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General Addition Formula for Meromorphic Functions Derived from Residue Theorem

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A meromorphic function is characterized by its singularity, and residues of the poles give us various information regarding the function. In the paper the algebraic relations of residues on some meromorphic functions are considered and a generic form of addition formula is induced from them. As the applications of that formula, the addition formula for rational function, hypergeometric function, elliptic function, and Riemann's Zeta function are studied. Also the general multiplication formula is discussed based on the previous discourse.

Key words: addition formula, multiplication formula, hypergeometric function, elliptic function, Riemann's Zeta function.

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Introduction

When $f(z)$ is holomorphic on the simply connected domain D surrounded with rectifiable curve, and $z = a$, b, and c are arbitrarily selected points in that domain, the following equation is satisfied according to Cauchy's residue theorem:

$$\oint_{\partial D} \frac{f(z)}{(z-a)(z-b)(z-c)}\, dz = 2\pi i \sum_{x=a,b,c} Res\,[\frac{f(z)}{(z-a)(z-b)(z-c)},\, x\,]$$
$$= 2\pi i \left(\frac{f(a)}{(a-b)(a-c)} + \frac{f(b)}{(b-a)(b-c)} + \frac{f(c)}{(c-a)(c-b)} \right). \tag{1}$$

Choosing c as $c = a + b$, the above equation is rewritten as

$$\oint\limits_{\partial D} \frac{f(z)}{(z-a)(z-b)(z-a-b)}\,dz = 2\pi i\left(\frac{f(a)}{(a-b)(-b)} + \frac{f(b)}{(b-a)(-a)} + \frac{f(a+b)}{ab}\right). \tag{2}$$

Hence,

$$f(a+b) = \frac{af(a)}{a-b} + \frac{bf(b)}{b-a} + \frac{ab}{2\pi i}\oint\limits_{\partial D} \frac{f(z)}{(z-a)(z-b)(z-a-b)}.dz. \tag{3}$$

Since a and b are chosen arbitrarily, the above equation is satisfied all over the domain D. Namely it gives the algebraic relation among $f(a)$, $f(b)$, $f(a+b)$, a, b, and the contour integral for any a and b ($a \neq b$) in D.
Now we suppose the case that the integral

$$\oint\limits_{\partial D} \frac{f(z)}{(z-a)(z-b)(z-a-b)}\,dz \tag{4}$$

converges to zero when the domain D is extended to the entire complex plane, and $f(z)$ is meromorphic and has the poles p_1, $p_2, \ldots, p_n$ in the entire complex plane (where n might be zero or infinity.) In this case the integral (4) is expressed as

$$\begin{aligned}&\oint\limits_{\partial D} \frac{f(z)}{(z-a)(z-b)(z-a-b)}\,dz \\ &= 2\pi i\left(\frac{f(a)}{(a-b)(-b)} + \frac{f(b)}{(b-a)(-a)} + \frac{f(a+b)}{ab}\right) \\ &+2\pi i \sum_{j=1,2,\ldots,n} Res\,[\frac{f(z)}{(z-a)(z-b)(z-c)}, p_j\,] \to 0.\end{aligned} \tag{5}$$

After taking the limitation, $f(a+b)$ is expressed as

$$f(a+b) = \frac{af(a) - bf(b)}{a-b} - ab\sum_{j=1}^{n} Res\,[\frac{f(z)}{(z-a)(z-b)(z-a-b)}, p_j\,]. \tag{6}$$

Here the equation (6) is called a generic form of addition formula, although strictly saying, it is not addition formula in the traditional meaning since it includes not only $f(a)$, $f(b)$, $f(a+b)$, but also function's arguments a, b. This formula suggests that the algebraic relation of $f(a+b)$, $f(a)$, $f(b)$, a, and b is determined solely by the poles of $f(z)$. For the meromorphic functions which do not satisfy the condition expressed with the integral (4) it is necessary to transform those functions appropriately so as to apply the generic form of addition formula. In the following sections, we discuss the feasibility of that formula on rational function, hypergeometric function, elliptic function, and Riemann's Zeta function.

Addition Formula for Rational Function

Although the addition theorem of rational functions is not new, here we see how the generic form of addition formula (6) works on some rational functions in the different manner, i.e., including function's arguments.
When $f(z)$ is a rational function, the integral (4) converges to zero if the order of $f(z)$ is less than two. Therefore the generic form of addition formula (6) can be applicable to the rational functions which have the order less than two. For example, applying the formula (6) to the function $f(z) = z$, we obtain

$$f(a+b) = \frac{af(a) - bf(b)}{a-b} \tag{7}$$

since $f(z) = z$ does not have any poles. As another example, the addition formula for $f(z) = \frac{1}{z}$ is derived from (6) as

$$f(a+b) = \frac{af(a) - bf(b)}{a-b} + \frac{1}{a+b} \tag{8}$$

since $f(z) = \frac{1}{z}$ has a pole at $z = 0$ and its residue is 1. Those formulae are easily verified by substituting z or $\frac{1}{z}$ for $f(z)$.

