Journal of Mathematical Physics, Analysis, Geometry
2010, vol. 6, No. 2, pp. 192–228

Properties of Characteristic Function of Commutative System of Unbounded Nonselfadjoint Operators

V.A. Zolotarev

Department of Higher Mathematics and Informatics
V.N. Karazin Kharkiv National University
4 Svobody Sq., Kharkiv, 61077, Ukraine

Mathematical Division, B. Verkin Institute for Low Temperature Physics and Engineering
National Academy of Sciences of Ukraine
47 Lenin Ave., Kharkiv, 61103, Ukraine

E-mail:Vladimir.O.Zolotarev@univer.kharkov.ua

Received April 1, 2009

A class of characteristic functions corresponding to commutative systems of unbounded nonselfadjoint operators is studied. The theorem on unitary equivalence is proved. The class of functions corresponding to these commutative systems of unbounded nonselfadjoint operators is described. There is obtained an analogue of the Hamilton–Caley theorem demonstrating that in the case of finite dimensionality of deficient subspaces there exists such a polynomial $P(\lambda_1, \lambda_2)$ that annihilates the resolvents $R_k = (A_k - \alpha I)^{-1}$; $P(R_1, R_2) = 0$.

Key words: commutative system, unbounded operators, characteristic function.

Mathematics Subject Classification 2000: 47A45.

In [1], M.S. Livšic introduced an effective method of study of unbounded nonselfadjoint operators which was further developed by A.V. Kuzhel [2, 3], A.V. Shtraus, E.R. Tsekanovsky, and Yu.L. Shmul'yan [5]. Another approach to the studying of unbounded nonselfadjoint operators based on the analysis of the boundary value space was developed in the works by V.A. Derkach and M.M. Malamud which resulted in the analytic formalism for studying the properties of Weyl functions. The dissipative Srödinger operator and its functional model was studied by B.S. Pavlov [7] and his disciples. In the previous work [11] the author suggested a method of study of commutative system of nonselfadjoint unbounded operators which was based on the concepts of commutative colligation and open system associated with it. The paper consists of three parts. The first one includes the necessary facts on the commutative systems of unbounded nonselfadjoint operators. In Section 2 the main properties of the characteristic function of commutative colligations are studied, the complete set of invariants of commutative system of unbounded nonselfadjoint operators is defined and the theorem on the unitary equivalence is proved. It turned out that the characteristic function, besides the traditional J-properties, must satisfy three additional relations, which are the corollary of the commutative property of the initial operator system. Section 3 is dedicated to the description of the class of functions that are characteristic for commutative colligations. An analogue of the Hamilton–Caley theorem is proved, namely, it is proved that in the case of the finiteness of the outer spaces there exists the polynomial $P(\lambda_1, \lambda_2)$ such that $P(R_1, R_2) = 0$, where $R_k = (A_k - \alpha I)^{-1}$ is the resolvent of A_k, $k = 1, 2$. It is determined that the polynomial $P(\lambda_1, \lambda_2)$ has the "involution" generated by the inversion with respect to some circle.

1. Preliminary Information

I. Recall the main definitions and statements about commutative systems of nonselfadjoint unbounded operators given in [11].

Definition 1 [11]. *Let a system of the linear unbounded operators $\{A_1, A_2\}$ be defined in a Hilbert space H such that: a) the domain $\mathfrak{D}(A_p)$ of the operator A_p is dense in H, $\overline{\mathfrak{D}(A_p)} = H$, $p = 1, 2$; b) every operator A_p is closed in H, $p = 1, 2$; c) there exists the nonempty domain $\Omega \subset \mathbb{C}\backslash\mathbb{R}$ such that the resolvents $R_p(\lambda) = (A_p - \lambda I)^{-1}$ are regular for all $\lambda \in \Omega$, $p = 1, 2$; d) at least in one point $\alpha \in \Omega$, the resolvents R_1 $(= R_1(\alpha))$, R_2 $(= R_2(\alpha))$ commute.*

And let the Hilbert spaces $E_\pm$ and the linear bounded operators $\psi_-\colon E_- \to H$, $\psi_+\colon H \to E_+$ and $\{\sigma_p^-\}_1^2$, $\{\tau_p^-\}_1^2$, $\{N_p\}_1^2$, $\Gamma\colon E_- \to E_-$; $\{\sigma_p^+\}_1^2$; $\{\tau_p^+\}_1^2$, $\left\{\tilde{N}_p\right\}_1^2$, $\tilde{\Gamma}\colon E_+ \to E_+$ be given, $\{\sigma_p^\pm\}_1^2$ and $\{\tau_p^\pm\}$ be selfadjoint. The family

$$\Delta = \Delta(\alpha) = \Bigg(\{\sigma_p^-\}_1^2; \{\tau_p^-\}_1^2; \{N_p\}_1^2; \Gamma; H \oplus E_-; \left\{\begin{bmatrix} A_p & \psi_- \\ \psi_+ & K \end{bmatrix}\right\}_1^2; \quad H \oplus E_+; \tilde{\Gamma}; \left\{\tilde{N}_p\right\}_1^2; \{\tau_p^+\}_1^2; \{\sigma_p^+\}_1^2\Bigg) \tag{1.1}$$

is said to be a commutative colligation if there exists such $\alpha \in \Omega$ that:

1) $2\,\mathrm{Im}\,\alpha \cdot N_p^* \psi_-^* \psi_- N_p = K^* \sigma_p^+ K - \sigma_p^-$; $2\,\mathrm{Im}\,\alpha \cdot \tilde{N}_p \psi_+ \psi_+^* \tilde{N}_p^* = K \tau_p^- K^* - \tau_p^+$;

2) the operators

$$\varphi_+^p = \psi_+ (A_p - \alpha I) : \mathfrak{D}(A_p) \to E_+,$$

$$(\varphi_-^p)^* = \psi_-^* \left(A_p^* - \bar{\alpha} I\right) : \mathfrak{D}\left(A_p^*\right) \to E_-$$

are such that:

3) $K^*\sigma_p^+\varphi_+^p + N_p^*\psi_-^*\left(A_p - \bar{\alpha}I\right) = 0$; $K\tau_p^-\left(\varphi_-^p\right)^* + \tilde{N}_p\psi_+\left(A_p^* - \alpha I\right) = 0$;

4) $2\operatorname{Im}\langle A_p h_p, h_p\rangle = \left\langle \sigma_p^+\varphi_+^p h_p, \varphi_+^p\right\rangle$; $\forall h_p \in \mathfrak{D}\left(A_p\right)$;

$$-2\operatorname{Im}\left\langle A_p^*\tilde{h}_p, \tilde{h}_p\right\rangle = \left\langle \tau_p^-\left(\varphi_-^p\right)^*\tilde{h}_p, \left(\varphi_-^p\right)^*\tilde{h}_p\right\rangle; \quad \forall \tilde{h}_p \in \mathfrak{D}\left(A_p^*\right), \tag{1.2}$$

where $p = 1, 2$. *And, moreover, the relations:*

5) $R_2\psi_- N_1 - R_1\psi_- N_2 = \psi_-\Gamma$; $\tilde{N}_1\psi_+ R_2 - \tilde{N}_2\psi_+ R_1 = \tilde{\Gamma}\psi_+$;

6) $\tilde{\Gamma}K - K\Gamma = i\left(\tilde{N}_1\psi_+\psi_- N_2 - \tilde{N}_2\psi_+\psi_- N_1\right)$;

7) $KN_p = \tilde{N}_p K$; *are true, where* $R_p = R_p(\alpha)$, $p = 1, 2$.

It is easy to show [11] that for every operator system satisfying the assumptions a)–d) there always exist Hilbert spaces $E_\pm$ and corresponding operators $\psi_\pm$; K; $\left\{\sigma_p^\pm\right\}_1^2$, $\left\{\tau_p^\pm\right\}_1^2$; $\{N_p\}_1^2$; $\left\{\tilde{N}_p\right\}_1^2$; Γ; $\tilde{\Gamma}$; such that the relations 1)–7) (1.2) hold.

In the studying of nonselfadjoint operators the open systems associated with the corresponding colligations play an important role [8, 10].

Denote a rectangle in $\mathbb{R}_+^2$ by $D = [0, T_1] \times [0, T_2]$, $0 < T_p < \infty$, $p = 1, 2$, and let $u_-(t)$ be a vector function in E_- defined as $t = (t_1, t_2) \in D$. The system of the relations

$$R_\Delta : \begin{cases} i\partial_1 h_1(t) + A_1 y_1(t) = \alpha\psi_- N_1 u_-(t); \\ y_1(t) = h_1(t) + \psi_- N_1 u_-(t) \in \mathfrak{D}\left(A_1\right); \\ i\partial_2 h_2(t) + A_2 y_2(t) = \alpha\psi_- N_2 u_-(t); \\ y_2(t) = h_2(t) + \psi_- N_2 u_-(t) \in \mathfrak{D}\left(A_2\right); \\ h_1(0) = h_1; \quad h_2(0) = h_2; \quad t = (t_1, t_2) \in D, \end{cases} \tag{1.3}$$

where $\partial_p = \partial/\partial t_p$, $p = 1, 2$, is said to be the open system $F_\Delta = \{R_\Delta, S_\Delta\}$ associated with the colligation Δ (1.1) and, moreover, the vector functions $y_1(t)$ and $y_2(t)$ are such that there exists $y(t)$ from H, and

$$y_1(t) = R_1 y(t); \quad y_2(t) = R_2 y(t). \tag{1.4}$$

Thus, the functions $\{y_p(t)\}_1^2$ have a common generator $y(t)$, and (1.4) implies

$$R_1 y_2(t) = R_2 y_1(t). \tag{1.5}$$

As for the initial data h_1 and h_2 in (1.3), we suppose

$$h_p = R_p y(0) - \psi_- N_p u_-(0), \quad p = 1, 2. \tag{1.6}$$

The mapping S_Δ is given by

$$S_\Delta : \quad u_+(t) = K u_-(t) - i\psi_+ y(t). \tag{1.7}$$

Consider the differential operators

$$L_p = i\partial_p + \alpha, \quad p = 1, 2. \tag{1.8}$$

Then the main equations (1.3) can be written in the following form:

$$\begin{cases} L_1 h_1(t) + y(t) = 0; \\ R_1 y(t) = h_1(t) + \psi_- N_1 u_-(t) \in \mathfrak{D}(A_1); \\ L_2 h_2(t) + y(t) = 0; \\ R_2 y(t) = h_2(t) + \psi_- N_2 u_-(t) \in \mathfrak{D}(A_2). \end{cases} \tag{1.9}$$

Thus, $L_1 h_1(t) = -y(t) = L_2 h_2(t)$. Therefore, taking into account (1.8) and (1.3), we obtain

$$\begin{cases} R_1 L_1 y(t) + y(t) = \psi_- N_1 L_1 u_-(t); \\ R_2 L_2 y(t) + y(t) = \psi_- N_2 L_2 u_-(t); \\ y(0) = y_0; \quad t = (t_1, t_2) \in D; \\ u_+(t) = K u_-(t) - i\psi_+ y(t). \end{cases} \tag{1.10}$$

If $y(t)$ satisfies relations (1.10), then $h_1(t)$, $h_2(t)$ (1.9) and, correspondingly, $y_1(t)$ and $y_2(t)$ (1.4) are uniquely found by this function.

Theorem 1.1 [11]. *The equation system (1.3) is consistent if the vector function $u_-(t)$ is the solution of the equation*

$$\{N_1 L_1 - N_2 L_2 + \Gamma L_1 L_2\} u_-(t) = 0 \tag{1.11}$$

on condition that (1.4), (1.6) hold and L_p are given by (1.8), $p = 1, 2$.

Theorem 1.2 [11]. *If (1.10) takes place for the vector function $y(t)$ and $u_-(t)$ is the solution of (1.11), then $u_+(t)$ (1.7) satisfies the equation*

$$\left\{\tilde{N}_1 L_1 - \tilde{N}_2 L_1 + \tilde{\Gamma} L_1 L_2\right\} u_+(t) = 0. \tag{1.12}$$

Theorem 1.3 [11]. *For the open system $F_\Delta = \{R_\Delta, S_\Delta\}$ (1.3), (1.7) associated with the colligation Δ (1.1), the conservation laws hold:*

$$\begin{aligned} &1) \quad \partial_1 \|h_p(t)\|^2 = \left\langle \sigma_p^- u_-(t), u_-(t)\right\rangle - \left\langle \sigma_p^+ u_+(t), u_+(t)\right\rangle, \quad p = 1, 2; \\ &2) \quad \partial_2 \left\{\left\langle \sigma_1^- L_1 u_-(t), L_1 u_-(t)\right\rangle - \left\langle \sigma_1^+ L_1 u_+(t), L_1 u_+(t)\right\rangle\right\} \\ &\quad = \partial_1 \left\{\left\langle \sigma_2^- L_2 u_-(t), L_2 u_-(t)\right\rangle - \left\langle \sigma_2^+ L_2 u_+(t), L_2 u_+(t)\right\rangle\right\}. \end{aligned} \tag{1.13}$$

Along with the open system $F_\Delta = \{R_\Delta, S_\Delta\}$ (1.3), (1.7), describing the evolution generated by $\{A_1, A_2\}$, consider also the dual situation corresponding to the dynamics set by the adjoint operator system $\{A_1^*, A_2^*\}$.

Let the vector function $\tilde{u}_+(t)$ in E_+ be specified in the rectangle $D = [0, T_1] \times [0, T_2]$ from $\mathbb{R}^2_+$, $t = (t_1, t_2) \in D$, $0 < T_p < \infty$; $p = 1, 2$. The equation system

$$R^+_\Delta : \begin{cases} i\partial_1 \tilde{h}_1(t) - A_1^* \tilde{y}_1(t) = -\bar{\alpha}\psi_+^* \tilde{N}_1^* \tilde{u}_+(t); \\ \tilde{y}_1(t) = \psi_+^* \tilde{N}_1^* \tilde{u}_+(t) - \tilde{h}_1(t) \in \mathfrak{D}\left(A_1^*\right); \\ i\partial_2 \tilde{h}_2(t) - A_2^* \tilde{y}_2(t) = -\bar{\alpha}\psi_+^* \tilde{N}_2^* \tilde{u}_+(t); \\ \tilde{y}_2(t) = \psi_+^* \tilde{N}_2^* u_+(t) - \tilde{h}_2(t) \in \mathfrak{D}\left(A_2^*\right); \\ \tilde{h}_1(T) = \tilde{h}_1; \quad \tilde{h}_2(T) = \tilde{h}_2; \quad t = (t_1, t_2) \in D, \end{cases} \tag{1.14}$$

where, as usual, $\partial_p = \partial/\partial t_p$, $p = 1, 2$, and $\tilde{y}_1(t)$ and $\tilde{y}_2(t)$ are such that

$$\tilde{y}_1(t) = R_1^* \tilde{y}(t); \quad \tilde{y}_2(t) = R_2^* \tilde{y}(t); \tag{1.15}$$

is said to be the dual open system $F^+_\Delta = \left\{R^+_\Delta, S^+_\Delta\right\}$, $F^+_\Delta = \left\{R^+_\Delta, S^+_\Delta\right\}$ associated with the colligation Δ (1.1). The vector functions $\left\{\tilde{y}_p(t)\right\}_1^2$ have the common generator $\tilde{y}(t) \in H$, besides,

$$R_1^* \tilde{y}_2(t) = R_2^* \tilde{y}_1(t). \tag{1.16}$$

The initial data $\tilde{h}_1$, $\tilde{h}_2$ of problem (1.14) are found from the equalities

$$\tilde{h}_p = \psi_+^* \tilde{N}_p^* u_+(T) - R_p^* \tilde{y}(T), \quad p = 1, 2. \tag{1.17}$$

The mapping S^+_Δ is given by

$$S^+_\Delta : \quad \tilde{u}_-(t) = K^* \tilde{u}_+(t) + i\psi_-^* \tilde{y}(t). \tag{1.18}$$

Consider the differential operators

$$L_p^+ = i\partial_p + \bar{\alpha}, \quad p = 1, 2. \tag{1.19}$$

Similarly to (1.10), we obtain that the vector function $\tilde{y}(t)$ satisfies the relations

$$\begin{cases} R_1^* L_1^+ \tilde{y}(t) + \tilde{y}(t) = \psi_+^* \tilde{N}_1^* L_1^+ \tilde{u}_+(t); \\ R_2^* L_2^* \tilde{y}(t) + \tilde{y}(t) = \psi_+^* \tilde{N}_2^* L_2^+ \tilde{u}_+(t); \\ \tilde{y}(T) = \tilde{y}_T; \quad t = (t_1, t_2) \in D; \\ \tilde{u}_-(t) = K^* \tilde{u}_+(t) + i\psi_-^* \tilde{y}(t). \end{cases} \tag{1.20}$$

Using (1.20), it is easy to obtain the analogues of Theorems 1.1–1.3.

Theorem 1.4 [11]. *The equation system (1.20) is consistent if $\tilde{u}_+(t)$ satisfies the equation*

$$\left\{\tilde{N}_1^* L_1^+ - \tilde{N}_2^* L_2^+ + \tilde{\Gamma}^* L_1^+ L_2^+\right\} \tilde{u}_+(t) = 0 \tag{1.21}$$

on condition that (1.15) and (1.17) take place.

Theorem 1.5 [11]. *Let $\tilde{y}(t)$ be the solution of (1.20) and let $\tilde{u}_+(t)$ satisfy equation (1.21), then for the vector function $\tilde{u}_-(t)$ (1.18)*

$$\left\{N_1^* L_1^+ - N_2^* L_2^+ + \Gamma^* L_1^+ L_2^+\right\} \tilde{u}_-(t) = 0 \tag{1.22}$$

takes place.

