Journal of Mathematical Physics, Analysis, Geometry
2010, vol. 6, No. 3, pp. 295–304

On the Conditions of Total Resonance of Liouville Type Hamiltonian Systems with n Degrees of Freedom

V.T. Lisitsa

Department of Mechanics and Mathematics, V.N. Karazin Kharkiv National University
4 Svobody Sq., Kharkiv, 61077, Ukraine
E-mail:lisitsa@univer.kharkov.ua

Received March 19, 2010

The completely singular dynamical systems of the Liouville type are studied. The motion paths of these systems are closed graphs if the Liouville tori are compact. The conditions under which a dynamical system of the Liouville type is strongly singular are obtained in the paper. These conditions have a form of the system of integral equations. It is proved that the obtained system is solvable.

Key words: Hamiltonian system, resonance, integrable system, action-angle variables.

Mathematics Subject Classification 2000: 70H06; 53B21.

1. Introduction

It is well known that many fundamental physical phenomena are described in the terms of the Hamiltonian formalism. The Hamiltonian systems integrable in the Liouville–Arnold terms [1] are an important class of the Hamiltonian systems. In the recent years, the interest towards the study of these systems renewed as new methods for the study of integrable systems [2] had appeared. Introduce the necessary definitions and recall the well-known facts.

Consider a Hamiltonian system. It is given by an even-dimensional manifold M^{2n} with nondegenerate exterior closed 2-form ω (the symplectic structure) and differentiable function H (the Hamiltonian) [1, 3].

Definition 1. *A differentiable vector field* $\operatorname{sgrad} H$ *given on* M^{2n} *is said to be the skew gradient of a function* H *if it satisfies the equality* $\omega\left(\bar{v}; \operatorname{sgrad} H\right) = \bar{v}(H)$ *for all differentiable vector fields* $\bar{v}$ *where* $\bar{v}(H)$ *is the derivative of the function* H *in the direction of* $\bar{v}$. *The vector field sgradH is called also the Hamiltonian vector field with Hamiltonian H [1, 3].*

Definition 2. *Let F and H be differentiable functions on symplectic manifold M^{2n} with symplectic structure ω. The differentiable function $\{F; H\} = \omega(\operatorname{sgrad} F; \operatorname{sgrad} H)$ is said to be the Poisson bracket of the functions F and H.*

Definition 3. *A differentiable function F is said to be the integral of the Hamiltonian system with Hamiltonian H if F is constant along the integral trajectories of the given Hamiltonian system.*

Note that the function F is an integral of the system with Hamiltonian H if and only if $\{F; H\} \equiv 0$. In particular, the Hamiltonian H is an integral.

Definition 4. *Two functions F_1 and F_2 are said to be involutive (or to be in involution) if $\{F_1; F_2\} \equiv 0$.*

Definition 5. *A Hamiltonian system with n degrees of freedom with Hamiltonian H is said to be completely integrable in the Liouville–Arnold terms if it has n functionally independent, pairwise involutive integrals, $H = F_1, F_2, \ldots, F_n$, [1, 3]. The independence of the integrals signifies that n forms of dF_i, $i = 1, \ldots, n$, are linearly independent.*

If the Hamiltonian system H is integrable in the Liouville – Arnold terms, then it is possible to introduce "action-angle" variables $(I; \varphi)$ so that the corresponding Hamiltonian differential equations have the form [1]

$$\begin{gathered} \frac{dI}{dt} = 0, \quad \frac{d\varphi}{dt} = \omega(I) = \frac{\partial H}{\partial I}, \\ I = I_1, \ldots, I_n; \quad \varphi = \varphi_1, \ldots, \varphi_n; \quad \omega = \omega_1, \ldots, \omega_n. \end{gathered} \tag{1}$$

The motion described by the equations of (1) is said to be conditionally periodic and the values $\omega_i = \dfrac{\partial H}{\partial I_i}$ are said to be the frequencies of a conditionally periodic motion.