Functional Equation for Hypergeometric Function

The addition theorems of some hypergeometric type functions such as Bessel function are obtained by applying Fourier-cosine expansion explicitly [2] although in some cases their expressions are analytic rather than algebraic.
Here we investigate the feasibility of the generic form of addition formula (6) to Gauss' Hypergeometric function

$${}_2F_1(a,b;c;z) = 1 + \frac{ab}{1\cdot c}\,z + \frac{a(a+1)b(b+1)}{1\cdot 2\cdot c(c+1)}\,z^2 + \dots . \tag{9}$$

The above expression converges in the unit disk [3]. Namely ${}_2F_1(a,b;c;\frac{1}{z})$ is determined uniquely on the entire complex plane except on the unit disk. Although ${}_2F_1(a,b;c;\frac{1}{z})$ is multi-valued in the unit disk when $1-c$, $c-a-b$ or $a-b$ is not integer owing to analytic continuation, the integral (4) converges to zero when the domain D is extended to the entire complex plane as ${}_2F_1(a,b;c;\frac{1}{z})$ converges

to 1 when $z \to \infty$. Therefore we can apply the generic form of addition formula (6) to ${}_2F_1(a,b;c;\frac{1}{z})$.
As mentioned above, ${}_2F_1(a,b;c;\frac{1}{z})$ is multi-valued on the unit disk when $1-c$, $c-a-b$ or $a-b$ is not integer and the series

$$ {}_2F_1(a,b;c;\frac{1}{z}) = 1 + \frac{ab}{1\cdot c}\,\frac{1}{z} + \frac{a(a+1)b(b+1)}{1\cdot 2\cdot c(c+1)}\,\frac{1}{z^2} + \ldots \tag{10} $$

does not converge there. However the integral

$$ \oint\limits_{\partial D} {}_2F_1(a,b;c;\frac{1}{z})\,dz $$

is easily calculated as long as the domain D includes the unit disk since the integral for every term in the series (10) is zero except the second term: $\frac{ab}{1\cdot c}\frac{1}{z}$ owing to the residue theorem, i.e.

$$ \oint\limits_{\partial D} {}_2F_1(a,b;c;\frac{1}{z})\,dz = 2\pi i\frac{ab}{c}. \tag{11} $$

Thus we consider $z=0$ as the pole of ${}_2F_1(a,b;c;\frac{1}{z})$ and its residue is $\frac{ab}{c}$. Applying the generic form of addition formula (6) to ${}_2F_1(a,b;c;\frac{1}{z})$, we obtain the following additive functional equation for ${}_2F_1(a,b;c;\frac{1}{z})$:

$$ \begin{aligned} {}_2F_1(a,b;c;\frac{1}{x+y}) & \\ &= \frac{x\cdot {}_2F_1(a,b;c;\frac{1}{x}) - y\cdot {}_2F_1(a,b;c;\frac{1}{y})}{x-y} \\ &\quad - xy\,Res\,[\frac{{}_2F_1(a,b;c;\frac{1}{z})}{(z-x)(z-y)(z-x-y)},\ z=0\,] \\ &= \frac{x\cdot {}_2F_1(a,b;c;\frac{1}{x}) - y\cdot {}_2F_1(a,b;c;\frac{1}{y})}{x-y} + \frac{ab}{c(x+y)}. \end{aligned} \tag{12} $$

This equation is valid for any x, y $(x \neq y)$, and $x+y$ outside of the unit disk. Note that it is not the addition formula for hypergeometric function itself, but for the hypergeometric function substituted $\frac{1}{z}$ for z.

Addition Formula for Elliptic Function

There have been known various kinds of addition theorem of elliptic functions since that of lemniscate function by Gauss [6]. Here we discuss the addition theorem which includes function's arguments as well as the case of rational functions mentioned in the previous section. Since the elliptic function is meromorphic and the integral (4) converges to zero when the domain D is extended to the entire complex plane, the application of generic form of addition formula (6) to elliptic function is straightforward.