Theorem 1.6 [11]. *For the dual open system $F_\Delta^+ = \left\{R_\Delta^+, S_\Delta^+\right\}$ (1.14)–(1.18), the conservation laws are true:*

$$\begin{aligned} &1) \quad \partial_p \left\|\tilde{h}_p(t)\right\|^2 = \left\langle \tau_p^- \tilde{u}_-(t), \tilde{u}_-(t)\right\rangle - \left\langle \tau_p^+ \tilde{u}_+(t), \tilde{u}_+(t)\right\rangle, \quad p = 1,2; \\ &2) \quad \partial_2 \left\{\left\langle \tau_1^- L_1^+ \tilde{u}_-(t), L_1^+ \tilde{u}_-(t)\right\rangle - \left\langle \tau_1^+ L_1^+ \tilde{u}_+(t), \tilde{u}_+(t)\right\rangle\right\} \\ &\quad = \partial_1 \left\{\left\langle \tau_2^- L_2^+ \tilde{u}_-(t), L_1^+ \tilde{u}_-(t)\right\rangle - \left\langle \tau_2^+ L_2^+ \tilde{u}_+(t), L_2^+ \tilde{u}_+(t)\right\rangle\right\}. \end{aligned} \tag{1.23}$$

O b s e r v a t i o n 1.1. The "external parameters" of the commutative colligation Δ (1.1) are not independent. Moreover, it is easy to show [11] that

$$\tilde{N}_p^* = \sigma_p^+ \tilde{N}_p^{-1} \tau_p^+; \quad N_p = \tau_p^- \left(N_p^*\right)^{-1} \sigma_p^-, \quad p = 1,2, \tag{1.24}$$

take place, besides, $\tilde{N}_p$ and N_p^* are boundedly invertible on the images of $\tau_p^+ E_+$ and $\sigma_p^- E_-$, $p = 1,2$, respectively.

2. Main Properties of Characteristic Functions

I. Suppose that the function $u_-(t)$ in (1.10) is a plane wave, $u_-(t) = e^{i\langle\lambda,t\rangle} u_-(0)$, where $\langle\lambda,t\rangle = \lambda_1 t_1 + \lambda_2 t_2$, $t = (t_1,t_2) \in D = [0,T_1]\times[0,T_2]$, and $\lambda = (\lambda_1,\lambda_2) \in \mathbb{C}^2$. And let $u_+(t)$ and $y(t)$ in (1.10) depend on t in a similar way, $u_+(t) = e^{i\langle\lambda,t\rangle} u_+(0)$, $y(t) = e^{i\langle\lambda,t\rangle} y(0)$. Then (1.10) yields

$$\begin{cases} y(0) = -\left(\lambda_1 - \alpha\right) T_{\lambda_1,\alpha} \psi_- N_1 u_-(0); \\ y(0) = -\left(\lambda_2 - \alpha\right) T_{\lambda_2,\alpha} \psi_- N_2 u_-(0); \\ u_+(0) = K u_-(0) - i\psi_+ y(0), \end{cases} \tag{2.1}$$

where $T_{\lambda_p,\alpha} = I + \left(\lambda_p - \alpha\right) R_p\left(\lambda_p\right)$, and $R_p\left(\lambda_p\right) = \left(A_p - \lambda_p I\right)^{-1}$ is the resolvent of A_p, $\lambda_p \in \Omega$, $p = 1,2$. The concordance of two different presentations for $y(0)$ (2.1) means that

$$\left(\lambda_1 - \alpha\right) T_{\lambda_1,\alpha} \psi_- N_1 u_-(0) = \left(\lambda_2 - \alpha\right) T_{\lambda_2,\alpha} \psi_- N_2 u_-(0).$$

Multiplying this equality by T_{α,λ_1}, T_{α,λ_2} and using 5) (1.2), we obtain the relation

$$\left\{\left(\lambda_1 - \alpha\right) N_1 - \left(\lambda_2 - \alpha\right) N_2 - \left(\lambda_1 - \alpha\right)\left(\lambda_2 - \alpha\right)\Gamma\right\} u_-(0) = 0, \tag{2.2}$$

which also follows from the consistency condition (1.11) with the function $u_-(t)$ depending on t in the chosen way. To every operator A_p of the commutative colligation Δ (1.1) there corresponds the characteristic function [11]

$$S_p(\lambda_p) \stackrel{\text{def}}{=} K + i(\lambda_p - \alpha)\psi_+ T_{\lambda_p,\alpha}\psi_- N_p \quad (p = 1, 2). \tag{2.3}$$

Theorem 2.1. *Let a point $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$ be such that for $u_-(0)$ (2.2) takes place, then*

$$S_1(\lambda_1)\, u_-(0) = S_2(\lambda_2)\, u_-(0). \tag{2.4}$$

The proof of the theorem follows from the last equation of (2.1).

If $u_-(0)$ satisfies equality (2.2), then (1.12) implies that the function $u_+(0) = S_1(\lambda_1)\, u_-(0)$ has the similar property

$$\left\{(\lambda_1 - \alpha)\tilde{N}_1 - (\lambda_2 - \alpha)\tilde{N}_2 - (\lambda_1 - \alpha)(\lambda_2 - \alpha)\tilde{\Gamma}\right\} u_+(0) = 0. \tag{2.5}$$

Theorem 2.2. *If the operators N_1 and $\tilde{N}_1$ of the commutative colligation Δ (1.1) are invertible, then for the characteristic function $S_1(\lambda_1)$ (2.3) the intertwining condition is true*

$$S_1(\lambda_1)\, N_1^{-1}\left[(\lambda_1 - \alpha)\Gamma + N_2\right] = \tilde{N}_1^{-1}\left[(\lambda_1 - \alpha)\tilde{\Gamma} + \tilde{N}_2\right] S_1(\lambda_1). \tag{2.6}$$

P r o o f. Equalities (2.2) and (2.5) imply

$$(\lambda_2 - \alpha)^{-1}(\lambda_1 - \alpha)\, u_-(0) = N_1^{-1}\left[(\lambda_1 - \alpha)\Gamma + N_2\right] u_-(0);$$

$$(\lambda_2 - \alpha)^{-1}(\lambda_1 - \alpha)\, u_+(0) = \tilde{N}_1^{-1}\left[(\lambda_1 - \alpha)\tilde{\Gamma} + \tilde{N}_2\right] u_+(0).$$

Multiplying the first equality by $S_1(\lambda_1)$ and taking into account that $u_+(0) = S_1(\lambda_1)\, u_-(0)$, we obtain relation (2.6). ∎

If $\dim E_\pm < \infty$, then the existence of non-trivial $u_-(0)$ and $u_+(0)$ satisfying (2.2) and (2.5), respectively, is possible if only $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$ belongs to the algebraic curves

$$\begin{aligned} \mathbb{Q} &= \left\{\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2 : \mathbb{Q}(\lambda_1, \lambda_2) = 0\right\}; \\ \tilde{\mathbb{Q}} &= \left\{\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2 : \tilde{\mathbb{Q}}(\lambda_1, \lambda_2) = 0\right\} \end{aligned} \tag{2.7}$$

given by the polynomials

$$\begin{aligned} \mathbb{Q}(\lambda_1, \lambda_2) &\stackrel{\text{def}}{=} \det\left[(\lambda_1 - \alpha)N_1 - (\lambda_2 - \alpha)N_2 - (\lambda_1 - \alpha)(\lambda_2 - \alpha)\Gamma\right]; \\ \tilde{\mathbb{Q}}(\lambda_1, \lambda_2) &\stackrel{\text{def}}{=} \det\left[(\lambda_1 - \alpha)\tilde{N}_1 - (\lambda_2 - \alpha)\tilde{N}_2 - (\lambda_1 - \alpha)(\lambda_2 - \alpha)\tilde{\Gamma}\right]. \end{aligned} \tag{2.8}$$

The intertwining condition (2.6) yields that the characteristic function $S_1(\lambda_1)$ (2.3) maps the root subspaces of the linear bundles $N_1^{-1}\left[(\lambda_1-\alpha)\Gamma+N_2\right]$ and $\tilde{N}_1\left[(\lambda_1-\alpha)\tilde{\Gamma}+\tilde{N}_2\right]$ one into another. If $S_1(\lambda_1)$ is invertible at least in one point of the holomorphy from Ω, then $\dim E_-=\dim E_+<\infty$ and the polynomials (2.8) coincide, $\mathbb{Q}(\lambda_1,\lambda_2)=\tilde{\mathbb{Q}}(\lambda_1,\lambda_2)$.

II. For the dual open system $F_\Delta^+=\left\{R_\Delta^+,S_\Delta^+\right\}$ (1.14)–(1.18), consider the case of $\tilde{u}_+(t)=e^{i\left\langle\bar{\lambda},t-T\right\rangle}u_+(T)$, where $\lambda=(\lambda_1,\lambda_2)\in\mathbb{C}^2$, $t=(t_1,t_2)\in D$, $\left\langle\bar{\lambda},t-T\right\rangle=\bar{\lambda}_1(t_1-T_1)+\bar{\lambda}_2(t_2-T_2)$. Suppose that $\tilde{y}(t)=e^{i\left\langle\bar{\lambda},t-T\right\rangle}\tilde{y}(T)$, $\tilde{u}_-(t)=e^{i\left\langle\bar{\lambda},t-T\right\rangle}\tilde{u}_-(T)$, then from (1.20) we obtain

$$\left\{\begin{array}{l}\tilde{y}(T)=-\left(\bar{\lambda}_1-\bar{\alpha}\right)T_{\lambda_1,\alpha}^*\psi_+^*\tilde{N}_1^*\tilde{u}_+(T);\\ y(T)=-\left(\bar{\lambda}_2-\bar{\alpha}\right)T_{\lambda_2,\alpha}^*\psi_+^*\tilde{N}_2^*\tilde{u}_+(T);\\ \tilde{u}_-(T)=K^*\tilde{u}_+(T)+i\psi_-^*\tilde{y}(T).\end{array}\right. \tag{2.9}$$

Double representation of $\tilde{y}(T)$ in (2.9) signifies that

$$\left\{\left(\bar{\lambda}_1-\bar{\alpha}\right)\tilde{N}_1^*-\left(\bar{\lambda}_2-\bar{\alpha}\right)\tilde{N}_2^*-\left(\bar{\lambda}_1-\bar{\alpha}\right)\left(\bar{\lambda}_2-\bar{\alpha}\right)\tilde{\Gamma}^*\right\}\tilde{u}_+(T)=0, \tag{2.10}$$

which is the corollary of the consistency condition (1.21).

Similarly to the statement of Theorem 2.1,

$$\overset{+}{S}_1(\lambda_1)\,\tilde{u}_+(T)=\overset{+}{S}_2(\lambda_2)\,\tilde{u}_+(T) \tag{2.11}$$

takes place on condition that $\tilde{u}_+(T)$ satisfies relation (2.10), where $\overset{+}{S}_p(\lambda_p)$ equals

$$\overset{+}{S}_p(\lambda_p)\overset{\text{def}}{=}K^*-i\left(\bar{\lambda}_p-\bar{\alpha}\right)\psi_-^*T_{\lambda_p,\alpha}^*\psi_+^*\tilde{N}_p^*,\quad p=1,2. \tag{2.12}$$

The functions $S_p(\lambda)$ (2.3) and $\overset{+}{S}_p(\lambda_p)$ (2.12) are linked to each other by the relations

$$N_p^*\,\overset{+}{S}_p(\lambda_p)=S_p^*(\lambda_p)\,\tilde{N}_p^*,\quad p=1,2. \tag{2.13}$$

The equality (1.22) implies that the vector function $\tilde{u}_-(T)=\overset{+}{S}_1(\lambda_1)\,\tilde{u}_+(T)$ satisfies the equality

$$\left\{\left(\bar{\lambda}_1-\bar{\alpha}\right)N_1^*-\left(\bar{\lambda}_2-\bar{\alpha}\right)N_2^*-\left(\bar{\lambda}_1-\bar{\alpha}\right)\left(\bar{\lambda}_2-\bar{\alpha}\right)\Gamma^*\right\}\tilde{u}_-(T)=0. \tag{2.14}$$

For $\overset{+}{S}_1(\lambda_1)$ (2.12), the intertwining property also holds

$$\overset{+}{S}_1(\lambda_1)\left(N_1^*\right)^{-1}\left[\left(\bar{\lambda}_1-\bar{\alpha}\right)\tilde{\Gamma}^*+\tilde{N}_2^*\right]=\left(N_1^*\right)^{-1}\left[\left(\bar{\lambda}_1-\bar{\alpha}\right)\Gamma^*+N_2^*\right]\overset{+}{S}_1(\lambda_1),$$

which follows from (2.6) if one takes into account (2.13). The algebraic curves corresponding to (2.10), (2.14) are the complex adjoints of the curves (2.7).

III. 1) (1.13) and 1) (1.23) imply

$$\frac{\sigma_1^- - S_1^*(w_1)\,\sigma_1^+ S_1(\lambda_1)}{i(\lambda_1 - \bar{w}_1)} = N_1^* \psi_-^* T_{w_1,\alpha}^* T_{\lambda_1,\alpha} \psi_- N_1;$$

$$\frac{\left(\overset{+}{S}_1(w_1)\right)^* \tau_1^- \overset{+}{S}_1(\lambda_1) - \tau_1^+}{i\left(\bar{\lambda}_1 - w_1\right)} = \tilde{N}_1 \psi_+ T_{w_1,\alpha} T_{\lambda_1,\alpha}^* \psi_+^* \tilde{N}_1^*; \tag{2.15}$$

$$\frac{S_1(\lambda_1) - S_1(w_1)}{i(\lambda_1 - w_1)} = \psi_+ T_{w_1,\alpha} T_{\lambda_1,\alpha} \psi_- N_1.$$

Define the operator function $K(\lambda, w)$ in $E_- \oplus E_+$

$$K(\lambda, w) \stackrel{\text{def}}{=} \begin{bmatrix} \dfrac{\sigma_1^- - S_1^*(w)\sigma_1^+ S_1(\lambda)}{i(\lambda - \bar{w})} & N_1^* \dfrac{\overset{+}{S}_1(\lambda) - \overset{+}{S}_1(w)}{i\left(\bar{w} - \bar{\lambda}\right)} \\ \tilde{N}_1 \dfrac{S_1(\lambda) - S_1(w)}{i(\lambda - w)} & \dfrac{\left(\overset{+}{S}_1(w)\right)^* \tau_1^- \overset{+}{S}_1(\lambda) - \tau_1^+}{i\left(\bar{\lambda} - w\right)} \end{bmatrix}. \tag{2.16}$$

It is obvious that the kernel $K(\lambda, w)$ (2.16) is positively defined [10, 11] as λ, $w \in \Omega$ since $K(\lambda, w) = \Pi^*(w)\Pi(\lambda)$, where $\Pi(\lambda) = \left[T_{\lambda,\alpha}\psi_- N_1, T_{\lambda,\alpha}^* \psi_+^* \tilde{N}_1^*\right]$ and $T_{\lambda,\alpha} = I + (\lambda - \alpha) R_1(\lambda)$.

A subspace $H_1 \subseteq H$ is said to be a reducing one [3, 5, 6] for a commutative system of the linear unbounded operators $\{A_1, A_2\}$ if there exists nonempty common domain of holomorphy Ω of resolvents $R_p(\lambda) = (A_p - \lambda I)^1$ such that in every point $\lambda \in \Omega$ $R_p(\lambda) P_1 = P_1 R_p(\lambda)$, $p = 1, 2$, where P_1 is the orthoprojector on H_1.

For the commutative colligation Δ (1.1), define the subspace H_1 in H:

$$H_1 = \text{span}\left\{R_2(w) R_1(\lambda) \psi_- u_- + R_2^*(\tilde{w})\, R_1^*\left(\tilde{\lambda}\right) \psi_+^* u_+ : u_\pm \in E_\pm; \lambda, w, \tilde{\lambda}, \tilde{w} \in \Omega\right\}. \tag{2.17}$$

Theorem 2.3. *Let the operators N_1 and $\tilde{N}_1$ be invertible. Then the subspace H_1 (2.17) reduces the commutative operator system $\{A_1, A_2\}$ of the colligation Δ (1.1), besides, the restriction of $\{A_1, A_2\}$ to $H_0 = H \ominus H_1$ is the commutative system of selfadjoint operators.*

P r o o f. The analyticity of the resolvents

$$R_p(\lambda) = \sum_{k=0}^{\infty} (\lambda - \alpha)^k R_p^{k+1}, \quad p = 1, 2, \tag{2.18}$$

in the neighborhood $U_\delta(\alpha) = \{\lambda \in \Omega : |\lambda - \alpha| < \delta\}$ of the point $\alpha \in \Omega$ yields that the subspace H_1 (2.17) generates the vectors

$$R_2^m R_1^n \psi_- u_- + (R_2^*)^p (R_1^*)^q \psi_+^* u_+,$$

where $u_\pm \in E_+$; m, n, p, $q \in \mathbb{Z}_+$. The equalities

$$\begin{aligned} R_2 \psi_- &= R_1 \psi_- N_2 N_1^{-1} + \psi_- \Gamma N_1^{-1}; \\ R_2^* \psi_+^* &= R_1^* \psi_+^* \tilde{N}_2^* \left(\tilde{N}_1^*\right)^{-1} + \psi_+^* \tilde{\Gamma}^* \left(\tilde{N}_1^*\right)^{-1} \end{aligned} \tag{2.19}$$

(taking into account 5) (1.2)) imply that the subspace H_1 (2.17) is given by

$$H_1 = \operatorname{span} \left\{ R_1^n \psi_- u_- + (R_1^*)^m \psi_+^* u_+ : u_\pm \in E_\pm; n, m \in \mathbb{Z}_+ \right\}. \tag{2.20}$$

It is easy to ascertain [11] that the subspace H_1 (2.20) reduces A_1, besides, the restriction of A_1 to $H_0 = H \ominus H_1$ is a selfadjoint operator. Therefore the operator $T_1 = I + i2 \operatorname{Im} \alpha R_1$, being the Caley transform of the operator A_1 restricted to H_0, is a unitary operator.