Consider the submanifolds $F_c = \{(q, p) : F_i = c_i\}$, where c_i are constants. The submanifolds F_c are the level surfaces of the integrals F_i, $i = 1, \ldots, n$. In the general case, the submanifolds F_c are homeomorphic to the product $T^{n-k} \times E^k$, where T^{n-k} is the torus of the dimension $n - k$, E^k is the Euclid space. If F_c are compact, then they are homeomorphic to the torus T^n (the Liouville–Arnold torus).

Note that in the general case the frequencies ω_i are rationally independent, i.e., the equality $k_1\omega_1 + \cdots + k_n\omega_n = 0$, where k_i are rational numbers, yields $k_1 = \cdots = k_n = 0$. Besides, if the level surfaces F_c are the tori T^n, then the trajectories of the conditionally periodic movement fill T^n everywhere densely.

There often arise special cases when some frequencies are rationally dependent. Besides, the dimensions of the tori filled with conditionally periodic trajectories decrease. If all the frequencies are pairwise rationally dependent, then the motion trajectories become closed curves. In this case the movement is completely singular.

Definition 6. *A completely singular Hamiltonian system is called the totally resonance Hamiltonian system.*

Completely singular systems are of interest from different points of view. First of all, completely singular systems have $(2n-1)$ of independent integrals of movement. Of course, not all of them are in the involution. If the Liouville tori are compact, they are circles, i.e., the trajectories of the Hamiltonian system are closed [4]. From the geometric point of view, the manifolds, all of whose geodesics are closed, belong to these systems. In [5], a completely integrable Hamiltonian system $\operatorname{sgrad} H$ with two degrees of freedom is considered and the following proposition is proved.

Consider the restriction H to a nonsingular leaf $Q^3 : \{x \in M^n \mid H(x) = c\}$. Then the following two statements are equivalent:

i) any trajectory on Q^3 is periodic;

ii) there exist two functionally independent functions f_1, f_2 on Q^3 that are Bott integrals of the system $v = \operatorname{sgrad} H$ on Q^3.

Here are some examples below:

1) **The Kepler problem**. In the spherical coordinates, the Kepler Hamiltonian of the motion has the form

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2\theta}\right) - \frac{k}{r}, \quad k > 0.$$

It is well known that the complete integral of the corresponding Hamilton–Jacobi equation for the given problem allows the separation of variables and has the form [4, 5]

$$w = \beta_\varphi \cdot \varphi + \int \sqrt{2h + \frac{2k}{r} - \frac{\beta_\theta^2}{r^2}}\,dr + \int \sqrt{\beta_\theta^2 - \frac{\beta_\varphi^2}{\sin^2\theta}}\,d\theta.$$

It is proved in [4] that all frequencies of the given motion are equal and the trajectories of this system are closed.

2) **The manifolds, all of whose geodesics are closed**. Consider the two-dimensional manifolds as an example. As known, all geodesics of the sphere S^2 are closed. In [8], the problems on the manifolds with closed geodesics are studied in detail. In particular, the surfaces of revolution, all of whose geodesics are closed, are analyzed. With a certain choice of coordinates, one can reduce the metric of the revolution surface to the form of

$$ds^2 = [f(\cos r)]^2 dr^2 + \sin^2 r d\theta^2, \quad 0 \le \theta \le 2\pi, \quad 0 \le r \le \pi.$$

The Hamiltonian of geodesic flow of the given surface is

$$H = \frac{1}{2}\left(\frac{p_1^2}{(f(\cos r))^2} + \frac{p_2^2}{\sin^2 r}\right).$$

The complete integral of the corresponding Hamilton–Jacobi equation allows the separation of variables and can be written as

$$w = \sqrt{2}\int f(\cos r)\sqrt{\frac{\beta_1}{\sin^2 r} + 2h}dr + \sqrt{2}\int \sqrt{-\beta_1}d\theta.$$

In this case, the condition for the rational dependence of frequencies is

$$\int\limits_c^{\pi-c} \frac{\sin c \cdot f(\cos r)}{\sin r\sqrt{\sin^2 r - \sin^2 c}}dr = \frac{p}{q}\pi,$$

for all $c \in \left(0; \frac{\pi}{2}\right)$, where p, q are coprime integer numbers. The last equality is known as the Darboux condition for closedness of all geodesics on the revolution surface [8, 9].