When an elliptic function $f(z)$ has the fundamental periods ω_1, ω_2, and poles p_1, p_2, $\ldots$, p_d on its fundamental parallelogram, any pole of $f(z)$ is expressed as

$$z = p_j + n\omega_1 + m\omega_2 \;\; (n, m : any\, integers,\; j : 1, 2, \ldots, d). \tag{13}$$

Therefore the following addition formula for the elliptic function is derived from (6):

$$f(x+y) = \frac{xf(x) - yf(y)}{x-y}$$
$$-xy \sum_{n,m=-\infty}^{\infty} \sum_{j=1}^{d} Res\,[\frac{f(z)}{(z-x)(z-y)(z-x-y)}, \, p_j + n\omega_1 + m\omega_2\,]. \tag{14}$$

This formula is valid for any x, y $(x \neq y)$ on the entire complex plane except on the poles.

Addition Formula and Functional Equation for Riemann's Zeta Function

Regarding the duplication formula for Riemann's Zeta function, it is known that $\zeta(2s)/\zeta(s)$ for arbitrary s is expressed as a Dirichlet series for Liouville function [7]. However we can not predict the algebraic relation of $\zeta(2s)$ and $\zeta(s)$ using that formula at this stage because of the nature of Liouville function. Here we discuss the feasibility of the generic form of addition formula (6) to Riemann's Zeta function for expressing algebraic additive functional relations specific to the function.

Riemann's Zeta function $\zeta(s)$ is a meromorphic function, therefore $\frac{1}{\zeta(s)}$ is also meromorphic. From Riemann's functional equation [4]

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s) \tag{15}$$

we know $\frac{1}{\zeta(s)}$ converges to zero rapidly when $Re(s) \leq 1$ and $|Im(s)| \to \infty$ except s is integer. As $\zeta(s)$ converges to a finite value when $Re(s) > 1$, the integral (4) for $f(z) = \frac{1}{\zeta(z)}$ converges to zero when the domain D is extended to the entire complex plane, and we can apply the generic form of addition formula (6) to $\frac{1}{\zeta(s)}$.

Since $\zeta(s)$ has trivial zeros and nontrivial zeros, $\frac{1}{\zeta(s)}$ has trivial poles and nontrivial poles accordingly. Regarding the trivial poles $z = -2, -4, -6, \ldots$, their residues are easily calculated. As each trivial pole is the first order, the residue is expressed as

$$Res\,[\frac{1}{\zeta(s)}, -2n] = \lim_{s\to -2n} \frac{s+2n}{\zeta(s)} = \frac{1}{\zeta'(s)}\,|_{s=-2n}. \tag{16}$$

Using Riemann's functional equation (15),

$$\begin{aligned} &\frac{1}{\zeta'(s)}\,|_{s=-2n} \\ &\quad = \frac{\pi}{\frac{\pi}{2}\cos\left(\frac{\pi s}{2}\right)(2\pi)^s\Gamma(1-s)\zeta(1-s) + \sin\left(\frac{\pi s}{2}\right)(...)}\,|_{s=-2n} \\ &\quad = \frac{(-1)^n \pi}{\frac{\pi}{2}(2\pi)^{-2n}\Gamma(2n+1)\zeta(2n+1)}. \end{aligned} \tag{17}$$

Therefore when the set of nontrivial poles is denoted as P, the addition formula for $\frac{1}{\zeta(s)}$ is derived from the generic form of addition formula (6) as

$$\begin{aligned} \frac{1}{\zeta(x+y)} &= \frac{\frac{x}{\zeta(x)} - \frac{y}{\zeta(y)}}{x-y} \\ &+ \sum_{n=1}^{\infty} \frac{(-1)^n 2(2\pi)^{2n} xy}{(2n+x)(2n+y)(2n+x+y)\Gamma(2n+1)\zeta(2n+1)} \\ &- xy \sum_{p\in P} Res\,[\frac{f(z)}{(z-x)(z-y)(z-x-y)}, p]. \end{aligned} \tag{18}$$

It is required for calculating $\zeta(x+y)$ from given values of $\zeta(x)$ and $\zeta(y)$ using the above addition formula to know the distribution of nontrivial poles (or nontrivial zeros for Zeta function) and their residues previously. Using the transformation similar to that applied to hypergeometric function in the previous section, we can induce the functional equation for Zeta function without referring to nontrivial zeros as the following.
Firstly we expand Zeta function as Laurent series about $z = 1$ as [5]

$$\zeta(s) = \frac{1}{s-1} + \gamma_0 - \frac{\gamma_1}{1!}(s-1) + \frac{\gamma_2}{2!}(s-1)^2 - \frac{\gamma_3}{3!}(s-1)^3 + \cdots, \tag{19}$$

where γ_j is Stieltjes constants defined as

$$\gamma_j = \lim_{m\to\infty}\left(\sum_{k=1}^{m} \frac{(\log k)^j}{k} - \frac{(\log m)^{j+1}}{j+1}\right). \tag{20}$$