It remains to prove that the subspace H_1 (2.20) also reduces the operator A_2 and that the restriction $A_2|_{H_0}$ is a selfadjoint operator. The equalities (2.19) imply that to prove the reducibility of the A_2 by the subspace H_1 (2.20), it is necessary to make sure that

$$R_2 (R_1^*)^m \psi_+^* u_+ \in H_1; \quad R_2^* R_1^n \psi_- u_- \in H_1$$

for all $u_\pm \in E_\pm$ and all n, $m \in \mathbb{Z}_+$. For instance, prove that

$$R_2^* R_1^n \psi_- u_- \in H_1, \quad \forall u_-, n \in \mathbb{Z}_+. \tag{2.21}$$

To do this, consider the following subspaces:

$$L^n = \operatorname{span} \{ R_2^* R_1^n \psi_- u_- : u_- \in E_- \}, \quad n \in \mathbb{Z}_+,$$

and let $L^n = L_1^n \oplus L_0^n$, where $L_q^n = P_q L^n$ and P_q is the orthoprojector on H_q, $q = 0, 1$. Since H_1 (2.20) reduces A_1, then $R_1^* L_q^n \subset H_q$ and $T_1^* L_q^n \subset H_q$, $q = 0, 1$, for all $n \in \mathbb{Z}_+$. Prove that $L_0^n = \{0\}$ for all $n \in \mathbb{Z}_+$, which signifies that inclusion (2.21) is true. The point 3) (1.2) yields that $T_1^* \psi_- = \psi_+^* \sigma_1^+ K N_1^{-1}$, therefore $T_1^* R_2^* \psi_- u_- = -R_2^* \psi_+^* \sigma_1^+ K N_1^{-1} u_- \in H_1$ and thus $T_1^* L_0^0 \subseteq L_1^0$. This implies that $L_0^0 = \{0\}$, in view of the unitarity of the restriction of T_1^* on H_0. Let, in view of the mathematical induction principle, $L_0^k = \{0\}$ be proved as $k = 0, 1, \dots, n$, now prove that $L_0^{n+1} = \{0\}$. Really,

$$T_1^* R_2^* R_1^{n+1} \psi_- = R_2^* T_1^* \frac{1}{i2\operatorname{Im}\alpha} (T_1 - I) R_1^n \psi_- = \frac{1}{i2\operatorname{Im}\alpha} R_2^* T_1^* T_1 R_1^n \psi_-$$

$$-\frac{1}{i2\operatorname{Im}\alpha} T_1^* R_2^* R_1^n \psi_- = \frac{1}{i} R_2^* B_1 R_1^n \psi_- + \frac{1}{i2\operatorname{Im}\alpha} R_2^* R_1^n \psi_- - \frac{1}{i2\operatorname{Im}\alpha} T_1^* R_2^* R_1^n \psi_-,$$

where $B_p = iR_p - iR_p^* + 2\operatorname{Im}\alpha R_p^* R_p$, $\tilde{B}_p = iR_p - iR_p^* + 2\operatorname{Im}\alpha R_p R_p^*$, $p = 1, 2$ [11]. Using $B_1 = \psi_+^* \sigma_1^+ \psi_-$ (see [11]), we obtain

$$T_1^* L_0^{n+1} \subset \operatorname{span}\{L_0^n + T_1^* L_0^n\} = \{0\},$$

and thus $L_0^{n+1} = \{0\}$ in view of unitarity of $T_1^*|_{H_0}$.

To complete the proof of the theorem, it remains to determine that the restriction of A_2 to H_0 is a selfadjoint operator. Prove that the operator $T_2 = I + i2\operatorname{Im}\alpha \cdot R_2$ restricted to H_0 is unitary. Since $B_2 H = \psi_+^* \sigma_2^+ \psi_+ H$ and $\tilde{B}_2 H = \psi_- \tau_2^- \psi_-^* H$ (see [11]) belong to H_1 (2.20), then $H_0 \subset \operatorname{Ker} B_2$ and $H_0 \subset \operatorname{Ker} \tilde{B}_2$, what guarantees that the restriction of T_2 to H_0 is unitary. ■

The colligation Δ (1.1) is said to be simple if $H = H_1$, where H_1 is given by (2.17).

Consider two commutative colligations Δ and $\hat{\Delta}$ (1.1) such that

$$E_\pm = \hat{E}_\pm; \quad \sigma_p^\pm = \hat{\sigma}_p^\pm; \quad \tau_p^\pm = \hat{\tau}_p^\pm; \quad N_p = \hat{N}_p; \quad \tilde{N}_p = \tilde{\hat{N}}_p, \quad p = 1,\, 2;$$

$$\Gamma = \hat{\Gamma}; \quad \tilde{\Gamma} = \tilde{\hat{\Gamma}}; \quad K = \hat{K};$$

and, moreover, $\alpha = \hat{\alpha} \in \Omega \cap \hat{\Omega} \neq \{\emptyset\}$. These colligations are said to be unitarily equivalent if there exists such a unitary operator $U\colon H \to \hat{H}$ that

$$\begin{gathered} UA_p = \hat{A}_p U; \quad U\mathfrak{D}(A_p) = \mathfrak{D}\left(\hat{A}_p\right); \quad UA_p^* = \hat{A}_p^* U; \\ U\mathfrak{D}\left(A_p^*\right) = \mathfrak{D}\left(\hat{A}_p^*\right) \quad (p = 1,\, 2); \quad U\psi_- = \hat{\psi}_-; \quad \hat{\psi}_+ U = \psi_+. \end{gathered} \tag{2.22}$$

It is easy to show that the characteristic functions of unitarily equivalent colligations Δ and $\hat{\Delta}$ coincide, $S_1(\lambda_1) = \hat{S}_1(\lambda_1)$ (2.3) for all $\lambda \in \Omega \cap \tilde{\Omega}$.

Theorem on the unitary equivalence 2.4. *Let Δ and $\hat{\Delta}$ (1.1) be two simple commutative colligations such that $E_\pm = \hat{E}_\pm$; $\sigma_p^\pm = \hat{\sigma}_p^\pm$; $\tau_p^\pm = \hat{p}^\pm$; $N_p = \hat{N}_p$; $\tilde{N}_p = \tilde{\{\hat{N}_p\}}$, $p = 1, 2$; $\Gamma = \hat{\Gamma}$; $\tilde{\Gamma} = \tilde{\hat{\Gamma}}$; and $\alpha = \hat{\alpha} \in \Omega \cap \hat{\Omega}$ ($\neq \emptyset$). Then if the operators N_1 and $\tilde{N}_1$ are invertible and in some neighborhood $U_\delta(\alpha)$ of point α the characteristic functions coincide, $S_1(\lambda_1) = \hat{S}_1(\lambda_1)$ (2.3), the colligations Δ and $\hat{\Delta}$ are unitarily equivalent.*

P r o o f. The coincidence of characteristic functions, $S_1(\lambda_1) = \hat{S}_1(\lambda_1)$, as $\lambda_1 \in U\delta(\alpha)$ and N_1, $\tilde{N}_1$ are unitary, and (2.15) imply that

$$\psi_-^* T_{w_1,\alpha} T_{\lambda_1,\alpha} \psi_- = \hat{\psi}_-^* \hat{T}_{w_1,\alpha}^* \hat{T}_{\lambda_1,\alpha} \hat{\psi}_-;$$

$$\psi_+ T_{w_1,\alpha} T_{\lambda_1,\alpha}^* \psi_+^* = \hat{\psi}_+ \hat{T}_{w_1,\alpha} \hat{T}_{\lambda_1,\alpha}^* \hat{\psi}_+^*;$$

$$\psi_+ T_{w_1,\alpha} T_{\lambda_1,\alpha} \psi_- = \hat{\psi}_+ \hat{T}_{w,\alpha} \hat{T}_{\lambda_1,\alpha} \hat{\psi}_-$$

for all λ_1, $w_1 \in U_\delta(\alpha)$. Taking into account holomorphy (2.18) of the resolvents $R_1(\lambda_1)$ and $\hat{R}_1(\lambda_1)$ in the neighborhood $U_\delta(\alpha)$, we can rewrite these equalities in the equivalent form

$$\begin{aligned} \psi_-^* (R_1^*)^m R_1^n \psi_- &= \hat{\psi}_-^* \left(\hat{R}_1^*\right)^m \hat{R}_1^n \hat{\psi}_-; \\ \psi_+ R_1^m (R_1^*)^n \psi_+^* &= \hat{\psi}_+ \hat{R}_1^m \left(\hat{R}_1^*\right)^n \hat{\psi}_+^*; \\ \psi_+ R_1^m R_1^n \psi_- &= \hat{\psi}_+ \hat{R}_1^m \hat{R}_1^n \hat{\psi}_- \end{aligned} \tag{2.23}$$

for all n, $m \in \mathbb{Z}_+$. Define the linear operator U: $H \to \hat{H}$

$$U R_1^n \psi_- u_- \stackrel{\text{def}}{=} \hat{R}_1^n \hat{\psi}_- u_-; \quad u\,(R_1^*)^m \psi_+^* u_+ \stackrel{\text{def}}{=} \left(\hat{R}_1^*\right)^m \hat{\psi}_+^* u_+, \tag{2.24}$$

where $u_\pm \in E_\pm$ and n, $m \in \mathbb{Z}_+$. The simplicity of the colligations Δ, $\hat{\Delta}$ and the invertibility of N_1, $\tilde{N}_1$ yield that the spaces H and $\hat{H}$ are given by (2.20), and thus the operator U (2.24) is unitary in view of (2.23). It is easy to prove (see [11]) that

$$U R_1 = \hat{R}_1 U; \quad U\psi_- = \hat{\psi}_-; \quad \hat{\psi}_+ U = \psi_+.$$

It remains to prove that $UR_2 = \hat{R}_2 U$. And since the application of the resolvent R_2 (R_2^*) to the vectors $R_1^n \psi_- u$ (correspondingly, to $(R_1^*)^m \psi_+^* u_+$) is expressed also in terms of these vectors, then it is obvious that

$$\left(UR_2 - \hat{R}_2 U\right) R_1^n \psi_- u_- = 0; \quad \left(UR_2^* - \hat{R}_2^* U\right) (R_1^*)^m \psi_+^* u_+ = 0, \tag{2.25}$$

for all $u_\pm \in E_+$ and all n, $m \in \mathbb{Z}_+$. Thus, it is necessary to prove that

$$\left(UR_2 - \hat{R}_2 U\right) (R_1^*)^m \psi_+^* u_+ = 0 \tag{2.26}$$

for all $u_+ \in E_+$ and all $m \in \mathbb{Z}_+$. It is easy to see that when $m = 0$

$$\hat{T}_1 \left(UR_2 - \hat{R}_2 U\right) \psi_+^* u_+ = \left(UR_2 - \hat{R}_2 U\right) T_1 \psi_+^* u_+$$

$$= -\left(UR_2 - \hat{R}_2 U\right) \psi_- \tau_1^- K^* \left(\tilde{N}_1^*\right)^{-1} u_+ = 0$$

in view of (2.25) and $T_1\psi_+^*\tilde{N}_1^* + \psi_-\tau_1^- K^* = 0$ (see 3) (1.2)). Prove that this implies (2.26) when $m = 0$. One can see that

$$0 = \hat{T}_1^* T_1 \left(UR_2 - \hat{R}_2 U\right)\psi_+^* = 2\operatorname{Im}\alpha \hat{B}_1 \left(UR_2 - \hat{R}_2 U\right)\psi_+^*$$

$$+ \left(UR_2 - \hat{R}_2 U\right)\psi_+^*,$$

and to prove (2.26) ($m = 0$), it is necessary to establish that

$$\hat{\psi}_+^* \sigma_1^+ \hat{\psi}_+ \left(UR_2 - \hat{R}_2 U\right)\psi_+^* = 0.$$

And the last,

$$\hat{\psi}_+ \left(UR_2 - \hat{R}_2 U\right)\psi_+^* = \psi_+ R_2 \psi_+^* - \hat{\psi}_+ \hat{R}_2 \hat{\psi}_+ = 0,$$

follows easily from the definition of U (2.24) and formulas (2.19), (2.23). Thus, relation (2.26) for $m = 0$ is proved.

Using the principle of mathematical induction, suppose that equality (2.26) is already proved for $m = n$; prove that it is also true for $m = n + 1$. It is easy to see that

$$\hat{T}_1 \left(UR_2 - \hat{R}_2 U\right)(R_1^*)^{n+1}\psi_+^* u_+ = \left(UR_2 - \hat{R}_2 U\right) T_1 R_1^* (R_1^*)^n \psi_+^* u_+$$

$$= \frac{1}{i2\operatorname{Im}\alpha}\left(UR_2 - \hat{R}_2 U\right) T_1 (T_1^* - I)(R_1^*)^n \psi_+^* u_+$$

$$= i\left(UR_2 - \hat{R}_2 U\right)\tilde{B}_1 (R_1^*)^n \psi_+^* u_+$$

$$= i\left(UR_2 - \hat{R}_2 U\right)\psi_- \tau_1^- \psi_-^* (R_1^*)^n \psi_+^* u_+ = 0$$

in view of the induction supposition and of (2.25). And since

$$0 = \hat{T}_1^* \hat{T}_1 \left(UR_2 - \hat{R}_2 U\right)(R_1^*)^{n+1}\psi_+^* u_+$$

$$= 2\operatorname{Im}\alpha \hat{B}_1 \left(UR_2 - \hat{R}_2 U\right)(R_1^*)^{n+1}\psi_+^* u_+ + \left(UR_2 - \hat{R}_2 U\right)(R_1^*)^{n+1}\psi_+^* u_+,$$

then to prove (2.26) for $m = n + 1$, it is sufficient to prove that

$$\hat{\psi}_+ \left(UR_2 - \hat{R}_2 U\right)(R_1^*)^{n+1}\psi_+^* = \psi_+ R_2 (R_1^*)^{n+1}\psi_+^* - \hat{\psi}_+ \hat{R}_2 \left(\hat{R}_1^*\right)^{n+1}\hat{\psi}_+^* = 0,$$

and this obviously follows from (2.23) and (2.19). ∎

Thus, the characteristic function $S_1(\lambda_1)$ (2.3) and the "external set of parameters" $\{\sigma_p^\pm\}_1^2$; $\{\tau_p^\pm\}_1^2$; $\{N_p\}_1^2$; $\left\{\tilde{N}_p\right\}_1^2$; Γ; $\tilde{\Gamma}$, on condition that the operators N_1 and $\tilde{N}_1$ are invertible, define the simple commutative colligation Δ (1.1) up to the unitary equivalence.

IV. Since $S_1(\lambda_1)$ (2.3) is the main analytic object, in terms of which the simple commutative colligation Δ (1.1) is characterized, here we describe the main properties of the function

$$S_1(\lambda) = K + i(\lambda - \alpha)\psi_+ T_{\lambda,\alpha}\psi_- N_1, \tag{2.27}$$

where for simplicity we denote $T_{\lambda,\alpha} = I + (\lambda - \alpha)R_1(\lambda)$.

Consider the generating vector function (2.1)

$$y = -(\lambda - \alpha)T_{\lambda,\alpha}\psi_- N_1 u_-, \tag{2.28}$$

where λ, $\alpha \in \Omega$, and $u_- \in E_-$. As in (1.4), using y (2.28) construct

$$y_1 = R_1 y = -(\lambda - \alpha)R_1(\lambda)\psi_- N_1 u_1 \in \mathfrak{D}(A_1). \tag{2.29}$$

Then the colligation relation 4) (1.2),

$$2\,\mathrm{Im}\,\langle A_1 y_1, y_1\rangle = \left\langle \sigma_1^+ \varphi_+^1 y_1, \varphi_+^1 y_1 \right\rangle, \tag{2.30}$$

when y_1 (2.29) is chosen in this way, signifies that

$$\frac{1}{i}\langle A_1 R_1(\lambda)\psi_- N_1 u_-, R_1(w)\psi_- N_1 \hat{u}_-\rangle - \frac{1}{i}\langle R_1(\lambda)\psi_- N_1 u_-, A_1 R_1(w)\psi_- N_1 \hat{u}_-\rangle$$
$$= \left\langle \sigma_1^+ \psi_+ T_{\lambda,\alpha}\psi_- N_1 u_-, \psi_+ T_{w,\alpha}\psi_- N_1 \hat{u}_- \right\rangle,$$

for all u_-, $\hat{u}_- \in E_-$ and all λ, w, $\alpha \in \Omega$. Using $A_1 R_1(\lambda) = \lambda R_1(\lambda) + I$, we obtain

$$\frac{1}{i} N_1^* \psi_-^* \left\{ R_1^*(w)\left[\lambda R_1(\lambda) + I\right] - \left[\bar{w} R_1^*(w) + I\right] R_1(\lambda) \right\} \psi_- N_1$$
$$= \frac{1}{(\lambda - \alpha)(\bar{w} - \bar{\alpha})}\left[S_1^*(w) - K^*\right]\sigma_1^+\left[S_1(\lambda) - K\right].$$

And, since

$$\left[S_1^*(w) - K^*\right]\sigma_1^+\left[S_1(\lambda) - K\right] = S_1^*(w)\sigma_1^+ S_1(\lambda) + K^*\sigma_1^+\left[K - S_1(\lambda)\right]$$
$$+ \left[K^* - S_1^*(w)\right]\sigma_1^+ K - K^*\sigma_1^+ K,$$

then, taking into account (2.27) and 1., 3. (1.2), we obtain the equality

$$\left[S_1^*(w) - K^*\right]\sigma_1^+\left[S_1(\lambda) - K\right] = S_1^*(w)\sigma_1^+ S_1(\lambda) - \sigma_1^-$$

$$+iN_1^*\psi_-^*\left\{(\lambda-\alpha)T_{\lambda,\bar\alpha}-(\bar w-\bar\alpha)\,T_{w,\bar\alpha}^*+(\alpha-\bar\alpha)\,I\right\}\psi_-N_1.$$

Therefore,

$$\frac{1}{(\lambda-\alpha)\,(\bar w-\bar\alpha)}\left(S_1^*(w)\sigma_1^+S_1(\lambda)-\sigma_1^-\right)=iN_1^*\psi_-^*\Big\{(\bar w-\lambda)\,R_1^*(w)R_1(\lambda)$$

$$+R_1(\lambda)-R_1^*(w)-\frac{1}{\bar w-\bar\alpha}\left(I+(\lambda-\bar\alpha)\,R_1(\lambda)\right)+\frac{1}{\lambda-\alpha}\left(I+(\bar w-\alpha)\,R_1^*(w)\right)$$

$$-\frac{\alpha-\bar\alpha}{(\lambda-\alpha)\,(\bar w-\bar\alpha)}I\Big\}\psi_-N_1.$$

After elementary calculations, we obtain the relation

$$S_1^*(w)\sigma_1^+S_1(\lambda)-\sigma_1^-=i\,(\bar w-\lambda)\,N_1^*\psi_-^*T_{w,\alpha}^*T_{\lambda,\alpha}\psi_-N_1$$

which exactly coincides with the first equality of (2.15).