2. Hamiltonian Systems of the Liouville Type

In [10, 11], the geodesic flows of the Liouville type with two degrees of freedom were studied and the conditions for closedness of all the trajectories, i.e., the conditions for complete singularity of these systems, were obtained.

The aim of the paper is to find the conditions for complete singularity of the Liouville type systems with n degrees of freedom.

A dynamical system is said to be of the Liouville type if its kinetic energy T and potential energy P are given by

$$T = \frac{1}{2}f\sum_{i=1}^{n}\varphi_i\left(q_i\right)q_i^2, \quad P = \frac{\psi}{f}, \qquad \text{where } f = \sum_{i=1}^{n} f_i\left(q_i\right),$$

$$\psi = \sum_{i=1}^{n}\psi_i\left(q_i\right).$$

Hereinafter we suppose that $\varphi_i > 0$. The Hamiltonian of this system is given by

$$H = \frac{1}{2f}\sum_{i=1}^{n}\frac{p_i^2}{\varphi_i} + \frac{\psi}{f}. \tag{2}$$

Consider the system with Hamiltonian (2) in the parallelepiped $0 \le q_i \le a_i$, $i = 1, \ldots, n$. Denote $g_i = hf_i - \psi_i$. Let $g_i(0) = g_i(a_i) = 0$, $g_{i0} = \max\limits_{q_i\in[0;a_i]} g_i = g_i\left(q_{i0}\right)$

(or $g_{i0} = \min\limits_{q_i \in [0;a_i]} g_i = g_i(q_{i0})$). Note that the functions g_i and g_{i0} depend on h. If the functions g_i are strictly monotonous on the intervals $(0;q_{i0})$ and $(q_{i0};a_i)$, then for every value of $c_i \in (0;g_{i0})$ there exist exactly two values of the variable q_i from the interval $(0,a_i)$, $q_{i1}(c_i)$, $q_{i2}(c_i) \in (0;a_i)$ such that $g_i(q_{i1}) = g_i(q_{i2}) = c_i$.

The following theorem is the main result of the paper.

Theorem. *Let there be a dynamical system with Hamiltonian (2). If the functions $g_i = hf_i - \psi_i$ satisfy the conditions $g_i(0) = g_i(a_i) = 0$ and are monotonous on the intervals $(0, q_{i0})$ and $(q_{i0}; a_i)$, then the conditions for total resonance of this Hamiltonian system are the following: there exists a constant $\lambda(h)$ depending on energy level h and rational numbers r_i such that*

$$\int\limits_{q_{i1}(c_i)}^{q_{i2}(c_i)} \sqrt{\frac{\varphi_i}{g_i - c_i}} dq_i = r_i \lambda(h), \quad i = 1, \ldots, n-1, \tag{3}$$

for all $c_i \in (0;g_{i0})$ (if $n > 2$, then the constants c_i satisfy the condition $c_1 + \cdots + c_n = 0$).

P r o o f. As known, the complete integral $w(q_1, \ldots, q_n; \beta_1, \ldots, \beta_{n-1}, h)$ of a Hamiltonian system of the Liouville type allows the separation of variables and is given by [7]

$$\begin{aligned} w = & \int \sqrt{2\varphi_1(\beta_1 + hf_1 - \psi_1)} dq_1 \\ & + \sum_{i=2}^{n-1} \int \sqrt{2\varphi_i(-\beta_{i-1} + \beta_i + hf_i - \psi_i)} dq_i \\ & + \int \sqrt{2\varphi_n(-\beta_{n-1} + hf_n - \psi_n)} dq_n. \end{aligned} \tag{4}$$

