_{\mathcal{R}} \widehat{f}(\rho_1, \rho_2, \ldots, \rho_n) \Phi_{-\frac{1}{2}+i\rho_1, -\frac{1}{2}+i\rho_2, \ldots, -\frac{1}{2}+i\rho_n}(u)\ (n!)\mathcal{N}\ d\sigma(\rho_1) \ldots d\sigma(\rho_n).$$

Remark 7. *The last equalities define $\mathcal{U}$ on a dense linear manifold of the functions with finite support on the set $\Delta_{\mathbb{D}}$.*

Lemma 10.

$$\mathcal{U} f_0 = \mathcal{N} \Delta(q^{-2\delta})^{-1} \left(\prod_{j=0}^{n-1} \frac{(q^{-2j}; q^2)_j}{(q^2; q^2)_j^2} q^{(j+1)^2-1} \right) \prod_{1 \le k < j \le n} (q^{-2i\rho_j} + q^{2i\rho_j} - q^{-2i\rho_k} - q^{2i\rho_k}), \tag{29}$$

where the constant $\mathcal{N}$ is defined in (15).

P r o o f.

$$\begin{aligned} (\mathcal{U} f_0)(\rho_1, \rho_2, \ldots, \rho_n) &= \mathcal{N} \Phi_{-\frac{1}{2}+i\rho_1, -\frac{1}{2}+i\rho_2, \ldots, -\frac{1}{2}+i\rho_n}(1, q^{-2}, \ldots, q^{-2(n-1)}) \\ &= \mathcal{N} \Delta(q^{-2\delta})^{-1} \sum_{\sigma \in S_n} \mathrm{sign}(\sigma)\ \Phi_{-\frac{1}{2}+i\rho_1}(1) \Phi_{-\frac{1}{2}+i\rho_2}(q^{-2}) \ldots \Phi_{-\frac{1}{2}+i\rho_n}(q^{-2(n-1)}). \end{aligned} \tag{30}$$

It can be verified that the last expression is a polynomial in the variables $q^{i\rho_1} + q^{-i\rho_1}, \ldots, q^{i\rho_n} + q^{-i\rho_n}$. It is antisymmetric, so

$$\prod_{1 \le k < j \le n} (q^{-2i\rho_j} + q^{2i\rho_j} - q^{-2i\rho_k} - q^{2i\rho_k}) \tag{31}$$

is a factor of (30). One can compare the degrees of the polynomials in the right-hand side of (30) and (31) as the elements of the graded algebra $\mathbb{C}[q^{i\rho_1} + q^{-i\rho_1}, q^{i\rho_2} + q^{-i\rho_2}, \ldots, q^{i\rho_n} + q^{-i\rho_n}]$. The degree of the polynomial (31) is $\frac{n(n-1)}{2}$. Since

$$\begin{aligned} \Phi_{-\frac{1}{2}+i\rho}(q^{-2k}) &= {}_3\Phi_2 \left(\begin{matrix} q^{1+i\rho}, q^{1-i\rho}, q^{-2k} \\ q^2, 0 \end{matrix} ; q^2, q^2 \right) \\ &= \sum_{j=0}^{k} \frac{(q^{1+i\rho}; q^2)_j\, (q^{1-i\rho}; q^2)_j\, (q^{-2k}; q^2)_j\, q^{2j}}{(q^2; q^2)_j^2}, \end{aligned}$$

then the degree of $\mathcal{U} f_0$ is $\frac{n(n-1)}{2}$, and it proves (29) up to a constant. This constant can be found by comparing the highest monomial coefficients in the lexicographic order. ■

Denote

$$\kappa(\rho_1,\rho_2,\ldots,\rho_n)$$
$$=\mathcal{N}\Delta(q^{-2\delta})^{-1}\left(\prod_{j=0}^{n-1}\frac{(q^{-2j};q^2)_j}{(q^2;q^2)_j^2}q^{(j+1)^2-1}\right)\prod_{1\le k<j\le n}(q^{-2i\rho_j}+q^{2i\rho_j}-q^{-2i\rho_k}-q^{2i\rho_k}). \tag{32}$$