Lemma 2.1. *If (2.30) holds for the operator A_1 of the commutative colligation Δ (1.1) on the vector functions y_1 (2.29), then the first formula in (2.15) is true for the characteristic function $S_1(\lambda)$ (2.27).*

Thus, the observance of the conservation law 1) (1.13), $p=1$, is adequate to the colligation relation (2.32) for the operator A_1.

Using (1.4), construct the vector function y_2 by the generating function y (2.28)

$$y_2=R_2y=-(\lambda-\alpha)R_2T_{\lambda,\alpha}\psi_-N_1u_-\in\mathfrak{D}\left(A_2\right),\tag{2.31}$$

where $u_-\in E_-$, and λ, $\alpha\in\Omega$. Write the colligation relation 4. (1.2) for y_2

$$2\,\mathrm{Im}\left\langle A_2y_2,y_2\right\rangle=\left\langle\sigma_2^+\varphi_+^2y_2,\varphi_+^2y_2\right\rangle,\tag{2.32}$$

which is equivalent to

$$\frac{1}{i}\left\langle A_2R_2T_{\lambda,\alpha}\psi_-N_1u_-,R_2T_{w,\alpha}\psi_-N_1\hat u_-\right\rangle$$

$$-\frac{1}{i}\left\langle R_2T_{\lambda,\alpha}\psi_-N_1u_-,A_2R_2T_{w,\alpha}\psi_-N_1\hat u_-\right\rangle$$

$$=\left\langle\sigma_2^+\psi_+T_{\lambda,\alpha}\psi_-N_1u_-,\psi_+T_{w,\alpha}\psi_-N_1\hat u_-\right\rangle$$

for all u_-, $\hat u_-\in E_-$ and all λ, w, $\alpha\in\Omega$. Since $A_2R_2=\alpha R_2+I$, the last equality yields

$$\frac{1}{(\lambda-\alpha)\,(\bar w-\bar\alpha)}\left[S_1^*(w)-K^*\right]\sigma_2^+\left[S_1(\lambda)-K\right]$$

$$=N_1^*\psi_-^*\left\{\frac{\alpha-\bar\alpha}{i}R_2^*T_{w,\alpha}^*T_{\lambda,\alpha}R_2+\frac{1}{i}R_2^*T_{w,\alpha}^*T_{\lambda,\alpha}-\frac{1}{i}T_{w,\alpha}^*T_{\lambda,\alpha}R_2\right\}\psi_-N_1.$$

It is obvious that

$$T_{\lambda,\alpha}R_2\psi_-N_1 = \frac{1}{\lambda-\alpha}\left[T_{\lambda,\alpha}\psi_-N_1L_\lambda - \psi_-N_2\right], \tag{2.33}$$

where L_λ is the linear bundle of the operators

$$L_\lambda = N_1^{-1}\left[(\lambda-\alpha)\Gamma + N_2\right]. \tag{2.34}$$

Using the form of function $S_1(\lambda)$ (2.3) and (2.35), we obtain

$$S_1^*(w)\sigma_2^+S_1(\lambda) - K^*\sigma_2^+S_1(\lambda) - K^*\sigma_2^+K - i(\lambda-\alpha)K^*\sigma_2^+\psi_+T_{\lambda,\alpha}\psi_-N_1$$
$$+i\left(\bar{w}-\bar{\alpha}\right)N_1^*\psi_-^*T_{w,\alpha}^*\psi_+^*\sigma_2^+K$$
$$= \frac{\alpha-\bar{\alpha}}{i}\left\{L_w^*N_1^*\psi_-^*T_{w,\alpha}^* - N_2^*\psi_-^*\right\}\left\{T_{\lambda,\alpha}\psi_-N_1L_\lambda - \psi_-N_2\right\}$$
$$+\frac{\lambda-\alpha}{i}\left\{L_w^*N_1^*\psi_-^*T_{w,\alpha}^* - N_2^*\psi_-^*\right\}\cdot T_{\lambda,\alpha}\psi_-N_1$$
$$-\frac{\bar{w}-\bar{\alpha}}{i}N_1^*\psi_-^*T_{w,\alpha}^*\left\{T_{\lambda,\alpha}\psi_-N_1L_\lambda - \psi_-N_2\right\}.$$

Denote by $K^{1,1}(\lambda,w)$ the left upper block of the kernel $K(\lambda,w)$ (2.16)

$$K^{1,1}(\lambda,w) = \frac{\sigma_1^- - S_1^*(w)\sigma_1^+S_1(\lambda)}{i\left(\lambda-\bar{w}\right)} = N_1^*\psi_-^*T_{w,\alpha}^*T_{\lambda,\alpha}\psi_-N_1. \tag{2.35}$$

Rewrite condition 3) (1.2) as $K^*\sigma_2^+\psi_+ + N_2^*\psi_-\left(I+\left(\alpha-\bar{\alpha}\right)R_2\right) = 0$. Then, taking into account (2.33), we have

$$K^*\sigma_2\psi_+T_{\lambda,\alpha}\psi_-N_1 = -N_2^*\psi_-^*\left(I+\left(\alpha-\bar{\alpha}\right)R_2\right)T_{\lambda,\alpha}\psi_-N_1 = -N_2^*\psi_-^*T_{\lambda,\alpha}\psi_-N_1$$
$$-\left(\alpha-\bar{\alpha}\right)N_2^*\psi_-^*T_{\lambda,\alpha}R_2\psi_-N_1 = -N_2^*\psi_-^*T_{\lambda,\alpha}\psi_-N_1 - \frac{\alpha-\bar{\alpha}}{\lambda-\alpha}N_2^*\psi_-^*T_{\lambda,\alpha}\psi_-N_1L_\lambda$$
$$+\frac{\alpha-\bar{\alpha}}{\lambda-\alpha}N_2^*\psi_-^*\psi_-N_2.$$

Therefore, using 1) (1.2), (2.35), after simple calculations we obtain the equality

$$\frac{i}{\alpha-\bar{\alpha}}\left\{S_1^*(w)\sigma_2^+S_1(\lambda) - \sigma_2^-\right\} = L_w^*K^{1,1}(\lambda,w)L_\lambda$$
$$-\frac{\lambda-\alpha}{\bar{\alpha}-\alpha}L_w^*K^{1,1}(\lambda,w) - \frac{\bar{w}-\bar{\alpha}}{\alpha-\bar{\alpha}}K^{1,1}(\lambda,w)L_\lambda. \tag{2.36}$$

The fact that this relation follows easily from the conservation law 2) (1.13) is an important observation. Really, let $u_\pm(t) = e^{i\langle\lambda,t\rangle}u_\pm(0)$ and $\hat{u}_\pm(t) = e^{i\langle w,t\rangle}\hat{u}_\pm(0)$, then 2) (1.13) implies

$$\left(\lambda_2-\bar{w}_2\right)\left(\lambda_1-\alpha\right)\left(\bar{w}_1-\bar{\alpha}\right)\left\{\left\langle\sigma_1^-u_-(0),\hat{u}_-(0)\right\rangle - \left\langle\sigma_1^+u_+(0),\hat{u}_+(0)\right\rangle\right\}$$

$$= (\lambda_1 - \bar{w}_1)(\lambda_2 - \alpha)(\bar{w}_2 - \bar{\alpha})\left\{\left\langle \sigma_2^- u_-(0), \hat{u}_-(0)\right\rangle - \left\langle \sigma_2^+ u_+(0); \hat{u}_+(0)\right\rangle\right\}.$$

And since

$$L_{\lambda_1} u_-(0) = \frac{\lambda_1 - \alpha}{\lambda_2 - \alpha} u_-(0); \quad L_{w_1} \hat{u}(0) = \frac{w_1 - \alpha}{w_2 - \alpha} \hat{u}_-(0); \tag{2.37}$$

in view of (2.2), then, taking into consideration $u_+(0) = S_1(\lambda) u_-(0)$; $\hat{u}_+(0) = S_1(w)\hat{u}_-(0)$, we obtain

$$(\lambda_2 - \bar{w}_2)\left\langle\left\{\sigma_1^- - S_1^*(w_1)\,\sigma_1^+ S_1(\lambda_1)\right\} L_{\lambda_1} u_-(0), L_{w_1}\hat{u}_-(0)\right\rangle$$
$$= (\lambda_1 - \bar{w}_1)\left\langle\left\{\sigma_2^- - S_1^*(w_1)\,\sigma_2^+ S_1(\lambda_1)\right\} u_-(0), \hat{u}_-(0)\right\rangle.$$

(2.37) implies

$$(\lambda_2 - \alpha)\, L_{\lambda_1} u_-(0) = (\lambda_1 - \alpha)\, u_-(0); \quad (w_2 - \alpha)\, L_{w_1} \hat{u}_-(0) = (w_1 - \alpha)\, \hat{u}_-(0).$$

Therefore, taking into consideration (2.35), we have

$$\Big\langle\Big\{(\alpha - \bar{\alpha})\, L_{w_1}^* K^{1,1}(\lambda_1, w_1)\, L_{\lambda_1} + (\lambda_1 - \alpha)\, L_w^* K^{1,1}(\lambda_1, w_1)$$
$$-\,(\bar{w}_1 - \bar{\alpha})\, K^{1,1}(\lambda_1, w_1)\Big\}\, u_-(0), \hat{u}_-(0)\Big\rangle$$
$$= i\left\langle\left\{S_1^*(w_1)\,\sigma_2^+ S_1(\lambda_1) - \sigma_2^-\right\} u_-(0), \hat{u}_-(0)\right\rangle,$$

which, in view of the arbitrariness of $u_-(0)$, $\hat{u}_-(0) \in E_-$, gives us (2.36).

Lemma 2.2. *Let the commutative colligation Δ (1.1) be given and N_1 be invertible. Then the colligation relation (2.32) for the operator A_2, where y_2 is given by (2.31), implies that the characteristic function $S_1(\lambda)$ (2.27) satisfies equality (2.36), besides, $K^{1,1}(\lambda, w)$ and L_λ are given by formulas (2.35) and (2.34), respectively. Moreover, relation (2.36) is equivalent to the conservation law 2) (1.13).*

Proceed to the consideration of colligation relations 4. (1.2) for the adjoint operators A_1^* and A_2^* of the commutative colligation Δ (1.1). Specify now the generating function (2.9)

$$\tilde{y} = -\left(\bar{\lambda} - \bar{\alpha}\right) T_{\lambda,\alpha}^* \psi_+^* \tilde{N}_1^* \tilde{u}_+, \tag{2.38}$$

where $\tilde{u}_+ \in E_+$, and λ, $\alpha \in \Omega$. According to (1.15), construct the vector function by $\tilde{y}$,

$$\tilde{y}_1 = R_1^* \tilde{y} = -\left(\bar{\lambda} - \bar{\alpha}\right) R_1^*(\lambda)\psi_+^* \tilde{N}_1^* \tilde{u}_+ \in \mathfrak{D}\left(A_1^*\right). \tag{2.39}$$

As in the previous case (see Lemma 1.1), it is easy to show that the colligation relation

$$-2\,\mathrm{Im}\left\langle A_1^* \tilde{y}_1, \tilde{y}_1\right\rangle = \left\langle \tau_1^- \left(\varphi_-^1\right)^* \tilde{y}_1, \left(\varphi_-^1\right)^* \tilde{y}_1\right\rangle, \tag{2.40}$$

for $\tilde{y}_1$ (2.39), implies that the block $K^{2,2}(\lambda, w)$ of the kernel $K(\lambda, w)$ (2.16) is given by

$$K^{2,2}(\lambda, w) = \frac{\left(\overset{+}{S}_1(w)\right)^* \tau_1^- \overset{+}{S}_1(\lambda) - \tau_1^+}{i(\lambda - w)} = \tilde{N}_1 \psi_+ T_{w,\alpha} T_{\lambda,\alpha}^* \psi_+^* \tilde{N}_1^*, \qquad (2.41)$$

besides, (2.12), (2.13)

$$\overset{+}{S}_1(\lambda) = (N_1^*)^{-1} S_1^*(\lambda) \tilde{N}_1^* = K^* - i\left(\bar{\lambda} - \bar{\alpha}\right) \psi_-^* T_{\lambda,\alpha}^* \psi_+^* \tilde{N}_1^*.$$

Lemma 2.3. *If for the operator A_1^* of the commutative colligation Δ (1.1) relation (2.40) is true on the vector functions $\tilde{y}_1$ (2.31), then for the block $K^{2,2}(\lambda, w)$ of the kernel $K(\lambda, w)$ (2.16) representation (2.41) takes place.*

By the generating function $\tilde{y}$ (2.38) according to (2.9), define $\tilde{y}_2$

$$\tilde{y}_2 = R_2^* \tilde{y} = -\left(\bar{\lambda} - \bar{\alpha}\right) R_2^* T_{\lambda,\alpha}^* \psi_+^* \tilde{N}_1^* \tilde{u}_+ \in \mathfrak{D}\left(A_2^*\right) \qquad (2.42)$$

and consider the colligation relation 4) (1.2) for A_2^*

$$-2 \operatorname{Im} \langle A_2^* \tilde{y}_2, \tilde{y}_2 \rangle = \left\langle \tau_2^- \left(\varphi_-^2\right)^* \tilde{y}_2, \left(\varphi_-^2\right)^* \tilde{y}_2 \right\rangle, \qquad (2.43)$$

where $\tilde{y}_2$ is given by (2.42). Applying similar considerations (see the proof of Lemma 2.2), it is easy to prove that

$$\frac{i}{\alpha - \bar{\alpha}} \left\{ \left(\overset{+}{S}_1(w)\right)^* \tau_2 - \overset{+}{S}_1(\lambda) - \tau_2^+ \right\} = \left(L_w^+\right)^* K^{2,2}(\lambda, w) L_\lambda^+$$
$$-\frac{\overline{\lambda} - \bar{\alpha}}{\alpha - \bar{\alpha}} \left(L_w^+\right)^* K^{2,2}(\lambda, w) - \frac{w - \alpha}{\bar{\alpha} - \alpha} K^{2,2}(\lambda, w) L_\lambda^+, \qquad (2.44)$$

where L_λ^+ is the linear bundle,

$$L_\lambda^+ = \left(\tilde{N}_1^*\right)^{-1} \left[\left(\bar{\lambda} - \bar{\alpha}\right) \tilde{\Gamma}^* + \tilde{N}_2^*\right], \qquad (2.45)$$

and $K^{2,2}(\lambda, w)$ are given by (2.41).

Lemma 2.4. *Let Δ (1.1) be a commutative colligation and $\tilde{N}_1$ be invertible. Then relation (2.43) for the operator A_2^* on the vectors $\tilde{y}_2$ (2.42) implies that (2.44) holds for the characteristic function $\overset{+}{S}_1(\lambda)$ (2.12), where $K^{2,2}(\lambda, w)$ and L_λ^+ are given by formulas (2.41) and (2.45), and $\overset{+}{S}_1(\lambda)$ is constructed by $S_1(\lambda)$ (2.3) by rule (2.13). Moreover, equality (2.44) is equivalent to the conservation law 2) (1.23).*

Note that the structure and properties of other blocks of the kernel $K(\lambda, w)$ (2.16) are also determined by the properties of the commutative system $\{A_1, A_2\}$ of the colligation Δ (1.1).