Note that the expressions under radicals must be positive. If the complete integral of the Hamilton–Jacobi equation $H\left(q_1, \ldots, q_n, \frac{\partial w}{\partial q_1}, \ldots, \frac{\partial w}{\partial q_n}\right) = h$ is found, then the canonical action variables can be defined by the formulas

$$I_i = \frac{1}{2\pi} \oint\limits_{\gamma_i} p_i dq_i = \frac{1}{2\pi} \oint\limits_{\gamma_i} \frac{\partial w(q;\beta)}{\partial q_i} dq_i, \tag{5}$$

where the integration is realized by the basic cycles of the Liouville–Arnold tori [1]. In our case, using (4), (5), we can find action variables

$$
\begin{aligned}
I_1 &= \frac{1}{2\pi}\oint\limits_{\gamma_1} \sqrt{2\varphi_1\left(\beta_1 + hf_1 - \psi_1\right)}dq_1,\\
I_i &= \frac{1}{2\pi}\oint\limits_{\gamma_i} \sqrt{2\varphi_i\left(-\beta_{i-1} + \beta_i + hf_i - \psi_i\right)}dq_i,\\
I_n &= \frac{1}{2\pi}\oint\limits_{\gamma_n} \sqrt{2\varphi_n\left(-\beta_{n-1} + hf_n - \psi_n\right)}dq_n, \quad i = 2,\dots,n-1.
\end{aligned} \tag{6}
$$

If we proceed to canonical "action–angle" variables, then the frequencies of conditionally periodic motion are given by $\omega_i = \dfrac{\partial h}{\partial I_i}$. The equation $\dfrac{\partial I_i}{\partial I_j} = \delta^i_j$, where δ^i_j is the Kronecker symbol, holds true by virtue of independence of canonical variables I_i, $I = 1,\dots,n$. In our case, we can put together the equation system

$$
\begin{cases}
\dfrac{\partial I_1}{\partial I_1} = 1 = \dfrac{\partial I_1}{\partial h}\cdot\dfrac{\partial h}{\partial I_1} + \dfrac{\partial I_1}{\partial \beta_1}\cdot\dfrac{\partial \beta_1}{\partial I_1}\\
\dfrac{\partial I_2}{\partial I_1} = 0 = \dfrac{\partial I_2}{\partial h}\cdot\dfrac{\partial h}{\partial I_1} + \dfrac{\partial I_2}{\partial \beta_1}\cdot\dfrac{\partial \beta_1}{\partial I_1} + \dfrac{\partial I_2}{\partial \beta_2}\cdot\dfrac{\partial \beta_2}{\partial I_1}\\
\dfrac{\partial I_3}{\partial I_1} = 0 = \dfrac{\partial I_3}{\partial h}\cdot\dfrac{\partial h}{\partial I_1} + \dfrac{\partial I_3}{\partial \beta_2}\cdot\dfrac{\partial \beta_2}{\partial I_1} + \dfrac{\partial I_3}{\partial \beta_3}\cdot\dfrac{\partial \beta_3}{\partial I_1}\\
\dots\\
\dfrac{\partial I_{n-1}}{\partial I_1} = 0 = \dfrac{\partial I_{n-1}}{\partial h}\cdot\dfrac{\partial h}{\partial I_1} + \dfrac{\partial I_{n-1}}{\partial \beta_{n-2}}\cdot\dfrac{\partial \beta_{n-2}}{\partial I_1} + \dfrac{\partial I_{n-1}}{\partial \beta_{n-1}}\cdot\dfrac{\partial \beta_{n-1}}{\partial I_1}\\
\dfrac{\partial I_n}{\partial I_1} = 0 = \dfrac{\partial I_n}{\partial h}\cdot\dfrac{\partial h}{\partial I_1} + \dfrac{\partial I_n}{\partial \beta_{n-1}}\dfrac{\partial \beta_{n-1}}{\partial I_1}.
\end{cases} \tag{7}
$$

System (7) may be considered as a linear equation system with n unknown $\dfrac{\partial h}{\partial I_1}$, $\dfrac{\partial \beta_i}{\partial I_1}$, $i = 1,\dots,n-1$. The main determinant of this system is the Jacobian $\Delta = \dfrac{\partial\left(I_1,\dots,I_n\right)}{\partial\left(h,\beta_1,\dots,\beta_{n-1}\right)} \neq 0$. By the Cramer rule, find

$$
\frac{\partial h}{\partial I_1} = \frac{\dfrac{\partial I_2}{\partial \beta_1}\cdot\dfrac{\partial I_3}{\partial \beta_2}\cdot\ldots\cdot\dfrac{\partial I_n}{\partial \beta_{n-1}}}{\Delta}. \tag{8}
$$

