Journal of Mathematical Physics, Analysis, Geometry
2010, vol. 6, No. 3, pp. 291–294

On a Question by A.M. Kagan

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Received January 11, 2010

There is given an example of probability distribution, not having Gaussian components, such that for any two independent identically distributed random variables ξ and η with this distribution and for all $a \neq 0$, $b \neq 0$ the distribution of the linear form $a\xi + b\eta$ has Gaussian components.

Key words: linear form, characteristic function, Gaussian distribution.

Mathematics Subject Classification 2000: 60E10.

A.M. Kagan posed the question, "Do there exist two independent random variables ξ and η such that the distribution of each of them does not have Gaussian components, but the distribution of the linear form $a\xi + b\eta$ has Gaussian components for any real numbers $a \neq 0$ and $b \neq 0$?". Let us formulate this question in terms of characteristic functions. Recall that the *characteristic function* of a random variable ξ with distribution P is the function defined for $t \in \mathbf{R}$ by the formula

$$\varphi_\xi(t) = E[e^{it\xi}] = \int_{-\infty}^{\infty} e^{itx} P(dx)\,.$$

The characteristic function φ_1 is called a *divisor* of the characteristic function φ if there exists a characteristic function φ_2 such that

$$\varphi(t) = \varphi_1(t)\varphi_2(t)$$

for any $t \in \mathbf{R}$. The characteristic function φ is called *indecomposable* if it is not equal to the function e^{iat} ($a \in \mathbf{R}$) and if each divisor of φ is equal to $\varphi(t)e^{ibt}$ or e^{ict}, where $b, c \in \mathbf{R}$. The characteristic function of the form $e^{-\sigma t^2 + iat}$, where $\sigma > 0$ and $a \in \mathbf{R}$, is said to be *Gaussian*. The factor e^{iat} is not essential in the problem under consideration. Therefore we can say that the characteristic function φ has a Gaussian divisor if $\varphi(t)e^{\sigma t^2}$ is a characteristic function for some positive number σ.

In terms of characteristic functions A. M. Kagan's question can be stated as, "Do there exist two characteristic functions $f(t)$ and $g(t)$, not having Gaussian divisors, such that the characteristic function $f(at)g(bt)$ has a Gaussian divisor for all numbers $a \neq 0$ and $b \neq 0$?". The aim of the paper is to give a positive answer to this question.

Theorem. *The characteristic function* $f(t) = (1-t^2)e^{-t^2/2}$ *does not have Gaussian divisors, but the characteristic function* $f(at)f(bt)$ *has Gaussian divisors for any nonzero* a *and* b.

R e m a r k 1. This theorem was proved for $a = b$. It is well known (see [1, Ch. 3, §§ 3 and 4]) that the characteristic function $f(t) = (1-t^2)e^{-t^2/2}$ of the probability density $(\sqrt{2\pi})^{-1}x^2e^{-x^2/2}$ is indecomposable, but the characteristic function $f^2(t)$ has the Gaussian divisor $e^{-t^2/4}$. Notice that the general case $a \neq 0$, $b \neq 0$ is not an immediate corollary of the particular case $a = b$.

R e m a r k 2. In terms of random variables the theorem means that *if* ξ *and* η *are the independent and identically distributed random variables with probability density* $(\sqrt{2\pi})^{-1}x^2e^{-x^2/2}$ *(and characteristic function* $(1-t^2)e^{-t^2/2}$ *), then the distribution of the linear form* $a\xi + b\eta$ *has Gaussian components for all* $a \neq 0$ *and* $b \neq 0$, *although the distributions of* ξ *and* η *do not have Gaussian components.*

To prove the theorem we need the following lemma.