Consider the obvious equality

$$\langle A_1 y_1, \tilde{y}_1 \rangle = \langle y_1, A_1^* \tilde{y}_1 \rangle, \tag{2.46}$$

assuming that y_1 and $\tilde{y}_1$ are given by (2.29) and (2.39), respectively. This implies

$$(\lambda - \alpha)(w - \alpha) \left\langle A_1 R_1(\lambda) \psi_- N_1 u_-, R_1^*(w) \psi_+^* \tilde{N}_1^* \tilde{u}_+ \right\rangle$$

$$= (\lambda - \alpha)(w - \alpha) \left\langle R_1(\lambda) \psi_- N_1 u_-, A_1^* R_1^*(w) \psi_+^* \tilde{N}_1^* \tilde{u}_1 \right\rangle,$$

and since $A_1 R_1(\lambda) = \lambda R_1(\lambda) + I$, then

$$\lambda \tilde{N}_1 \psi_+ (T_{w,\alpha} - I)(T_{\lambda,\alpha} - I) \psi_- N_1 + (\lambda - \alpha) \tilde{N}_1 \psi_+ (T_{w,\alpha} - I) \psi_- N_1$$

$$= w \tilde{N}_1 \psi_+ (T_{w,\alpha} - I)(T_{\lambda,\alpha} - I) \psi_- N_1 + (w - \alpha) \tilde{N}_1 \psi_+ (T_{\lambda,\alpha} - I) \psi_- N_1.$$

Therefore,

$$(\lambda - w) \tilde{N}_1 \psi_+ T_{w,\alpha} T_{\lambda,\alpha} \psi_- N_1 = (\lambda - \alpha) \tilde{N}_1 \psi_+ T_{\lambda,\alpha} \psi_- N_1 - (w - \alpha) \tilde{N}_1 \psi_+ T_{w,\alpha} \psi_- N_1.$$

Taking into account the form of $S_1(\lambda)$ (2.27), we have

$$K^{2,1}(\lambda, w) = \tilde{N}_1 \frac{S_1(\lambda) - S_1(w)}{i(\lambda - w)} = \tilde{N}_1 \psi_+ T_{w,\alpha} T_{\lambda,\alpha} \psi_- N_1. \tag{2.47}$$

Lemma 2.5. *If (2.46) holds for the operator A_1 of the commutative colligation Δ (1.1), then the block $K^{2,1}(\lambda, w)$ of the kernel $K(\lambda, w)$ (2.16) has representation (2.47).*

Study the similar to (2.46) equality for A_2

$$\langle A_2 y_2, \tilde{y} \rangle = \langle y_2, A_2^* \tilde{y}_2 \rangle, \tag{2.48}$$

assuming that y_2 and $\tilde{y}_2$ are given by formulas (2.31) and (2.42). In view of $A_2 R_2 = \alpha R_2 + I$, it is easy to see that (2.48) leads to the relation

$$\tilde{N}_1 \psi_+ R_2 T_{w,\alpha} T_{\lambda,\alpha} \psi_- N_1 = \tilde{N}_1 \psi_+ T_{w,\alpha} T_{\lambda,\alpha} R_2 \psi_- N_1.$$

We can write this equality in the following way:

$$(\lambda - \alpha) \left[\tilde{L}_w \psi_+ T_{w,\alpha} - \tilde{N}_1^{-1} N_2 \psi_+ \right] \cdot T_{\lambda,\alpha} \psi_- N_1$$

$$= (w - \alpha) \psi_+ T_{w,\alpha} \left[T_{\lambda,\alpha} \psi_- N_1 L_\lambda - \psi_- N_2 \right],$$

where

$$\tilde{L}_\lambda = \tilde{N}_1^{-1}\left[(\lambda-\alpha)\tilde{\Gamma}+\tilde{N}_2\right]. \tag{2.49}$$

It is obvious that $\tilde{L}_\lambda$ (2.49) and L_λ^+ (2.45) satisfy the relation

$$L_\lambda^+ = \left(\tilde{N}_1^*\right)^{-1}\tilde{L}_\lambda^*\tilde{N}_1^*. \tag{2.50}$$

Using the definition of $S_1(\lambda)$ (2.27) and the last formula of (2.15), we obtain

$$\frac{\lambda-\alpha}{\lambda-w}\tilde{L}_w\left[S_1(\lambda)-S_1(w)\right]-\tilde{N}_1^{-1}\tilde{N}_2S_1(\lambda)$$

$$=\frac{w-\alpha}{\lambda-w}\left[S_1(\lambda)-S_1(w)\right]L_\lambda-S_1(w)N_1^{-1}N_2.$$

This easily implies that

$$(w-\alpha)\left[\tilde{L}_\lambda S_1(\lambda)-S_1(\lambda)L_\lambda\right]=(\lambda-\alpha)\left[\tilde{L}_wS_1(w)-S_1(w)L_w\right],$$

and the above implies that the relation

$$\frac{1}{\lambda-\alpha}\left[\tilde{L}_\lambda S_1(\lambda)-S_1(\lambda)L_\lambda\right]=C$$

is constant and does not depend on λ. Thus,

$$\tilde{L}_\lambda S_1(\lambda)-S_1(\lambda)L_\lambda=(\lambda-\alpha)C,$$

which is impossible when $C\neq 0$, because the coefficient of the expression $\tilde{L}_\lambda S_1(\lambda)-S_1(\lambda)L_\lambda$ when $(\lambda-\alpha)$ is equal to zero

$$\tilde{N}_1^{-1}\tilde{\Gamma}K+i\tilde{N}_1^{-1}\tilde{N}_2\psi_+\psi_-N_1-KN_1^{-1}\Gamma-i\psi_+\psi_-N_2=0$$

in view of 6) and 7) (1.2). Thus, $C=0$, and we again come to the intertwining condition (2.6).

Lemma 2.6. *Let Δ (1.1) be a commutative colligation and the operators N_1, $\tilde{N}_1$ be invertible. Then equality (2.48) for A_2 implies the intertwining condition (2.6) for the characteristic function $S_1(\lambda)$ (2.27).*

Summarizing the statements of Lemmas 2.1–2.6, we obtain the following theorem.

Theorem 2.5. *Let* Δ *(1.1) be a commutative colligation and the operators* N_1, $\tilde{N}_1$ *be invertible. Then the characteristic function* $S_1(\lambda)$ *(2.27) satisfies the relations:*

$$\begin{aligned}
&1) \quad S_1(\lambda)L_\lambda = \tilde{L}_\lambda S_1(\lambda);\\
&2) \quad \frac{i}{\alpha-\bar{\alpha}}\left\{S_1^*(w)\sigma_2^+ S_1(\lambda)-\sigma_2^-\right\} = L_w^* K^{1,1}(\lambda,w)L_\lambda\\
&\quad -\frac{\lambda-\alpha}{\bar{\alpha}-\alpha}L_w^*K^{1,1}(\lambda,w)-\frac{\bar{w}-\bar{\alpha}}{\alpha-\bar{\alpha}}K^{1,1}(\lambda,w)L_\lambda;\\
&3) \quad \frac{i}{\alpha-\bar{\alpha}}\left\{\left(\overset{+}{S}_1(w)\right)\tau_2^-\overset{+}{S}_2(\lambda)-\tau_2^+\right\} = \left(L_w^+\right)^* K^{2,2}(\lambda,w)L_\lambda^+\\
&\quad -\frac{\bar{\lambda}-\bar{\alpha}}{\alpha-\bar{\alpha}}\left(L_w^+\right)^*K^{2,2}(\lambda,w)-\frac{w-\alpha}{\bar{\alpha}-\alpha}K^{2,2}(\lambda,w)L_\lambda^+,
\end{aligned} \tag{2.51}$$

where L_λ, $\tilde{L}_\lambda$, *and* L_λ^+ *are the linear bundles of operators (2.34), (2.49) and (2.45); and* $K^{p,s}(\lambda,w)$ *are the corresponding blocks of the kernel* $K(\lambda,w)$ *(2.16). Moreover,* $\overset{+}{S}(\lambda)$ *is defined from* $S_1(\lambda)$ *by formula (2.13), and* L_λ^+ *and* $\tilde{L}_\lambda$ *(2.49) are linked to each other by relation (2.50).*

O b s e r v a t i o n 2.1. The colligation relations (2.30), (2.40), and (2.46) for the operators A_1 and A_1^* of the commutative colligation Δ (1.1) have the "metric nature" and give the well-known (2.15) representations for the blocks $K^{p,s}(\lambda,w)$ of the positively defined kernel $K(\lambda,w)$ (2.16). Similar relations (2.32), (2.46), and (2.48) for the operators A_2 and A_2^* of the commutative colligation Δ (1.1) lead to the new nontrivial conditions for the characteristic function $S_1(\lambda)$ (2.27) that should be considered as a corollary of commutativity of the operators A_1 and A_2. Note that the equalities 2) and 3) of (2.51) follow from the conservation laws 2) (1.13), (1.23) and also have the sensible interpretation in terms of conditions (1.2) of the colligation Δ (1.1).

O b s e r v a t i o n 2.2. Between the "external parameters" of the colligation Δ (1.1) besides the colligation relations 1)–7) (1.2) there exist additional relations. In particular, assuming in 2) and 3) (2.51) that $\lambda = w = \alpha$, we obtain

$$K^*\sigma_2^+K-\sigma_2^- = N_2^*\left(N_1^*\right)^{-1}\left\{K^*\sigma_1^+K-\sigma_1^-\right\}N_1^{-1}N_2;$$

$$K\tau_2^-K^*-\tau_2^+ = \tilde{N}_2\tilde{N}_1^{-1}\left\{K\tau_1^-K^*-\tau_1^+\right\}\left(\tilde{N}_1^*\right)^{-1}\tilde{N}_2^*.$$

Probably, these are not the only possible conditions of dependance between "external parameters" of the colligation Δ (1.1).

3. Theorem of Existence and Analogue of Hamilton–Caley Theorem

I. In this section, we prove the theorem of the existence, namely, describe the properties which the operator function $S_1(\lambda_1)$ from E_- into E_+ must satisfy to be a characteristic function of some colligation Δ (1.1). Moreover, we prove that in the case of the finite dimension of E_- and E_+ there exists such a polynomial $P(\lambda_1, \lambda_2)$ that "annihilates" A_1 and A_2.

In H_1 (2.17), define the vector functions

$$F(\lambda, u_-) = T_{\lambda,\alpha}\psi_- N_1 u_-; \quad \tilde{F}(\lambda, u_+) = T^*_{\lambda,\alpha}\psi^-_+ \tilde{N}^*_1 u_+, \tag{3.1}$$

where $u_\pm \in E_\pm$; λ, $\alpha \in \Omega$; and $T_{\lambda,\alpha} = I + (\lambda - \alpha)R_1(\lambda)$. Obviously, the linear span of the $F(\lambda, u_-)$ and $\tilde{F}(\lambda, u_+)$, on condition of the invertibility of N_1 and $\tilde{N}_1$, generates the whole H_1.

Theorem 3.1. *Let there be given the commutative colligation Δ (1.1), the operators N_1 and $\tilde{N}_1$ of which are invertible. Then the resolvents $\{R_1, R_2\}$ and $\{R^*_1, R^*_2\}$ act on the vector functions $F(\lambda, u_-)$ and $\tilde{F}(\lambda, u_+)$ (3.1) in the following way:*

$$\begin{aligned}
&1)\quad R_1 F(\lambda, u_-) = \frac{F(\lambda, u_-) - F(\alpha, u_-)}{\lambda - \alpha};\\
&2)\quad R_2 F(\lambda, u_-) = \frac{F(\lambda, L_\lambda u_-) - F(\alpha, L_\alpha u_-)}{\lambda - \alpha};\\
&3)\quad R^*_1 F(\lambda, u_-) = \frac{F(\lambda, u_-) + \tilde{F}\Big(\alpha, \left(\tilde{N}^*_1\right)^{-1} \sigma^+_1 S_1(\lambda) u_-\Big)}{\lambda - \bar{\alpha}};\\
&4)\quad R^*_2 F(\lambda, Q_\lambda u_-) = F(\lambda, L_\lambda u_-) + \tilde{F}\Big(\alpha, \left(\tilde{N}^*_1\right)^{-1} \sigma^+_2 S_1(\lambda) u_-\Big);\\
&5)\quad R^*_1 \tilde{F}(\lambda, u_+) = \frac{\tilde{F}(\lambda, u_+) - \tilde{F}(\alpha, u_+)}{\bar{\lambda} - \bar{\alpha}};\\
&6)\quad R^*_2 \tilde{F}(\lambda, u_+) = \frac{\tilde{F}\left(\lambda, L^+_\lambda u_+\right) - \tilde{F}(\alpha, L^+_\alpha u_+)}{\bar{\lambda} - \bar{\alpha}};\\
&7)\quad R_1 \tilde{F}(\lambda, u_+) = \frac{\tilde{F}(\lambda, u_+) + F\Big(\alpha, N_1^{-1}\tau^-_1 \overset{+}{S}_1(\lambda) u_+\Big)}{\bar{\lambda} - \alpha};\\
&8)\quad R_2 \tilde{F}\left(\lambda, Q^+_\lambda u_+\right) = \tilde{F}\left(\lambda, L^+_\lambda u_+\right) + F\Big(\alpha, N_1^{-1}\tau^-_2 \overset{+}{S}_1(\lambda) u_+\Big)
\end{aligned} \tag{3.2}$$

for all $u_\pm \in E_\pm$ and all λ, $\alpha \in \Omega$, besides, the linear bundles Q_λ and Q^+_λ are given by

$$\begin{aligned}
&Q_\lambda = (\lambda - \alpha)I + (\alpha - \bar{\alpha})L_\lambda; \quad L_\lambda = N_1^{-1}\left[(\lambda - \alpha)\Gamma + N_2\right];\\
&Q^+_\lambda = \left(\bar{\lambda} - \bar{\alpha}\right)I + (\bar{\alpha} - \alpha)L^+_\lambda; \quad L^+_\lambda = \left(\tilde{N}^*_1\right)^{-1}\left[\left(\bar{\lambda} - \bar{\alpha}\right)\tilde{\Gamma}^* + \tilde{N}^*_2\right],
\end{aligned} \tag{3.3}$$

$S_1(\lambda)$ and $\overset{+}{S}_1(\lambda)$ are given by (2.3) and (2.12).

P r o o f. The proof of 1) (3.2) follows easily from the Hilbert identity for resolvents

$$\begin{aligned} R_1 F(\lambda, u_-) &= R_1 T_{\lambda,\alpha}\psi_- N_1 u_- = [R_1(\alpha) + (\lambda-\alpha) R_1(\alpha) R_1(\lambda)]\,\psi_- N_1 u_- \\ &= R_1(\lambda)\psi_- N_1 u_- = \frac{T_{\lambda,\alpha} - I}{\lambda - \alpha}\psi_- N_1 u_- = \frac{F(\lambda, u_-) - F(\alpha, u_-)}{\lambda - \alpha}. \end{aligned}$$

Relation 5) (3.2) is proved in a similar way. It is easy to see that the equalities 2) and 6) (3.2) follow from formulas 5) (1.2). Since the relations 3) and 7) (3.2) have the dual nature, it is sufficient to prove one of them, for instance, 7) (3.2). It is obvious that

$$R_1 \tilde{F}(\lambda, u_+) = R_1\left(I + \left(\bar\lambda - \bar\alpha\right) R_1^*(\lambda)\right)\psi_+^* \tilde{N}_1^* u_+ = R_1 \psi_+^* \tilde{N}_1^* u_+$$
$$+ \left(\bar\lambda - \bar\alpha\right) R_1 R_1^* T_{\lambda,\alpha}^* \psi_+^* \tilde{N}_1^* u_+$$

since $R_1 T_{\lambda,\alpha} = R_1(\lambda)$. $(\alpha - \bar\alpha) R_1 R_1^* = i\psi_- \tau_1^- \psi_-^* + R_1 - R_1^*$ imply that

$$R_1 \tilde{F}(\lambda, u_+) = R_1 \psi_+^* \tilde{N}_1^* u_+ + \frac{\bar\lambda - \bar\alpha}{\alpha - \bar\alpha}\left\{i\psi_- \tau_1^- \psi_-^* + R_1 - R_1^*\right\} T_{\lambda,\alpha}^* \psi_+^* \tilde{N}_1^* u_+$$

$$= R_1 \psi_+^* \tilde{N}_1^* u_+ + \frac{1}{\alpha - \bar\alpha}\psi_- \tau_1^- \left[K^* - \overset{+}{S}_1(\lambda)\right] u_+ + \frac{\bar\lambda - \bar\alpha}{\alpha - \bar\alpha} R_1 T_{\lambda,\alpha}^* \psi_+^* \tilde{N}_1^* u_+ -$$

in view of the definition of $\overset{+}{S}_1(\lambda)$ (2.12) and 5) (3.2). Hence,

$$\left(\alpha - \bar\lambda\right) R_1 \tilde{F}(\lambda, u_+) = -\psi_- \tau_1^- \overset{+}{S}_1(\lambda) u_+ - T_{\lambda,\alpha}^* \psi_+^* \tilde{N}_1^* u_+$$

since $(\alpha - \bar\alpha) R_1 \psi_+^* \tilde{N}_1^* + \psi_- \tau_1^- K^* + \psi_+^* \tilde{N}_1^* = 0$, in view of condition 3) (1.2) of the colligation Δ (1.1). Thus, formula 7) (3.2) is proved. Prove that formulas 4) and 8) (3.2) are true. Prove, for instance, equality 4). To do this, use the fact that $R_2^* = R_2 - (\alpha - \bar\alpha) R_2^* R_2 + i\psi_+^* \sigma_2^+ \psi_+$. It is easy to see that

$$R_2^* T_{\lambda,\alpha}\psi_- N_1 = R_2 T_{\lambda,\alpha}\psi_- N_1 - (\alpha - \bar\alpha) R_2^* R_2 T_{\lambda,\alpha}\psi_- N_1 + i\psi_+^* \sigma_2^+ \psi_+ T_{\lambda,\alpha}\psi_- N_1$$

$$= \frac{1}{\lambda - \alpha}\Big\{T_{\lambda,\alpha}\psi_- N_1 L_2 - \psi_- N_2 - (\alpha - \bar\alpha) R_2^* \left[T_{\lambda,\alpha}\psi_- N_1 L_\lambda - \psi_- N_2\right]$$
$$+ \psi_+^* \sigma_2^+ \left[S_1(\lambda) - K\right]\Big\}$$

in virtue of 2) (3.2) and the definition of $S_1(\lambda)$ (2.12). Taking now into account $\psi_+^* \sigma_2^+ K + [I + (\bar\alpha - \alpha) R_2^*]\,\psi_- N_2 = 0$, we obtain

$$R_2^* T_{\lambda,\alpha}\psi_- N_1 \left\{(\lambda - \alpha) I + (\alpha - \bar\alpha) L_\lambda\right\} = T_{\lambda,\alpha}\psi_- N_1 L_\lambda + \psi_+^* \sigma_2^+ S_1(\lambda).$$

This equality exactly coincides with 4) (3.2). ■

Corollary 3.1. *If the suppositions of Theorem 3.1 hold, then the formulas*

$$T_{\lambda,\alpha}F\left(\alpha,u_-\right)=F\left(\lambda,u_-\right); \quad T^*_{\lambda,\alpha}\tilde{F}\left(\alpha,u_+\right)=\tilde{F}\left(\lambda,u_+\right) \qquad (3.4)$$

take place for all $u_\pm\in E_\pm$ and all λ, $\alpha\in\Omega$.

II. Proceed now to the description of the class of functions formed by the characteristic functions $S_1(\lambda)$ (2.3) of the commutative colligation Δ (1.1).

Theorem 3.2. *Let the commutative colligation Δ (1.1) be given and the operators N_1 and $\tilde{N}_1$ be boundedly invertible. Suppose that the operators*

$$\left(K^*\sigma_2^+K-\sigma_2^-\right)^{-1} \text{ and } \left(K\tau_2^-K^*-\tau_2^+\right)^{-1}$$

exist and are bounded in E_- and E_+, respectively.

Then there exists a neighborhood $U_\delta(\alpha)=\{\lambda\in\mathbb{C}:|\lambda-\alpha|<\delta\}$ of the point α such that the linear bundles Q_λ, L_λ and Q_λ^+, L_λ^+ (3.3) are invertible for all $\lambda\in U_\delta(\alpha)$.