Similarly, can be found

$$
\frac{\partial h}{\partial I_i} = \frac{(-1)^{i+1}\dfrac{\partial I_1}{\partial \beta_1}\cdot\dfrac{\partial I_2}{\partial \beta_2}\cdots\dfrac{\partial I_{i-1}}{\partial \beta_{i-1}}\cdot\dfrac{\partial I_{i+1}}{\partial \beta_i}\cdots\dfrac{\partial I_n}{\partial \beta_{n-1}}}{\Delta}. \tag{9}
$$

Consider the case of a completely singular system. This implies that all the frequencies are pairwise rationally dependent, i.e.,

$$\omega_1 = r_1\omega_2, \quad \omega_2 = r_2\omega_3, \dots, \omega_{n-1} = r_{n-1}\omega_n, \tag{10}$$

where r_i are rational numbers.

Since $\omega_i = \dfrac{\partial h}{\partial I_i}$, then using (8), (9), and (10) find that

$$\frac{\partial I_1}{\partial \beta_1} = -r_1\frac{\partial I_2}{\partial \beta_1}, \quad \frac{\partial I_2}{\partial \beta_2} = -r_2\frac{\partial I_3}{\partial \beta_2}, \dots, \frac{\partial I_{n-1}}{\partial \beta_{n-1}} = -r_{n-1}\frac{\partial I_n}{\partial \beta_{n-1}}. \tag{11}$$

Equations (6) and (11) yield the conditions for complete singularity of the Hamiltonian system of the Liouville type

$$\begin{cases} \oint\limits_{\gamma_1} \sqrt{\dfrac{\varphi_1}{\beta_1 + hf_1 - \psi_1}}dq_1 = r_1 \oint\limits_{\gamma_1} \sqrt{\dfrac{\varphi_2}{\beta_2 - \beta_1 + hf_2 - \psi_2}}dq_2 \\ \dots \\ \oint\limits_{\gamma_i} \sqrt{\dfrac{\varphi_i}{\beta_i - \beta_{i-1} + hf_i - \psi_i}}dq_i \\ \qquad = r_i \oint\limits_{\gamma_{i+1}} \sqrt{\dfrac{\varphi_{i+1}}{\beta_{i+1} - \beta_i + h\cdot f_{i+1} - \psi_{i+1}}}dq_{i+1} \\ \dots \\ \oint\limits_{\gamma_{n-1}} \sqrt{\dfrac{\varphi_{n-1}}{\beta_{n-1} - \beta_{n-2} + hf_{n-1} - \psi_{n-1}}}dq_{n-1} \\ \qquad = r_{n-1} \oint\limits_{\gamma_n} \sqrt{\dfrac{\varphi_n}{-\beta_{n-1} + hf_n - \psi_n}}dq_n. \end{cases} \tag{12}$$

Consider a movement of the given Hamiltonian system on a surface of the fixed energy level h. If we introduce the designations $c_1 = -\beta_1$, $c_2 = -\beta_2+\beta_1, \dots, c_{i+1} = -\beta_{i+1} + \beta_i$, $c_n = \beta_{n-1}$, then we get

$$\oint\limits_{\gamma_i} \sqrt{\frac{\varphi_i}{g_i - c_i}}dq_i = r_i \oint\limits_{\gamma_{i+1}} \sqrt{\frac{\varphi_{i+1}}{g_{i+1} - c_{i+1}}}dq_{i+1}, \quad i = 1, \dots, n-1. \tag{13}$$