Lemma. *For every* $\gamma > 0$ *the following three functions*

$$\varphi_{1,\gamma}(t) = e^{-\gamma^2t^2/2}, \quad \varphi_{2,\gamma}(t) = t^2e^{-\gamma^2t^2/2}, \quad \varphi_{3,\gamma}(t) = t^4e^{-\gamma^2t^2/2}$$

are Fourier transforms of the functions

$$p_{1,\gamma}(x) = \frac{1}{\sqrt{2\pi}\gamma}e^{-\frac{x^2}{2\gamma^2}}, \quad p_{2,\gamma}(x) = \frac{1}{\sqrt{2\pi}\gamma}\Big(\frac{1}{\gamma^2} - \frac{x^2}{\gamma^4}\Big)e^{-\frac{x^2}{2\gamma^2}},$$
$$p_{3,\gamma}(x) = \frac{1}{\sqrt{2\pi}\gamma}\Big(3\frac{1}{\gamma^4} - 6\frac{x^2}{\gamma^6} + \frac{x^4}{\gamma^8}\Big)e^{-\frac{x^2}{2\gamma^2}},$$

respectively. Therefore,

$$\varphi_{k,\gamma}(t) = \int_{-\infty}^{\infty} e^{itx}p_{k,\gamma}(x)\,dx \quad (k = 1, 2, 3). \tag{1}$$

Equality (1) for $k = 1$ is a direct consequence of the equality

$$e^{-t^2/2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{itx}e^{-x^2/2}\,dx.$$

Multiplying by t^2 both sides of equality (1) for $k = 1$ and integrating twice by parts, we get equality (1) for $k = 2$. Analogously, equality (1) for $k = 3$ follows from equality (1) for $k = 2$. ∎

P r o o f of the theorem. Since the characteristic function $(1 - t^2)e^{-t^2/2}$ is even, we may suppose that $a > 0$ and $b > 0$. The characteristic function $f(t) = (1-t^2)e^{-t^2/2}$ is indecomposable, hence it does not have Gaussian divisors. Our aim is to prove that the characteristic function

$$f(at)f(bt) = (1 - a^2t^2)(1 - b^2t^2) \exp\big(-(a^2 + b^2)t^2/2\big)$$

has Gaussian divisors for all $a > 0$ and $b > 0$. Therefore, for every $a > 0$ and $b > 0$ we have to find γ^2, $0 < \gamma^2 < a^2 + b^2$, such that

$$\varphi_\gamma(t) := \big(1 - (a^2 + b^2)t^2 + a^2b^2t^4\big) \exp\big(-\gamma^2t^2/2\big) \tag{2}$$

is a characteristic function. It is sufficient to prove that if $\gamma^2 \in (0, a^2 + b^2)$ and γ^2 is sufficiently close to $a^2 + b^2$, then the function $\varphi_\gamma(t)$ is a Fourier transform of a probability density $p_\gamma(x)$. It follows from the lemma that the function $\varphi_\gamma(t)$ (see (2)) is a Fourier transform of the function

$$p_\gamma(x) = p_{1,\gamma}(x) - (a^2 + b^2)p_{2,\gamma}(x) + a^2b^2p_{3,\gamma}(x) = \frac{1}{\sqrt{2\pi}\gamma}e^{-\frac{x^2}{2\gamma^2}}Q_\gamma(x)\,,$$

where

$$Q_\gamma(x) = 1 - (a^2 + b^2)\Big(\frac{1}{\gamma^2} - \frac{x^2}{\gamma^4}\Big) + a^2b^2\Big(3\frac{1}{\gamma^4} - 6\frac{x^2}{\gamma^6} + \frac{x^4}{\gamma^8}\Big)\,. \tag{3}$$

We will prove that the polynomial $Q_\gamma(x)$ is positive for all $x \in \mathbf{R}$ if a positive number γ^2 is less than a^2+b^2 and sufficiently close to a^2+b^2. We put $a = r\cos\theta$, $b = r\sin\theta$ ($r > 0$, $0 < \theta < \pi/2$). Then $a^2 + b^2 = r^2$, and we can rewrite (3) as follows:

$$Q_\gamma(x) = \Big(1 - \frac{r^2}{\gamma^2} + \frac{3}{4}\cdot\frac{r^4\sin^2 2\theta}{\gamma^4}\Big) + \Big(\frac{r^2}{\gamma^4} - \frac{3}{2}\cdot\frac{r^4\sin^2 2\theta}{\gamma^6}\Big)x^2 + \frac{1}{4}\cdot\frac{r^4\sin^2 2\theta}{\gamma^8}x^4\,.$$