P r o o f. Prove that the operators Q_λ and L_λ are boundedly invertible in some neighborhood $U_\delta(\alpha)$ of the point α (the proof is similar for Q_λ^+ and L_λ^+). The point 2) (2.54) implies that

$$i\left\{K^*\sigma_2^+S_1(\lambda)-\sigma_2^-\right\}=N_2^*\left(N_1^*\right)^{-1}K^{1,1}(\lambda,\alpha)\cdot Q_\lambda. \qquad (3.5)$$

Prove that the invertibility of $K^*\sigma_2^+K-\sigma_2^-$ necessitates the bounded invertibility of the operator $\left\{K^*\sigma_2^+S_1(\lambda)-\sigma_2^-\right\}$ in some neighborhood $U_\delta(\alpha)$ of the point α. Since

$$K^*\sigma_2^+S_1(\lambda)-\sigma_2^-=K^*\sigma_2^+K-\sigma_2^-+i(\lambda-\alpha)K^*\sigma_2^+\psi_+T_{\lambda,\alpha}\psi_-N_1,$$

then the series

$$\left\{K^*\sigma_2^+S_1(\lambda)-\sigma_2^-\right\}^{-1}$$
$$=\left\{K^*\sigma_2^+K-\sigma_2^-\right\}^{-1}\cdot\sum_{p=0}^{\infty}(\lambda-\alpha)^p\left[-iK^*\sigma_2^+\psi_+T_{\lambda,\alpha}\psi_-N_1\left\{K^*\sigma_2^+K-\sigma_2^-\right\}^{-1}\right]^p$$

converges uniformly when $|\lambda-\alpha|\ll 1$ in virtue of holomorphy of $S_1(\lambda)$ (2.3) in the point $\lambda=\alpha$. Thus, the operator $\left\{K^*\sigma_2^+S_1(\lambda)-\sigma_2^-\right\}$ is boundedly invertible in some neighborhood $U_\delta(\alpha)$ of the point α.

Let $C=i\left\{K^*\sigma_2^+S_1(\lambda)-\sigma_2^-\right\}$, $A=N_2^*\left(N_1^*\right)^{-1}K^{1,1}(\lambda,\alpha)$ and $B=Q_\lambda$, then equality (3.5) in this notation is $C=A\cdot B$. The bounded invertibility of C implies

$$n\left\|u_-\right\|\le\left\|Cu_-\right\|, \quad 0<n<\infty,$$

for all $u_- \in E_-$, and since the operator A is bounded when $\lambda \in U_\delta(\alpha)$, then

$$n\,\|u_-\| \le \|Cu_-\| \le \|A\| \cdot \|Bu_-\|\,.$$

Therefore, for B the estimation

$$m\,\|u_-\| \le \|Bu_-\|$$

is true for all $u_- \in E_-$, where $m = n \cdot \|A\|^{-1} > 0$. Thus, the invertibility of the linear bundle Q_λ (3.3) for $\lambda \in U_\delta(\alpha)$ is proved.

(3.3) implies that

$$Q_\lambda - (\lambda - \alpha)I = (\alpha - \bar{\alpha})\,L_\lambda;$$

and for the invertibility of L_λ it is necessary to establish that $(Q_\lambda - (\lambda-\alpha)I)^{-1}$ exists and it is bounded when $\lambda \in U_\delta(\alpha)$. The last obviously follows from the uniform convergence of the series

$$(Q_\lambda - (\lambda-\alpha)I)^{-1} = Q_\lambda^{-1}\left(I - (\lambda-\alpha)Q_\lambda^{-1}\right)^{-1} = \sum_{p=0}^{\infty}(\lambda-\alpha)^p\left[Q_\lambda^{-1}\right]^{p+1}.$$

The theorem is proved. ■

O b s e r v a t i o n 3.1. The invertibility of the bundles Q_λ and L_λ (Q_λ^+ and L_λ^+) in the point $\lambda = \alpha$ implies that the operator N_2 ($\tilde{N}_2^*$) is boundedly invertible. Thus, Theorem 3.2 yields that the invertibility of the expressions

$$K^*\sigma_2^+ K - \sigma_2^-; \quad K\tau_2^- K^* - \tau_2^*$$

ensures the existence of the bounded inverse of the operators N_2 and $\tilde{N}_2^*$.

Proceed to the definition of the class of operator functions generated by characteristic functions $S_1(\lambda)$ (2.3) of the commutative colligations Δ (1.1).

Class $\Omega_\alpha(\sigma,\tau,N,\Gamma)$**.** *Let $E_\pm$ be Hilbert spaces, $\alpha \in \mathbb{C}\backslash\mathbb{R}_+$, and, moreover, suppose that in E_-, correspondingly in E_+, the linear bounded operators*

$$\begin{array}{llll} \{\sigma_p^-\}_1^2; & \{\tau_p^-\}_1^2; & \{N_p\}_1^2; & \Gamma: \quad E_- \to E_-; \\ \{\sigma_p^+\}_1^2; & \{\tau_p^+\}_1^2; & \left\{\tilde{N}_p\right\}_1^2; & \tilde{\Gamma}: \quad E_+ \to E_+ \end{array} \tag{3.6}$$

are specified, where $\left\{\sigma_p^\pm\right\}_1^2$ and $\left\{\tau_p^\pm\right\}_1^2$ are selfadjoint, and N_1 and $\tilde{N}_1$ are invertible.

An operator function $S(\lambda)$: $E_- \to E_+$ is said to belong to the class $\Omega_\alpha(\sigma,\tau,N,\Gamma)$ if:

1) the function $S(\lambda)$ is holomorphic in some neighborhood $U_\delta(\alpha) = \{\lambda \in \mathbb{C} : |\lambda - \alpha| < \delta\}$ of a point α and $S(\alpha) \neq 0$;

2) the kernel $K(\lambda, w)$ *(2.16) constructed by the functions* $S(\lambda)$ *and* $S^{+}(\lambda) = (N_1^*)^{-1} S^*(\lambda)\tilde{N}_1^*$ *is Hermitian positive for all* λ, $w \in U_\delta(\alpha)$*;*

3) the operator function $S(\lambda)$ *satisfies relations (2.51), where the linear bundles* L_λ *and* L_λ^+ *are constructed by using formulas (3.3) and* $\tilde{L}_\lambda = \tilde{N}_1^{-1}\left(L_\lambda^+\right)^* \tilde{N}_1$*;*

4) the operators $\left\{K^*\sigma_2^+ K - \sigma_2^-\right\}$ *and* $\left\{K\tau_2^- K^* - \tau_2^+\right\}$ *are boundedly invertible, where* $K = S(\alpha)$*;*

5) for the operator family (3.6), (1.24) and $S(\alpha)N_1 = \tilde{N}_1 S(\alpha)$ *take place.*

It is obvious that the characteristic function $S_1(\lambda)$ (2.3) belongs to the class $\Omega_\alpha(\sigma, \tau, N, \Gamma)$.

Theorem of existence 3.3. *Let the operator function* $S(\lambda)$*:* $E_- \to E_+$ *belong to the class* $\Omega_\alpha(\sigma, \tau, N, \Gamma)$*. Then there exists a commutative colligation* Δ *(1.1) such that the characteristic function* $S_1(\lambda)$ *(2.3) of the operator* A_1 *coincides with* $S(\lambda)$*,* $S_1(\lambda) = S(\lambda)$ *for all* $\lambda \in U_\delta(\alpha)$*.*

P r o o f. Consider the family of "δ-functions" $e_\lambda f$ assuming that every $e_\lambda f$ has the support concentrated in the point $\lambda \in U_\delta(\alpha)$ and possesses the value $f = (u_-, u_+) \in E_- \oplus E_+$. The formal linear combinations

$$\sum_{k=1}^{N} e_{\lambda_k} f_k,$$

where $\lambda_k \in U_\delta(\alpha)$, $f_k \in E_- \oplus E_+$, $1 \le k \le N$, $N \in \mathbb{Z}_+$, constitute the linear manifold L on which we, by means of the kernel $K(\lambda, w)$ (2.16), define the Hermitian nonnegative bilinear form

$$\langle e_\lambda f, e_w g\rangle_K \stackrel{\text{def}}{=} \langle K(\lambda, w) f, g\rangle_{E_- \oplus E_+}. \tag{3.7}$$

As a result of closure of the linear span L by norm generated by form (3.7) and of factorization by the kernel of this metric, we obtain the Hilbert space H_K [9].

Specify the linear operators K: $E_- \to E_+$, ψ_-: $E_- \to H_K$, ψ_+^*: $E_+ \to H_K$ using the formulas

$$K = S(\alpha); \quad \psi_- u_- = e_\alpha N_1^{-1} u_-; \quad \psi_+^* u_+ = e_\alpha \left(\tilde{N}_1^*\right)^{-1} u_+ \tag{3.8}$$

and prove that relations 1) (1.2) take place for K, ψ_-, ψ_+ (3.8). Taking into account the form of the block $K^{1,1}(\lambda, w)$ of the kernel $K(\lambda, w)$ (5.16), we have

$$\left\langle \psi_- N_1 u_-, \psi_- N_1 u_-'\right\rangle_K = \left\langle e_\alpha u_-, e_\alpha u_-'\right\rangle_K = \left\langle K^{1,1}(\alpha, \alpha) u_-, u_-'\right\rangle$$

$$= \left\langle \frac{\sigma_1^- - K^*\sigma_1^+ K}{i\left(\alpha - \bar{\alpha}\right)} u_-, u_-'\right\rangle,$$

which proves that $2\operatorname{Im}\alpha N_1^*\psi_-^*\psi_-N_1 = K^*\sigma_1^+K-\sigma_1^-$. To prove $2\operatorname{Im}\alpha\cdot N_2^*\psi_-^*\psi_-N_2 = K^*\sigma_2^+K-\sigma_2^-$, consider

$$\left\langle \psi_-N_2u_-,\psi_-N_2u'_-\right\rangle_K = \left\langle e_\alpha N_1^{-1}N_2u_-,e_\alpha N_1^{-1}N_2u'_-\right\rangle_K$$

$$= \left\langle N_2^*\cdot\left(N_1^*\right)^{-1}K^{1,1}(\alpha,\alpha)N_1^{-1}N_2u_-,u'_-\right\rangle = \frac{i}{\alpha-\bar\alpha}\left\langle\left(K^*\sigma_2^+K-\sigma_2^-\right)u_-,u'_-\right\rangle$$

in view of 2) (2.51). The relations $2\operatorname{Im}\alpha\tilde N_p\psi_+\psi_+^*\tilde N_p^* = K\tau_p^-K^*-\tau_p^+$, $p=1,2$, are proved in the similar way taking into account the form of the block $K^{2,2}(\lambda,w)$ of the kernel $K(\lambda,w)$ and equality 3) (2.51).

It is easy to show that

$$\begin{aligned}\psi_-^*e_\lambda f &= \left(N_1^*\right)^{-1}\frac{\sigma_1^- - K^*\sigma_1^+S(\lambda)}{i\left(\lambda-\bar\lambda\right)}u_- + \frac{\overset{+}{S}(\lambda)-K^*}{i\left(\bar\alpha-\bar\lambda\right)}u_+;\\ \psi_+e_\lambda f &= \frac{S(\lambda)-K}{i(\lambda-\alpha)}u_- + \tilde N_1^{-1}\cdot\frac{K\tau_1^-\overset{+}{S}(\lambda)-\tau_1^+}{i\left(\bar\lambda-\alpha\right)}u_=.\end{aligned}\tag{3.9}$$

As in (3.2), define the action of the resolvents $\{R_1,R_2\}$ and $\{R_1^*,R_2^*\}$ in H_K by using the formulas:

$$R_1e_\lambda f = e_\lambda\left(\frac{u_-}{\lambda-\alpha},\frac{u_+}{\bar\lambda-\alpha}\right)+e_\alpha\left(\frac{N_1^{-1}\tau_1^-\overset{+}{S}(\lambda)}{\bar\lambda-\alpha}u_+ - \frac{u_-}{\lambda-\alpha},0\right);$$

$$R_1^*e_\lambda f = e_\lambda\left(\frac{u_-}{\lambda-\bar\alpha},\frac{u_+}{\bar\lambda-\bar\alpha}\right)+e_\alpha\left(0,\frac{\left(\tilde N_1^*\right)^{-1}\sigma_1^+S(\lambda)u_-}{\lambda-\bar\alpha} - \frac{u_+}{\bar\lambda-\bar\alpha}\right);\tag{3.10}$$

$$R_2e_\lambda f = e_\lambda\left(\frac{L_\lambda u_-}{\lambda-\alpha},L_\lambda^+\left(Q_\lambda^+\right)^{-1}u_+\right)+e_\alpha\left(N_1^{-1}\tau_2^-\overset{+}{S}(\lambda)\left(Q_\lambda^+\right)^{-1}u_+ - \frac{L_\alpha u_-}{\lambda-\alpha},0\right);$$

$$R_2^*e_\lambda f = e_\lambda\left(L_\lambda\left(Q_\lambda\right)^{-1}u_-,\frac{L_\lambda^+u_+}{\bar\lambda-\bar\alpha}\right)+e_\alpha\left(0,\left(\tilde N_1^*\right)^{-1}\sigma_2^+S(\lambda)Q_\lambda^{-1}u_- - \frac{L_\alpha^+u_+}{\bar\lambda-\bar\alpha}\right),$$

where $\lambda\in U_\delta(\alpha)$, $f=(u_-,u_+)\in E_-\oplus E_+$, and the linear bundles L_λ, Q_λ, L_λ^+, Q_λ^+ are given by (3.3). Prove that relations 3) (1.2) are true. So, to prove $K^*\sigma_1^+\varphi_+^1 + N_1^*\psi_-^*\left(A_1-\bar\alpha I\right)=0$, write this equality in the following way: $\psi_+^*\sigma_1^+K+\varphi_-N_1+\left(\bar\alpha-\alpha\right)R_1^*\psi_-N_1=0$. Then, taking into account (3.8) and (3.10), we obtain

$$\psi_+^*\sigma_1^+Ku_- + \psi_-N_1u_- + \left(\bar\alpha-\alpha\right)R_1^*\psi_-N_1u_- = e_\alpha\left(\tilde N_1^*\right)^{-1}\sigma_1^+Ku_- + e_\alpha u_-$$

$$+\left(\bar\alpha-\alpha\right)e_\alpha\frac{u_-}{\alpha-\bar\alpha}+\left(\bar\alpha-\alpha\right)\cdot e_\alpha\frac{\left(\tilde N_1^*\right)^{-1}\sigma_1^+K}{\alpha-\bar\alpha}u_- = 0,$$

q.e.d. $K\tau_1^-\left(\varphi_-^1\right)^* + \tilde{N}_1\psi_+\left(A_1^* - \alpha I\right) = 0$ is proved in a similar way. Rewrite the equality $K\tau_2^-\left(\varphi_-^2\right)^* + \tilde{N}_2\psi_+\left(A_2^* - \alpha I\right) = 0$ as $\psi_-\tau_2^- K^* + \psi_+^*\tilde{N}_2^* + (\alpha - \bar{\alpha})\, R_2\psi_+^*\tilde{N}_2^* = 0$. Then, using (3.8), we have

$$e_\alpha N_1^{-1}\tau_2^- K^* u_+ + e_\alpha\left(\tilde{N}_1^*\right)^{-1}\tilde{N}_2^* u_+ + (\alpha - \bar{\alpha})\, R_2 e_\alpha\left(\tilde{N}_1^*\right)^{-1}\tilde{N}_2^* u_+$$

$$= e_\alpha N_1^{-1}\tau_2^- K^* u_+ + e_\alpha\left(\tilde{N}_1^*\right)^{-1}\tilde{N}_2^* u_+ - R_2 e_\alpha Q_\alpha^+ u_+ = 0$$

since $Q_\alpha^+ = (\bar{\alpha} - \alpha)\, L_\alpha^+ = (\bar{\alpha} - \alpha)\left(\tilde{N}_1^*\right)^{-1}\tilde{N}_2^*$ in view of the definition of Q_λ^+ and L_λ^+ (3.3) and (3.10). $K^*\sigma_2^+\varphi_+^2 + N_2^*\psi_-^*\left(A_2 - \bar{\alpha} I\right) = 0$ is proved in exactly the same way.

It is obvious that the intertwining condition $S(\lambda)L_\lambda = \tilde{L}_\lambda S(\lambda)$ 1) (2.51) and $KN_1 = \tilde{N}_1 K$, definition of the class $\Omega_\alpha(\sigma, \tau, N, \Gamma)$, yield $KN_2 = \tilde{N}_2 K$, which proves 7) (1.2).

Note that

$$T_{\lambda,\alpha} e_\alpha u_- = e_\lambda u_-; \quad T_{\lambda,\alpha}^* e_\alpha u_+ = e_\lambda u_+ \tag{3.11}$$

take place for all λ, $\alpha \in \Omega$ and all $u_\pm \in E_\pm$. In fact, (3.10) imply $(\lambda - \alpha)R_1 e_\lambda u_- = e_\lambda u_- - e_\alpha u_-$, and thus $T_{\alpha,\lambda} e_\lambda u_- = e_\alpha u_-$, which proves the first equality of (3.11).

To prove the first condition in 5) (1.2), consider

$$T_{\lambda,\alpha}\left[R_2\psi_- N_1 u_- - R_1\psi_- N_2 - \psi_-\Gamma u_-\right] = R_2 T_{\lambda,\alpha} e_\alpha u_- - R_1 T_{\lambda,\alpha} e_\alpha N_1^{-1} N_2 u_-$$

$$-T_{\lambda,\alpha} e_\alpha N_1^{-1}\Gamma u_- = R_2 e_\lambda u_- - R_1 e_\lambda N_1^{-1} N_2 u_- - e_\lambda N_1^{-1}\Gamma u_-$$

$$= \frac{1}{\lambda - \alpha}\left\{e_\lambda L_\lambda u_- - e_\alpha L_\alpha u_-\right\} - \frac{1}{\lambda - \alpha}\left\{e_\lambda N_1^{-1} N_2 u_- - e_\alpha N_1^{-1} N_2 u_-\right\}$$

$$-e_\lambda N_1^{-1}\Gamma u_- = \frac{1}{\lambda - \alpha} e_\lambda\left\{L_\lambda - N_1^{-1} N_2 - (\lambda - \alpha) N_1^{-1}\Gamma\right\} = 0.$$

Taking into account the invertibility of $T_{\lambda,\alpha}$, $T_{\lambda,\alpha}T_{\alpha,\lambda} = I$, we obtain the required. The proof of the second equality in 5) (1.2) is of a similar nature, besides, it is necessary to use the second relation of (3.11).