It should be noted that the constants c_i satisfy the condition $c_1 + \dots + c_n = 0$ and cannot have the same sign. Let $g_{i0} = \max g_i$ when $q_i \in [0, a_i]$, $g_{i0} = g_i(q_{i0})$, where $q_{i0} \in (0, a_i)$). By the supposition of the theorem, the functions g_i are strictly monotonous on the intervals $(0; q_{i0})$ and $(q_{i0}; a_i)$. In this case, for every value of $c_i \in (0; g_{i0})$ there exist exactly two values of the variable q_i from the interval $(0, a_i)$, $q_{i1}(c_i)$, $q_{i2}(c_i) \in (0; a_i)$ such that $g_i(q_{i1}) = g_i(q_{i2}) = c_i$. Now, transform $\oint\limits_{\gamma_i} \sqrt{\frac{\varphi_i}{g_i - c_i}} dq_i$. Since the integrand must be nonnegative, $g_i - c_i > 0$, then $q_i \in (q_{i1}(c_i); q_{i2}(c_i))$. While going through the basic closed cycle of γ_i, q_i varies from q_{i1} to q_{i2}, and then from q_{i2} to q_{i1}. Then

$$\oint\limits_{\gamma_i} \sqrt{\frac{\varphi_i}{g_i - c_i}} dq_i = 2 \int\limits_{q_{i1}(c_i)}^{q_{i2}(c_i)} \sqrt{\frac{\varphi_i}{g_i - c_i}} dq_i, \quad c_i \in (0, g_{i0}). \tag{14}$$

Represent the integral from the right-hand side of (14) in the form

$$\int\limits_{q_{i1}(c_i)}^{q_{i2}(c_i)} \sqrt{\frac{\varphi_i}{g_i - c_i}} dq_i = \int\limits_{q_{i1}}^{q_{i0}} \sqrt{\frac{\varphi_i}{g_i - c_i}} dq_i + \int\limits_{q_{i0}}^{q_{i2}} \sqrt{\frac{\varphi_i}{g_i - c_i}} dq_i. \tag{15}$$

Since the function g_i is monotonous on the intervals $(q_{i1}; q_{i0})$ and $(q_{i0}; q_{i2})$, then we can change a variable into the intervals mentioned above. Let θ_{i1} be the inverse function to g_i on the interval (q_{i1}, q_{i0}). Then $q_i = \theta_{i1}(g_i)$, $dq_i = \theta'_{i1} dg_i$ on $(q_{i1}; q_{i0})$. In this case, the first term in the right-hand side of (15) is given by

$$\int\limits_{q_{i1}(c_i)}^{q_{i0}} \sqrt{\frac{\varphi_i}{g_i - c_i}} dq_i = \int\limits_{c_i}^{g_{i0}} \sqrt{\frac{\tilde{\varphi}_{i1}}{g_i - c_i}} \theta'_{i1} dg_i \text{ where } \tilde{\varphi}_{i1} = \varphi_i(\theta_{i1}(g_i)).$$

If θ_{i2} is the inverse function to g_i on $(g_{i0}; g_{i2})$, then in a similar way we can find

$$\int\limits_{q_{i0}}^{q_{i2}(c_i)} \sqrt{\frac{\varphi_i}{g_i - c_i}} dq_i = \int\limits_{g_{i0}}^{c_i} \sqrt{\frac{\tilde{\varphi}_{i2}}{g_i - c_i}} \theta'_{2i} dg_i = -\int\limits_{c_i}^{g_{i0}} \sqrt{\frac{\tilde{\varphi}_{i2}}{g_i - c_i}} \theta'_{i2} dg_i.$$

Denoting $F_i(g_i) = \sqrt{\tilde{\varphi}_{i1}}\theta'_{i1} - \sqrt{\tilde{\varphi}_{i2}}\theta'_{i2}$ gives

$$\int\limits_{q_{i1}(c_i)}^{q_{i2}(c_i)} \sqrt{\frac{\varphi_i}{g_i - c_i}} dq_i = \int\limits_{c_i}^{g_{i0}} \frac{F_i(g_i)}{\sqrt{g_i - c_i}} dg_i. \tag{16}$$