We denote $\delta := \sin^2 2\theta$. Then $0 < \delta < 1$, and we can represent $Q_\gamma(x)$ in the form

$$Q_\gamma(x) = \Big(1 - \frac{r^2}{\gamma^2} + \frac{3}{4}\delta\frac{r^4}{\gamma^4}\Big) + \Big(\frac{r^2}{\gamma^2} - \frac{3}{2}\delta\frac{r^4}{\gamma^4}\Big)\frac{x^2}{\gamma^2} + \frac{1}{4}\delta\frac{r^4}{\gamma^4}\frac{x^4}{\gamma^4}\,.$$

Let us denote $s := \dfrac{r^2}{\gamma^2}$ and $y := \dfrac{x^2}{\gamma^2}$. Therefore, $s > 1$ and $y \geqslant 0$. The polynomial $Q_\gamma(x)$ can be rewritten as follows:

$$Q_\gamma(x) = \Big(1 - s + \frac{3}{4}\delta s^2\Big) + \Big(s - \frac{3}{2}\delta s^2\Big)y + \delta s^2y^2 =: \varkappa_{s,\delta}(y)\,.$$

We prove that the inequality $\min\{\varkappa_{s,\delta}(y) : y \geqslant 0\} > 0$ is valid for every $\delta \in (0,1)$ if $s > 1$ and s is sufficiently close to 1. Since the coefficients of the polynomial $\varkappa_{s,\delta}(y)$ depend continuously on s, it remains to verify that $\min\{\varkappa_{1,\delta}(y) : y \geqslant 0\} > 0$ for every $\delta \in (0,1)$. Let us consider the polynomial

$$\psi_\delta(y) := \frac{\varkappa_{1,\delta}(y)}{\delta} = \frac{3}{4} + \Big(\frac{1}{\delta} - \frac{3}{2}\Big)y + y^2$$

and show that $\min\{\psi_\delta(y) : y \geqslant 0\} > 0$ for every $\delta \in (0,1)$. If $0 < \delta \leqslant 2/3$, then $\psi_\delta(y) \geqslant 3/4$ for all $y \geqslant 0$. If $2/3 < \delta < 1$, then $\psi_\delta(y)$ has a minimum at the point $y_\delta = \frac{1}{2}\left(\frac{3}{2} - \frac{1}{\delta}\right) > 0$. Therefore,

$$\min_{y \geqslant 0} \psi_\delta(y) = \psi_\delta(y_\delta) = \frac{3}{4} - \Big(\frac{3}{4} - \frac{1}{2\delta}\Big)^2 \geqslant \frac{3}{4} - \Big(\frac{3}{4} - \frac{1}{2}\Big)^2 = \frac{11}{16}\,.$$

Hence $\min\{\varkappa_{1,\delta}(y) : y \geqslant 0\} \geqslant \delta\frac{11}{16}$ for every $\delta \in (0,1)$. The theorem is proved. ■

R e m a r k 3. A simpler example can be given if in A.M. Kagan's question not require the random variables ξ and η to be independent. Let us consider the characteristic function of two variables

$$f(t,s) = \big(e^{-t^2/2} + e^{-s^2/2}\big)/2$$

which is the characteristic function of the mixture with weights 1/2 of standard Gaussian distributions concentrated on the coordinate axes $\{y = 0\}$ and $\{x = 0\}$. Let (ξ, η) be a random vector with this distribution. We assert that the characteristic functions $f(t,0)$ and $f(0,s)$ of the coordinates ξ and η of random vector (ξ, η) do not have Gaussian divisors, but the characteristic function $f(at, bt)$ of the linear form $a\xi + b\eta$ has Gaussian divisors for all $a \neq 0$ and $b \neq 0$. Indeed, the distribution with the characteristic function $f(t,0) = (e^{-t^2/2} + 1)/2$ is a mixture with weights 1/2 of standard Gaussian distribution and the degenerate distribution concentrated at the point 0. Since this distribution has an atom, it does not have absolutely continuous components, so it does not have Gaussian components. However the characteristic function $f(at,bt) = (e^{-a^2t^2/2} + e^{-b^2t^2/2})/2$ has Gaussian divisors for any $a \neq 0$ and $b \neq 0$.

I am thankful to G.M. Feldman who attracted my attention to A.M. Kagan's question.

References

[1] *Ju.V. Linnik and I.V. Ostrovskiĭ,* Decomposition of Random Variables and Vectors. Nauka, Moscow, 1972. (Engl. transl.: Amer. Math. Soc., Providence, RI, 1977.)