Since $A_p R_p = \alpha R_p + I$, then

$$\frac{1}{i}\left\langle A_p R_p e_\lambda f, R_p e_w g\right\rangle_K - \frac{1}{i}\left\langle R_p e_\lambda f, A_p R_p e_w g\right\rangle_K$$

$$= \frac{1}{i}\left\langle\left(\alpha R_p + I\right) e_\lambda f, R_p e_w g\right\rangle_K - \frac{1}{i}\left\langle R_p e_\lambda f, \left(\alpha R_p + I\right) e_w g\right\rangle_K$$

$$= \left\langle\left\{\frac{\alpha - \bar{\alpha}}{i} R_p^* R_p - i R_p^* + i R_p\right\} e_\lambda f, e_w g\right\rangle_K$$

for $p = 1$, 2. Therefore, to prove 4. (1.2), we have to prove that

$$B_p = iR_p - iR_p^* + \frac{\alpha - \bar{\alpha}}{i} R_p^* R_p = \psi_+^* \sigma_p^+ \psi_+, \quad p = 1, 2, \tag{3.12}$$

where R_p and R_p^* are given by (3.10), and ψ_+^* and ψ_+ are given by formulas (3.8) and (3.9), respectively. To prove (3.12) when $p = 1$, consider

$$B_1 e_\lambda f = iR_1 e_\lambda f - iR_1^* e_\lambda f + \frac{\alpha - \bar{\alpha}}{i} R_1^* R_1 e_\lambda f$$

$$= e_\lambda \left(\frac{iu_-}{\lambda - \alpha}, \frac{iu_+}{\bar{\lambda} - \alpha} \right) + e_\alpha \left(\frac{iN_1^{-1} \tau_1^- \overset{+}{S}(\lambda) u_+}{\bar{\lambda} - \alpha} - \frac{iu_-}{\lambda - \alpha}, 0 \right)$$

$$+ e_\lambda \left(\frac{-iu_-}{\lambda - \bar{\alpha}}, \frac{-iu_+}{\bar{\lambda} - \bar{\alpha}} \right) + e_\alpha \left(0, \frac{-i \left(\tilde{N}_1^* \right)^{-1} \sigma_1^+ S(\lambda) u_-}{\lambda - \bar{\alpha}} + \frac{iu_+}{\bar{\lambda} - \bar{\alpha}} \right)$$

$$+ \frac{\alpha - \bar{\alpha}}{i} R_1^* \left\{ e_\lambda \left(\frac{u_-}{\lambda - \alpha}, \frac{u_+}{\bar{\lambda} - \alpha} \right) + e_\alpha \left(\frac{N_1^{-1} \tau_1^- \overset{+}{S}(\lambda) u_+}{\bar{\lambda} - \alpha} - \frac{u_-}{\lambda - \alpha}, 0 \right) \right\}$$

$$= e_\alpha \left(\frac{iN_1^{-1} \tau_1^- \overset{+}{S}(\lambda) u_+}{\bar{\lambda} - \alpha} - \frac{iu_-}{\lambda - \alpha}, \frac{-i \left(\tilde{N}_1^* \right)^{-1} \sigma_1^+ S(\lambda) u_-}{\lambda - \bar{\alpha}} + \frac{iu_+}{\bar{\lambda} - \bar{\alpha}} \right)$$

$$+ \frac{\alpha - \bar{\alpha}}{i} e_\alpha \left(0, \frac{\left(\tilde{N}_1^* \right)^{-1} \sigma_1^+ S(\lambda)}{\lambda - \bar{\alpha}} \cdot \frac{u_-}{\lambda - \alpha} - \frac{1}{\bar{\lambda} - \bar{\alpha}} \cdot \frac{u_+}{\bar{\lambda} - \alpha} \right)$$

$$+ \frac{\alpha - \bar{\alpha}}{i} e_\alpha \left(\frac{1}{\alpha - \bar{\alpha}} \left[\frac{N_1^{-1} \tau_1^- \overset{+}{S}(\lambda) u_+}{\bar{\lambda} - \alpha} - \frac{u_-}{\lambda - \alpha} \right], 0 \right)$$

$$+ \frac{\alpha - \bar{\alpha}}{i} e_\alpha \left(0, \frac{\left(\tilde{N}_1^* \right)^{-1} \sigma_1^+ K}{\alpha - \bar{\alpha}} \cdot \left\{ \frac{N_1^{-1} \tau_1^- \overset{+}{S}(\lambda) u_+}{\bar{\lambda} - \alpha} - \frac{u_-}{\lambda - \alpha} \right\} \right)$$

$$= e_\alpha \left(0, \left(\tilde{N}_1^* \right)^{-1} \sigma_1^+ \frac{S(\lambda) - K}{i(\lambda - \alpha)} u_- + \tilde{N}_1^{*-1} \frac{\sigma_1^+ \tilde{N}_1^{-1} K \tau_1^- \overset{+}{S}(\lambda) - \tilde{N}_1^*}{i \left(\bar{\lambda} - \alpha \right)} u_+ \right).$$

And if one takes into account 5) of the definition of class $\Omega_\alpha(\sigma, \tau, N, \Gamma)$, $\tilde{N}_1^* = \sigma_1^+ \tilde{N}_1^{-1} \tau_1^+$ (1.24), one can get

$$B_1 e_\lambda f = e_\alpha \left(0, \left(\tilde{N}_1^* \right)^{-1} \sigma_1^+ \left\{ \frac{S(\lambda) - K}{i(\lambda - \alpha)} u_- + \tilde{N}_1^{-1} \frac{K \tau_1^- \overset{+}{S}(\lambda) - \tau_1^+}{i \left(\bar{\lambda} - \alpha \right)} u_+ \right\} \right)$$

$$= \psi_+^* \sigma_1^+ \psi_+ e_\lambda f$$

in view of the definition of ψ_+^* (3.8) and (3.9), which proves (3.12) when $p = 1$.

Analogously, (3.12) is proved for B_2.

Using $A_p^* R_p^* = \bar{\alpha} R_p^* + I$, we obtain

$$-\frac{1}{i} \left\langle A_p^* R_p^* e_\lambda f, R_p^* e_w g \right\rangle + \frac{1}{i} \left\langle R_p^* e_\lambda f, A_p^* R_p^* e_w g \right\rangle$$

$$= \left\langle \left\{ \frac{\alpha - \bar{\alpha}}{i} R_p R_p^* + i R_p - i R_p^* \right\} e_\lambda f, e_w g \right\rangle, \ p = 1, 2,$$

and thus to prove the second relation of 4. (1.2), it is sufficient to prove that

$$\tilde{B}_p = i R_p - i R_p^* + \frac{\alpha - \bar{\alpha}}{i} R_p R_p^* = \psi_- \tau_p^- \psi_-^*, \quad p = 1, 2, \tag{3.13}$$

where R_p and R_p^* are given by formulas (3.10), ψ_- and ψ_-^* are given by (3.8), (3.9). The proof of formulas (3.13) is similar to that of (3.12).

Since $A_p R_p = \alpha R_p + I$ and $A_p^* R_p^* = \bar{\alpha} R_p^* + I$, (3.10) implies

$$\begin{aligned}
A_1 R_1 e_\lambda f &= e_\lambda \left(\frac{\lambda u_-}{\lambda - \alpha}, \frac{\bar{\lambda} u_+}{\bar{\lambda} - \alpha} \right) + e_\alpha \left(\frac{N_1^{-1} \tau_1^- \overset{+}{S}(\lambda) u_+}{\bar{\lambda} - \alpha} - \frac{u_-}{\lambda - \alpha}, 0 \right); \\
A_1^* R_1^* e_\lambda f &= e_\lambda \left(\frac{\lambda u_-}{\lambda - \bar{\alpha}}, \frac{\bar{\lambda} u_+}{\bar{\lambda} - \bar{\alpha}} \right) + e_\alpha \left(0, \frac{\left(\tilde{N}_1^* \right)^{-1} \sigma_1^+ S(\lambda) u_-}{\lambda - \alpha} - \frac{u_+}{\bar{\lambda} - \bar{\alpha}} \right);
\end{aligned}$$

$$\begin{aligned}
A_2 R_2 e_\lambda f &= e_\lambda \left(\left(\frac{\alpha L_\lambda}{\lambda - \alpha} + I \right) u_-, \left(\alpha L_\lambda^+ \left(Q_\lambda^+ \right)^{-1} + I \right) u_+ \right) \\
&\quad + e_\alpha \left(N_1^{-1} \tau_2^- \overset{+}{S}(\lambda) \left(Q_\lambda^+ \right)^{-1} u_+ - \frac{L_\alpha u_-}{\lambda - \alpha}, 0 \right); \\
A_2^* R_2^* e_\lambda f &= e_\lambda \left(\left(\alpha L_\lambda Q_\lambda^{-1} + I \right) u_-, \left(\frac{\alpha L_\lambda^+}{\bar{\lambda} - \bar{\alpha}} + I \right) u_+ \right) \\
&\quad + e_\alpha \left(0, \left(\tilde{N}_1^* \right)^{-1} \sigma_2^+ S(\lambda) Q_\lambda^{-1} u_- - \frac{L_\alpha^+ u_+}{\bar{\lambda} - \bar{\alpha}} \right)
\end{aligned} \tag{3.14}$$

for all $\lambda \in U_\delta(\alpha)$ and all $f = (u_-, u_+) \in E_- \oplus E_+$, besides, the existence of Q_λ^{-1} and $\left(Q_\lambda^+ \right)^{-1}$ follows from 4) of the definition of class $\Omega_\alpha(\sigma, \tau, N, \Gamma)$. Specify the operator A_1 in H_K

$$A_1 e_\lambda f = e_\lambda \left(\lambda u_-, \bar{\lambda} u_+ \right). \tag{3.15}$$

Then (3.14) and (3.10) imply that

$$e_\lambda f = R_1 \left(A_1 - \lambda I \right) e_\lambda f = R_1 e_\lambda \left((\lambda - \alpha) u_-, \left(\bar{\lambda} - \alpha \right) u_+ \right)$$

$$= e_\lambda f + e_\alpha \left(N_1^{-1} \tau_1^- \overset{+}{S}(\lambda) u_+ - u_-, 0 \right).$$

Therefore, the domain $\mathfrak{D}(A_1)$ of the operator A_1 (3.15) is given by

$$\mathfrak{D}(A_1) = \left\{ \sum_{p=1}^{N} e_{\lambda_p} f_p \in H_K : \lambda_p \in U_\delta(\alpha); f_p = \left(u_-^p, u_+^p\right) \in E_- \oplus E_+; \right. \\ \left. u_-^p = N_1^{-1} \tau_1^- \overset{+}{S}(\lambda_p) u_+^p; 1 \le p \le N; N \le \infty \right\}. \tag{3.16}$$

Similar considerations show that the adjoint operator A_1^* equals

$$A_1^* e_\lambda f = e_\lambda \left(\lambda u_-, \bar{\lambda} u_+ \right), \tag{3.15*}$$

and its domain $\mathfrak{D}(A_1^*)$ is represented by

$$\mathfrak{D}(A_1^*) = \left\{ \sum_{p=1}^{N} e_\lambda f_p \in H_K : \lambda_p \in U_\delta(\alpha); f_p = \left(u_-^p, u_+^p\right) \in E_- \oplus E_+; \right. \\ \left. u_+^p = \left(\tilde{N}_1^*\right)^{-1} \sigma_1^+ S(\lambda_p) u_-^p; 1 \le p \le N; N \le \infty \right\}. \tag{3.16*}$$

It is easy to establish that the operator A_1^* (3.15*), (3.16*) is the adjoint of A_1 (3.15), (3.16). By (3.14) specify the operator A_2 in H_K

$$A_2 e_\lambda f = e_\lambda \left(\left(\alpha + (\lambda - \alpha) L_\lambda^{-1}\right) u_-, \left(\alpha + \left(L_\lambda^+\right)^{-1} Q_\lambda^+\right) u_+ \right), \tag{3.17}$$

besides, the existence of the inverse of L_λ and L_λ^+ again follows from 4) of the definition of class $\Omega_\alpha(\sigma, \tau, N, \Gamma)$. Taking now into account (3.10) and (3.14), we have

$$e_\lambda f = R_2 (A_2 - \alpha I) e_\lambda f = R_2 e_\lambda \left((\lambda - \alpha) L_\lambda^{-1} u_-, \left(L_\lambda^+\right)^{-1} Q_\lambda^+ u_+ \right)$$

$$= e_\lambda f + e_\alpha \left(N_1^{-1} \tau_2^- \overset{+}{S}(\lambda) \left(L_\lambda^+\right)^{-1} u_+ - L_\alpha L_\lambda^{-1} u_-, 0 \right).$$

Thus, the domain $\mathfrak{D}(A_2)$ of the operator A_2 represents

$$\mathfrak{D}(A_2) = \left\{ \sum_{p=1}^{N} e_{\lambda_p} f_p \in H_K : \lambda_p \in U_\delta(\alpha); f_p = \left(u_-^p, u_+^p\right) \in E_- \oplus E_+; \right. \\ \left. u_-^p = L_{\lambda_p} N_2^{-1} \tau_2^- \overset{+}{S}(\lambda_p) \left(L_{\lambda_p}^+\right)^{-1} u_+^p; 1 \le p \le N; N \le \infty \right\}. \tag{3.18}$$

It is easy to show that the adjoint A_2^* of the operator A_2 (3.17), (3.18) is given by

$$A_2^* e_\lambda f = e_\lambda \left(\left(\bar{\alpha} + L_\lambda^{-1} Q_\lambda\right) u_-, \left(\bar{\alpha} + \left(\bar{\lambda} - \bar{\alpha}\right) \left(L_\lambda^+\right)^{-1}\right) u_+ \right), \tag{3.17*}$$

and its domain equals

$$\mathfrak{D}\left(A_2^*\right) = \left\{ \sum_{p=1}^{n} e_{\lambda_p} f_p \in H_K : \lambda_p \in U_\delta(\alpha); f_p = \left(u_-^p, u_+^p\right) \in E_- \oplus E_+; \right.$$
$$\left. u_+^p = L_{\lambda_p}^+ \left(\tilde{N}_2^*\right)^{-1} \sigma_2^+ S\left(\lambda_p\right) L_{\lambda_p}^{-1} u_-^p; 1 \le p \le N; N \le \infty \right\}. \tag{3.18*}$$

Construct now the commutative colligation

$$\Delta_K = \left(\left\{\sigma_p^-\right\}; \left\{\tau_p^-\right\}_1^2; \left\{N_p\right\}_1^2; \Gamma; H_K \oplus E_-; \left\{ \begin{bmatrix} A_p & \psi_- \\ \psi_+ & K \end{bmatrix} \right\}_1^2; \right.$$
$$\left. H_K \oplus E_+; \tilde{\Gamma}; \left\{\tilde{N}_p\right\}_1^2; \left\{\tau_p^+\right\}_1^2; \left\{\sigma_p^+\right\}_1^2 \right), \tag{3.19}$$

where K, ψ_-, ψ_+, and $\{A_1, A_2\}$ are given correspondingly by formulas (3.8), (3.9) and (3.15)–(3.18), $\Omega = U_\delta(\alpha)$.