Transforming s to $\frac{1}{s}+1$, we obtain the following expression:

$$\zeta(\frac{1}{s}+1) = s + \gamma_0 - \frac{\gamma_1}{1!}\frac{1}{s} + \frac{\gamma_2}{2!}\frac{1}{s^2} - \frac{\gamma_3}{3!}\frac{1}{s^3} + \cdots . \tag{21}$$

It follows that the pole of $\zeta(\frac{1}{s}+1)$ is $s=0$ and its residue is $-\gamma_1$. When putting $f(z) = \zeta(\frac{1}{z}+1)$, the integral (4) converges to zero as the highest order of the above series is one. Therefore we can apply the generic form of addition formula (6) to $\zeta(\frac{1}{z}+1)$ as

$$\zeta(\frac{1}{x+y}+1) = \frac{x\zeta(\frac{1}{x}+1) - y\zeta(\frac{1}{y}+1)}{x-y} - \frac{\gamma_1}{x+y}. \tag{22}$$

This functional equation is satisfied by any x, y ($x \neq y$) on the entire complex plane except the origin.

General Multiplication Formula for Meromorphic Functions

By putting $c = ab$ in the equation (1) instead of $c = a + b$ the generic form of multiplication formula for meromorphic functions is induced with the manner similar to that discussed in Introduction. In this case, the equation (1) is rewritten as

$$\oint_{\partial D} \frac{f(z)}{(z-a)(z-b)(z-ab)}\,dz$$
$$= 2\pi i \left(\frac{f(a)}{(a-b)(1-b)a} + \frac{f(b)}{(b-a)(1-a)b} + \frac{f(ab)}{(b-1)(a-1)ab} \right), \tag{23}$$

where $f(z)$ is holomorphic on the domain D and a, b are arbitrarily selected points in D. Therefore when $f(z)$ is meromorphic on the entire complex plane and its poles are p_1, p_2, $\ldots$, p_n, the following generic form of multiplication formula:

$$f(ab) = \frac{(a-1)bf(a) - (b-1)af(b)}{a-b}$$
$$-(a-1)(b-1)ab \sum_{j=1,2,\ldots,n} Res\,[\frac{f(z)}{(z-a)(z-b)(z-ab)}, p_j\,] \tag{24}$$

is deduced under the condition that the integral

$$\oint_{\partial D} \frac{f(z)}{(z-a)(z-b)(z-ab)}\,dz \tag{25}$$

converges to zero when D is extended to the entire complex plane. Since this condition is actually equivalent with that of the integral (4), multiplicative formulae or functional equations for rational function, hypergeometric function, elliptic

function, and Zeta function are deduced from the discussion parallel with that for deducing addition formulae for those functions.
In the case of hypergeometric function, the following multiplicative functional equation is deduced from the generic form of multiplication formula (24):

$$
\begin{aligned}
{}_2F_1(a,b;c;\frac{1}{xy}) \\
&= \frac{(x-1)y\cdot {}_2F_1(a,b;c;\frac{1}{x}) - (y-1)x\cdot {}_2F_1(a,b;c;\frac{1}{y})}{x-y} \\
&+ \frac{ab(x-1)(y-1)}{cxy}. \qquad (26)
\end{aligned}
$$

Similarly, the multiplication formula for elliptic function is expressed as

$$
\begin{aligned}
f(xy) &= \frac{(x-1)yf(x) - (y-1)xf(y)}{x-y} - (x-1)(y-1)xy \\
&\times \sum_{n,m=-\infty}^{\infty} \sum_{j=1}^{d} Res\,[\frac{f(z)}{(z-x)(z-y)(z-xy)}, p_j + n\omega_1 + m\omega_2\,], \qquad (27)
\end{aligned}
$$

and the multiplicative functional equation for Zeta function is

$$
\zeta(\frac{1}{xy}+1) = \frac{(x-1)y\zeta(\frac{1}{x}+1) - (y-1)x\zeta(\frac{1}{y}+1)}{x-y} - \frac{\gamma_1(x-1)(y-1)}{xy}, \qquad (28)
$$

where the definitions of ω_1, ω_2, p_j, and γ_1 are the same as those defined in the previous sections.

Summary

The generic form of addition and multiplication formula for the meromorphic functions satisfying a certain condition were induced from the residue theorem. Those formulae include function's arguments, and are determined solely by function's singularities.
For the functions which do not satisfy the said condition it is often possible to induce the additive or multiplicative functional equation instead of addition or multiplication formula by applying the generic form of addition or multiplication formula to the functions modified in order to satisfy the said condition.
As practical cases, the addition formula for some rational functions, elliptic function, Riemann's Zeta function, and the additive functional equation for hypergeometric function, Zeta function were deduced. Also the multiplication formula or multiplicative functional equation for elliptic function, hypergeometric function, and Zeta function were shown.

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