Notice that the function $\kappa(\rho_1,\rho_2,\ldots,\rho_n)$ is positive on $\mathcal{R}$. Consider the operator

$$\mathcal{F}=\frac{1}{\kappa(\rho_1,\rho_2,\ldots,\rho_n)}\,\mathcal{U}$$

and the measure

$$d\Sigma(\rho_1,\rho_2,\ldots,\rho_n)=\kappa(\rho_1,\rho_2,\ldots,\rho_n)^2(n!)\mathcal{N}(d\sigma(\rho_1)\ldots d\sigma(\rho_n))|_{\mathcal{R}} \tag{33}$$

on the set $\mathcal{R}$ (the constant $\mathcal{N}$ is defined in (15)).

The following proposition is the consequence of Lemmas 9 and 10.

Proposition 13. *The pairwise commuting bounded selfadjoint operators $\mathcal{L}_k^{\mathrm{radial}}$, $k=1,2,\ldots,n$, are unitary equivalent to the operators of multiplication by*

$$\frac{e_k(a(-\frac{1}{2}+i\rho_1),a(-\frac{1}{2}+i\rho_2),\ldots,a(-\frac{1}{2}+i\rho_n))}{\kappa(\rho_1,\rho_2,\ldots,\rho_n)},\qquad k=1,2,\ldots,n,$$

(respectively) in the Hilbert space $L^2(\mathcal{R},d\Sigma)$. The unitary equivalence is provided by the mapping

$$\mathcal{F}:L^2(\Delta_{\mathbb{D}},d\nu_q)\to L^2(\mathcal{R},d\Sigma),$$

$$\mathcal{F}:\ f(u)\mapsto\widehat{f}(\rho_1,\rho_2,\ldots,\rho_n)$$
$$=\frac{1}{\kappa(\rho_1,\rho_2,\ldots,\rho_n)}\int\limits_{\Delta_{\mathbb{D}}}\Phi_{-\frac{1}{2}+i\rho_1,-\frac{1}{2}+i\rho_2,\ldots,-\frac{1}{2}+i\rho_n}(u)f(u)d\nu_q(u), \tag{34}$$

where $\Phi_{l_1,l_2,\ldots,l_n}(u)$ are defined in (28), and the measure $d\nu_q(u)$ is defined in (24).

The inverse mapping is

$$\mathcal{F}^{-1}:\widehat{f}(\rho_1,\rho_2,\ldots,\rho_n)$$
$$\mapsto\int\limits_{\mathcal{R}}\widehat{f}(\rho_1,\rho_2,\ldots,\rho_n)\Phi_{-\frac{1}{2}+i\rho_1,-\frac{1}{2}+i\rho_2,\ldots,-\frac{1}{2}+i\rho_n}(u)\,d\Sigma(\rho_1,\rho_2,\ldots,\rho_n).$$

Remark 8. *The cyclic vector $f_0 \in L^2(\Delta_{\mathbb{D}}, d\nu_q)$ is mapped into $1 \in L^2(\mathcal{R}, d\Sigma)$ by $\mathcal{F}$.*

Remark 9. *For the convenience we use the variables $\rho_1, \rho_2, \ldots, \rho_n$ in the image of $\mathcal{F}$. Notice that if we change the variables*

$$z_k = \frac{e_k(a(-\frac{1}{2} + i\rho_1), a(-\frac{1}{2} + i\rho_2), \ldots, a(-\frac{1}{2} + i\rho_n))}{\kappa(\rho_1, \rho_2, \ldots, \rho_n)},$$

we get that $\mathcal{F}$ maps $\mathcal{L}_k^{\mathrm{radial}}$ into the operator of multiplication by the independent variable z_k.

The measure $d\Sigma$ on $\mathcal{R}$ defined in (33) is a sought-for radial part of the Plancherel measure.

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