Finally, prove that the characteristic function $S_1(\lambda)$ of the operator A_1 (3.15), (3.16) of the colligation Δ_K coincides with $S(\lambda)$. (3.8) and (3.11) imply

$$T_{\lambda,\alpha}\psi_- N_1 u_- = e_\lambda \left(u_-, 0\right).$$

Using the form of the operator ψ_+ (3.9), we have

$$\psi_+ T_{\lambda,\alpha}\psi_- N_1 u_- = \frac{S(\lambda) - K}{i(\lambda - \alpha)} u_-$$

and thus $S_1(\lambda) = S(\lambda)$. ∎

III. Formulas (3.10) imply

$$R_1 e_\lambda u_- = \frac{e_\lambda u_- - e_\alpha u_-}{\lambda - \alpha};$$
$$R_2 e_\lambda u_- = e_\lambda N_1^{-1} \Gamma u_- + \frac{e_\lambda N_1^{-1} N_2 u_- - e_\alpha N_1^{-1} N_2 u_-}{\lambda - \alpha}.$$

Therefore, if on the subspace in H_K,

$$H_K^- = \operatorname{span}\left\{e_\lambda u_- : \lambda \in U_\delta(\alpha); u_- \in E_-\right\}, \tag{3.20}$$

the linear bounded operators $N_p e_\lambda u_- \overset{\text{def}}{=} e_\lambda N_p u_-$, $p = 1, 2$, and $\Gamma e_\lambda u_- \overset{\text{def}}{=} e_\lambda \Gamma u_-$ are given, then it is obvious that

$$\left\{N_1 R_2 - N_2 R_1 - \Gamma\right\} f_- = 0 \tag{3.21}$$

for all $f_- \in H_K^-$. Consider also the action of the resolvents R_1 and R_2 on the elements of another subspace in H_K,

$$H_K^+ = \operatorname{span}\left\{e_\lambda u_+ : \lambda \in U_\delta(\alpha); u_+ \in E_+\right\}. \tag{3.22}$$

Then (5.69) implies

$$\begin{aligned} \left(\bar\lambda - \alpha\right) R_1 e_\lambda u_+ &= e_\lambda u_+ + e_\alpha N_1^{-1} \tau_1^- \overset{+}{S}(\lambda) u_+; \\ R_2 e_\lambda Q_\lambda^+ u_+ &= e_\lambda L_\lambda^+ u_+ + e_\alpha N_1^{-1} \tau_2^- \overset{+}{S}(\lambda) u_+. \end{aligned} \tag{3.23}$$

Therefore

$$R_2 e_\lambda Q_\lambda^+ u_+ - R_1 e_\lambda \left(\bar\lambda - \alpha\right) L_\lambda^+ u_+ = e_\alpha N_1^{-1}\left(\tau_2^- \overset{+}{S}(\lambda) - \tau_1^- \overset{+}{S}(\lambda) L_\lambda^+\right) u_+. \tag{3.24}$$

Taking into account the form of the linear bundles Q_λ^+ and L_λ^+ (3.3), transform the left-hand side of the equality

$$\begin{aligned} R_2 e_\lambda Q_\lambda^+ u_+ - \left(\bar\lambda - \alpha\right) R_1 e_\lambda L_\lambda^+ u_+ &= \left(\bar\lambda - \alpha\right) R_2 e_\lambda u_+ + (\alpha - \bar\alpha) R_2 e_\lambda u_+ \\ &+ (\bar\alpha - \alpha)\left(\bar\lambda - \alpha\right) R_2 e_\lambda \left(\tilde N_1^*\right)^{-1} \tilde\Gamma^* u_+ + |\alpha - \bar\alpha|^2 R_2 e_\lambda \left(\tilde N_1^*\right) \tilde\Gamma^* u_+ \\ &+ (\bar\alpha - \alpha) R_2 e_\lambda \left(\tilde N_1^*\right) \tilde N_2^* u_+ - \left(\bar\lambda - \alpha\right)^2 R_1 e_\lambda \left(\tilde N_1^*\right)^{-1} \tilde\Gamma^* u_+ \\ &- \left(\bar\lambda - \alpha\right)(\alpha - \bar\alpha) R_1 e_\lambda \left(\tilde N_1^*\right)^{-1} \tilde\Gamma^* u_+ - \left(\bar\lambda - \alpha\right) R_1 e_\lambda \left(\tilde N_1^*\right)^{-1} \tilde N_2^* u_+ \\ &= \left(\bar\lambda - \alpha\right)\Big\{R_2 e_\lambda u_+ + (\bar\alpha - \alpha) R_2 e_\lambda \left(\tilde N_1^*\right)^{-1} \tilde\Gamma^* u_+ \\ &- (\alpha - \bar\alpha) R_1 e_\lambda \left(\tilde N_1^*\right)^{-1} \tilde\Gamma^* u_+ - R_1 e_\lambda \left(\tilde N_1^*\right)^{-1} \tilde N_2^* u_+\Big\} \\ &+ (\alpha - \bar\alpha) R_2 \left[\left(\bar\lambda - \alpha\right) R_1 e_\lambda u_+ - e_\alpha N_1^{-1} \tau_1^- \overset{+}{S}(\lambda) u_+\right] \\ &+ |\alpha - \bar\alpha|^2 R_2 \left[\left(\bar\lambda - \alpha\right) R_1 e_\lambda \left(\tilde N_1^*\right)^{-1} \tilde\Gamma^* u_+ - e_\alpha N_1^{-1} \tau_1^- \overset{+}{S}(\lambda) \left(\tilde N_1^*\right)^{-1} \tilde\Gamma^* u_+\right] \\ &+ (\bar\alpha - \alpha) R_2 \left[\left(\bar\lambda - \alpha\right) R_1 e_\lambda \left(\tilde N_1^*\right)^{-1} \tilde N_2^* u_+ - e_\alpha N_1^{-1} \tau_2^- \overset{+}{S}(\lambda) \left(\tilde N_1^*\right)^{-1} \tilde N_2^* u_+\right] \\ &- \left(\bar\lambda - \alpha\right)\left[e_\lambda \left(\tilde N_1^*\right)^{-1} \tilde\Gamma^* u_+ + e_\alpha N_1^{-1} \tau_1^- \overset{+}{S}(\lambda) \left(\tilde N_1^*\right)^{-1} \tilde\Gamma^* u_+\right] \end{aligned}$$

in view of the first relation in (3.23). Define the linear operators $\tilde N_1^*$, $\tilde N_2^*$, and $\tilde\Gamma^*$ in H_K^+ (3.22), $\tilde N_p^* e_\lambda u_+ \overset{\text{def}}{=} e_\lambda \tilde N_p^* u_+$, $p = 1, 2$, and $\tilde\Gamma^* e_\lambda u_+ \overset{\text{def}}{=} e_\lambda \tilde\Gamma^* u_+$.

Then, in view of the invariancy of the subspace H_K^- (3.20) with respect to the resolvents R_1 and R_2, we have

$$R_2 e_\lambda Q_\lambda^+ u_+ - \left(\bar{\lambda}-\alpha\right) R_1 e_\lambda L_\lambda^+ u_+ = \left(\bar{\lambda}-\alpha\right) \Big\{ R_2 + (\bar{\alpha}-\alpha)\left(\tilde{N}_1^*\right)^{-1} \tilde{\Gamma}^* R_2$$
$$+ (\bar{\alpha}-\alpha)\left(\tilde{N}_1^*\right)^{-1} \tilde{\Gamma}^* R_1 - \left(\tilde{N}_1^*\right)^{-1} \tilde{N}_2^* R_1 + (\alpha-\bar{\alpha}) R_1 R_2 + |\alpha-\bar{\alpha}|^2$$
$$\times \left(\tilde{N}_1^*\right)^{-1} \tilde{\Gamma}^* R_1 R_2 + (\bar{\alpha}-\alpha)\left(\tilde{N}_1^*\right)^{-1} \tilde{N}_2^* R_1 R_2 - \left(\tilde{N}_1^*\right)^{-1} \tilde{\Gamma}^* \Big\} e_\lambda u_+ + f_-,$$

where $f_- \in H_K^-$ (3.20). Thus, we can be finally written relation (3.24) as

$$\left(\bar{\lambda}-\alpha\right)\left(\tilde{N}_1^*\right)^{-1} \Big\{ \tilde{N}_1^* R_2 \left(I + (\alpha-\bar{\alpha}) R_1\right) - \tilde{N}_2^* R_1 \left(I + (\alpha-\bar{\alpha}) R_2\right)$$
$$-\tilde{\Gamma}^* \left(I + (\alpha-\bar{\alpha}) R_1\right)\left(I + (\alpha-\bar{\alpha}) R_2\right) \Big\} e_\lambda u_+ = g_-,$$

where $g_- \in H_K^-$ (3.20). Taking into account (3.21), we have

$$\{N_1 R_2 - N_2 R_1 - \Gamma\} \cdot \left(\tilde{N}_1^*\right)^{-1} \Big\{ \tilde{N}_1^* R_2 \left(I + (\alpha-\bar{\alpha}) R_1\right)$$
$$-\tilde{N}_2^* R_1 \left(I + (\alpha-\bar{\alpha}) R_2\right) - \tilde{\Gamma}^* \left(I + (\alpha-\bar{\alpha}) R_1\right)\left(I + (\alpha-\bar{\alpha}) R_2\right) \Big\} e_\lambda u_+ = 0. \tag{3.25}$$

Let $\dim E_\pm = n_\pm < \infty$. By using $\mathbb{Q}(\lambda_1,\lambda_2)$ and $\tilde{\mathbb{Q}}(\lambda_1,\lambda_2)$ (2.8), define the following polynomials:

$$\mathbb{Q}_-(\lambda_1,\lambda_2) = (\lambda_1,\lambda_2)^{n_-}\, \mathbb{Q}\left(\frac{1}{\lambda_1}+\alpha, \frac{1}{\lambda_2}+\alpha\right) = \det\left[N_1\lambda_2 - N_2\lambda_1 + \Gamma\right];$$

$$\tilde{\mathbb{Q}}_+(\lambda_1,\lambda_2) = (\lambda_1\lambda_2)^{n_+}\, \overline{\tilde{\mathbb{Q}}\left(\frac{1+(\bar{\alpha}-\alpha)\bar{\lambda}_1}{\bar{\lambda}_1}+\alpha, \frac{1+(\bar{\alpha}-\alpha)\bar{\lambda}_2}{\bar{\lambda}_2}+\alpha\right)}$$
$$= \det\Big[\tilde{N}_1^*\lambda_2\left(1+(\alpha-\bar{\alpha})\lambda_1\right) - \tilde{N}_2^*\lambda_1\left(1+(\alpha-\bar{\alpha})\lambda_2\right) \tag{3.26}$$
$$-\tilde{\Gamma}^*\left(1+(\alpha-\bar{\alpha})\lambda_1\right)\left(1+(\alpha-\bar{\alpha})\lambda_2\right)\Big].$$

Using $\mathbb{Q}_-(\lambda_1,\lambda_2)$ and $\tilde{\mathbb{Q}}_+(\lambda_1,\lambda_2)$ (3.26), construct the polynomial

$$\mathbb{P}(\lambda_1,\lambda_2) \stackrel{\text{def}}{=} \mathbb{Q}_-(\lambda_1,\lambda_2)\cdot\tilde{\mathbb{Q}}_+(\lambda_1,\lambda_2). \tag{3.27}$$

Formulate an analogue of the Hamilton–Caley theorem for a commutative system of unbounded operators $\{A_1, A_2\}$.

Theorem 3.4. *Let a simple commutative colligation* Δ *(1.1) be given such that* $\dim E_{\pm} = n_{\pm} < \infty$, *and the operators* N_1, $\tilde{N}_1$ *are boundedly invertible. Moreover, suppose that relations (1.24) are true, and* $\left\{K^*\sigma_2^+K - \sigma_2^-\right\}$, $\left\{K\tau_2^-K^* - \tau_2^+\right\}$ *are invertible. Then the resolvents* $\{R_1, R_2\}$ *of the main operators* $\{A_1, A_2\}$ *of the colligation* Δ *(1.1) annihilate the polynomial*

$$\mathbb{P}\left(R_1, R_2\right) = 0, \tag{3.28}$$

where $\mathbb{P}\left(\lambda_1, \lambda_2\right)$ *is given by (3.27) and constructed by the polynomials* $\mathbb{Q}\left(\lambda_1, \lambda_2\right)$ *and* $\tilde{\mathbb{Q}}\left(\lambda_1, \lambda_2\right)$ *(2.8) using formulas (3.26).*

P r o o f. Since

$$\mathbb{Q}_-\left(\lambda_1, \lambda_2\right) I_{E_-} = B\left(\lambda_1, \lambda_2\right)\left\{N_1\lambda_2 - N_2\lambda_1 - \Gamma\right\},$$

where $B\left(\lambda_1, \lambda_2\right)$ is a matrix-valued polynomial of λ_1, λ_2, then (3.21) implies

$$\mathbb{Q}_-\left(R_1, R_2\right) f_- = 0$$

for all $f_- \in H_K^-$ (3.20). Similar considerations, by using (3.25), show that

$$\mathbb{P}\left(R_1, R_2\right) f_+ = 0$$

for all $f_+ \in H_K^+$ (3.22). And since the closed linear span $H_K^{\pm}$ generates the whole space H_K, we finally obtain

$$\mathbb{P}\left(R_1, R_2\right) f = 0$$

for all $f \in H_K$. Application of Theorem 2.4 finishes the proof. ■

Suppose that the characteristic function $S_1(\lambda)$ (2.3) is such that $S_1(\alpha)$ is invertible, then the intertwining condition 1) (2.51) implies that $n = n_- = n_+$ and $\mathbb{Q}\left(\lambda_1, \lambda_2\right)$. Therefore polynomial (3.27) in this case is given by

$$\mathbb{P}\left(\lambda_1, \lambda_2\right) = \mathbb{Q}_-\left(\lambda_1, \lambda_2\right) \cdot \mathbb{Q}_+\left(\lambda_1, \lambda_2\right), \tag{3.29}$$

where $\mathbb{Q}_{\pm}\left(\lambda_1, \lambda_2\right)$ are defined by the polynomial $\mathbb{Q}\left(\lambda_1, \lambda_2\right)$ (2.8) using formulas (3.26). Let

$$\bar{w}_p = \frac{\lambda_p}{1 + \left(\alpha - \bar{\alpha}\right)\lambda_p}, \quad p = 1, 2. \tag{3.30}$$

Then it is obvious that the inverse transform to (3.30) is equal to

$$\lambda_p = \frac{\bar{w}_p}{1 + \left(\bar{\alpha} - \alpha\right)\bar{w}_p} = \overline{\frac{w_p}{1 + \left(\alpha - \bar{\alpha}\right)w_p}}, \quad p = 1, 2. \tag{3.31}$$

It is easy to see that

$$\mathbb{P}\left(\lambda_1,\lambda_2\right)=\mathbb{P}\left(\frac{\bar{w}_1}{1+\left(\bar{\alpha}-\alpha\right)\bar{w}_1},\frac{\bar{w}_2}{1+\left(\bar{\alpha}-\alpha\right)\bar{w}_2}\right)$$

$$=\det\left[N_1\frac{\bar{w}_2}{1+\left(\bar{\alpha}-\alpha\right)\bar{w}_2}-N_2\frac{\bar{w}_1}{1+\left(\bar{\alpha}-\alpha\right)\bar{w}_2}-\Gamma\right]\cdot\det\left\{\frac{1}{1+\left(\bar{\alpha}-\alpha\right)\bar{w}_1}\right.$$

$$\left.\times\frac{1}{1+\left(\bar{\alpha}-\alpha\right)\bar{w}_2}\cdot\left[N_1^*\bar{w}_2-N_2^*\bar{w}_1-\Gamma^*\right]\right\}=\left(1+\left(\bar{\alpha}-\alpha\right)\bar{w}_1\right)^{-2n}\left(1\right.$$

$$\left.+\left(\bar{\alpha}-\alpha\right)\bar{w}_2\right)^{-2n}\overline{\mathbb{P}\left(w_1,w_2\right)}.$$

Thus, polynomial (3.29) has the "antiholomorphic involution"

$$\left(1+\left(\bar{\alpha}-\alpha\right)\bar{w}_1\right)^{2n}\left(1+\left(\bar{\alpha}-\alpha\right)\bar{w}_2\right)^{2n}\mathbb{P}\left(\frac{\bar{w}_1}{1+\left(\bar{\alpha}-\alpha\right)\bar{w}_1},\frac{\bar{w}_2}{1+\left(\bar{\alpha}-\alpha\right)\bar{w}_2}\right)$$

$$=\overline{\mathbb{P}\left(w_1,w_2\right)} \tag{3.32}$$

relatively to transform (3.30).

As known, the broken-linear transformation (3.30) is a holomorphic transformation of the generalized circle into the circle and its boundary (circle) is invariant relatively to this transformation. To find this circle, multiply both sides of the equality

$$\lambda=\frac{\bar{\lambda}}{1+\left(\bar{\alpha}-\alpha\right)\bar{\lambda}}$$

by $\bar{\alpha}-\alpha$. Then, after elementary transformations, we obtain

$$1+\left(\alpha-\bar{\alpha}\right)\lambda=\frac{1}{1+\left(\bar{\alpha}-\alpha\right)\bar{\lambda}}.$$

This signifies that $\xi=1+\left(\alpha-\bar{\alpha}\right)\lambda$ satisfies the relation $\xi=\frac{1}{\bar{\xi}}$, therefore ξ belongs to the unit circle $\mathbb{T}$. Thus, we obtain the circle

$$\mathbb{T}_\alpha=\left\{\lambda=\frac{\xi-1}{\alpha-\bar{\alpha}}\in\mathbb{C}:|\xi|=1\right\}, \tag{3.33}$$

the radius r of which is equal to $r=|2\operatorname{Im}\alpha|^{-1}$ and the center $\mathbb{T}_\alpha$ (3.33) is in the point i (when $\alpha\in\mathbb{C}_+$) or in the point $-i$ (when $\alpha\in\mathbb{C}_-$). It is obvious that the transformation (3.30) written in the form

$$1+\left(\alpha-\bar{\alpha}\right)w=\frac{1}{1+\left(\bar{\alpha}-\alpha\right)\bar{\lambda}}$$

represents the inversion relatively to the circle $\mathbb{T}_\alpha$ (3.33).

Theorem 3.5. *The polynomial $\mathbb{P}\left(\lambda_1,\lambda_2\right)$ (3.29) has the antiholomorphic involution (3.32) given by inversion (3.30) relative to the circle $\mathbb{T}_\alpha$ (3.33).*

References

[1] *M.S. Livšic*, On One Class of Linear Operators in Hilbert Space. — *Mat. Sb.* **19/61** (1946), No. 2, 236–260.

[2] *A.V. Kuzhel*, On the Reduction of Unbounded Nonselfadjoint Operators to the Triangular Form. — *DAN SSSR* **119** (1958), No. 5, 868–871. (Russian)

[3] *A. Kuzhel*, Characteristic Functions and Models of Nonselfadjoint Operator. Kluwer Acad. Publ., Dordrecht, London, 1996.

[4] *A.V. Shtraus*, Characteristic Functions of Linear Operators. — *Izv. AN SSSR, Ser. Mat.* **24** (1960), No. 1, 43–74. (Russian)

[5] *E.R. Tsekanovskiy and Yu.L. Shmul'yan*, Matters of the Expansion Theory of Unbounded Operators in Framed Hilbert Spaces. — *Itogi Nauki. Mat. Analiz VINITI* **14** (1977), 59–100. (Russian)

[6] *V.A. Derkach and M.M. Malamud*, Generalized Resolvents and the Boundary Value Problems for Hermitian Operators with Gaps. — *J. Funct. Anal.* **95** (1991), No. 1, 1–95.

[7] *B.S. Pavlov*, Spectral Analysis of Dissipative Shrödinger Operator in Terms of Functional Model. — *Itogi Nauki. Sovr. Probl. Mat., Fund. Napravl. VINITI* **65** (1991), 95–163. (Russian)

[8] *M.S. Livšic, N. Kravitsky, A. Markus, and V. Vinnikov*, Theory of Commuting Nonselfadjoint Operators. Kluwer Acad. Publ., Dordrecht, London, 1996.

[9] *V.A. Zolotarev*, Time Cones and Functional Model on Riemann Surface. — *Mat. Sb.* **181** (1990), No. 7, 965–994. (Russian)

[10] *V.A. Zolotarev*, Analitic Methods of Spectral Representations of Nonselfadjoint and Nonunitary Operators. MagPress, Kharkov, 2003. (Russian)

[11] *V.A. Zolotarev*, On Commutative Systems of Nonselfadjoint Unbounded Linear Operators. — *J. Math. Phys., Anal., Geom.* **5** (2009), 275–279.