If $c_i < 0$, then we can repeat all previous arguments, but in this case $g_{i0} = \min\limits_{q_i \in [0; a_i]} g_i = g_i(q_{i0}))$, $F_i(x) < 0$, and it is necessary to substitute the expression under radical by $c_i - g_i$. Taking into account equality (16), the conditions for the resonance can be written as

$$\int\limits_{c_i}^{g_{i0}} \frac{F_i(x)}{\sqrt{x - c_i}} dx = r_i \int\limits_{c_{i+1}}^{g_{i+10}} \frac{F_{i+1}(x)}{\sqrt{x - c_{i+1}}} dx, \quad i = 1, \ldots, n-1, \tag{17}$$

where $c_i \in (0; g_{i0})$. When $i = 1$, the equation of system (18) is

$$\int_{c_1}^{g_{10}} \frac{F_1(x)}{\sqrt{x - c_1}} dx = r_1 \int_{c_2}^{g_{20}} \frac{F_2(x)}{\sqrt{x - c_2}} dx. \qquad (18)$$

If the left-hand side of equation (18) put $\lambda(h)$ to some function, then for the function $F_1(x)$ one obtains the Abel equation

$$\lambda(h) = \int_{c_1}^{g_{i0}} \frac{F_1(x)}{\sqrt{x - c_1}} dx. \qquad (19)$$

From equations (18) and (19), it follows that

$$\int_{c_2}^{g_{20}} \frac{F_2(x)}{\sqrt{x - c_2}} = r_1 \lambda(h).$$

Analogously, it follows that

$$\int_{c_i}^{g_{i0}} \frac{F_i(x)}{\sqrt{x - c_i}} dx = r_i \lambda(h), \qquad (20)$$

where r_i are rational numbers. The solutions of equations (18)–(20) can be found explicitly in [12]. Thus, system (3) of the integral equations is solvable. From equations (16), (20) there follows the statement of the theorem.

The theorem is proved.

Note that in terms of [13] the statement of the theorem means that all numbers of rotation of the given Hamiltonian system are rational.

References

[1] *V.I. Arnold*, Mathematical Methods of Classical Mechanics. Nauka, Moscow, 1989. (Russian)

[2] *A.M. Perelomov*, Integrable Systems of Classical Mechanics and Lie Algebras. Nauka, Moscow, 1990. (Russian)

[3] *A.T. Fomenko*, Symplectic Geometry. Methods and Applications. MGU, Moscow, 1988. (Russian)

[4] *N.N. Nekhoroshev*, Action-Angle Variables and Their Generalizations. — *Tr. Moskow Math. Soc.* **26** (1972), 181–198. (Russian)

[5] *B.S. Kruglikov*, Existence of a Pair of Additional Bott Integrals for a Resonance Hamiltonian System with Two Degrees of Freedom. — *Tr. Mat. Inst. Steklov.* **205** (1994), 109–112. (Russian)

[6] *G. Goldsteyn*, Classical Mechanics. Nauka, Moscow, 1975. (Russian)

[7] *A.I. Lurye*, Analitical Mechanics. GIFML, Moscow, 1961. (Russian)

[8] *A. Besse*, Manifolds with Closed Geodesics. Mir, Moscow, 1981. (Russian)

[9] *J. Darboux*, Leçons sur la Théorie Générale des Surfaces. Chelsea. **3**. (3-éme éd.), 1972.

[10] *V.N. Kolokoltsov*, Geodesic Flows on Two-Dimensional Manifolds with an Additional First Integral that is Polinomial in the Velocities. — *Izv. AN SSSR, Ser. Math.* **46** (1982), No. 5, 994–1010. (Russian)

[11] *V.N. Kolokoltsov,* New Examples of Manifolds with Closed Geodesics. — *Vestnik Moscow Univ. Ser. I: Math., Mech.* No. 4 (1984), 80–82.

[12] *S.G. Michlin*, Lectures on Linear Integral Equations. GIFML, Moscow, 1959. (Russian)

[13] *A.V. Bolsinov and A.T. Fomenko*, Integrable Hamiltonian Systems. **1**. Udmurt. Univ., Izhevsk, 1999. (Russian)