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On Singular Limit and Upper Semicontinuous Family of Attractors of Thermoviscoelastic Berger Plate

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A system of partial differential equations with integral terms which take into account hereditary effects is considered. The system describes a behaviour of thermoviscoelastic plate with Berger's type of nonlinearity. The hereditary effect is taken into account both in the temperature variable and in the bending one. The main goal of the paper is to analyze the passage to the singular limit when memory kernels collapse into the Dirac mass. In particular, it is proved that the solutions to the system with memory are close in some sense to the solutions to the corresponding memory-free limiting system. Besides, the upper semicontinuity of the family of attractors with respect to the singular limit is obtained.

Key words: materials with memory, attractors, upper semicontinuity.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Our main goal in this paper is to study asymptotic behaviour of the following system of integrodifferential equations arising in the plate theory:

$$\begin{cases} u_{tt} + (1 + h(0))\Delta^2 u + \int_0^\infty h'(s)\Delta^2 u(t-s)ds \\ \qquad\qquad + \left(\Gamma - \int_\Omega |\nabla u|^2 \, d\mathbf{x}\right) \Delta u + \Delta v = p(\mathbf{x}), \\ v_t - \int_0^\infty k(s)\Delta v(t-s)ds - \Delta u_t = 0, \\ \qquad\qquad \mathbf{x} = (x_1, x_2) \in \Omega \subset \mathbb{R}^2, t > 0 \end{cases} \tag{1.1}$$

with initial data

$$u(t, \mathbf{x})|_{t\le 0} = u_0(-t, \mathbf{x}), \;\; v(t, \mathbf{x})|_{t\le 0} = v_0(-t, \mathbf{x}).$$

Here we consider a thin plate of uniform thickness. When the plate is unloaded and in null equilibrium, its middle surface occupies a region $\Omega \subset \mathbb{R}^2$ of the plane $\{x_3 = 0\}$; $u(t, \mathbf{x})$ is a vertical component of displacement of the corresponding point in the middle surface. The presence of nonlocal term $\left(\Gamma - \int_\Omega |\nabla u|^2 \, d\mathbf{x}\right)$ is explained by peculiarities of derivation of equation due to Berger's approach (see [2]). In the first equation it is taken into account that the material is homogeneous, isotropic and viscous, so a convolution integral with scalar kernel $h(s)$ appears. The function $v(t, \mathbf{x})$ is a temperature variation field and thus it satisfies one of the variants of heat equation. Here we consider the heat equation according to Gurtin–Pipkin Law (see [16]), with the convolution integral with the scalar kernel $k(s)$ instead of usual Furier Law, which has two main shortcomings. First, it is unable to take into account the memory effects. Second, it predicts that a thermal disturbance at one point of the body is instantly felt everywhere in the body. Also, we note that for the sake of simplicity we put all the other physical constants equal to one.

The functions $h(s)$ and $k(s)$ are kernels which are the smooth, decreasing and convex functions defined on $[0, +\infty)$. The functions $h(s)$ and $k(s)$ vanish at infinity. In what follows we study the properties of the problem when $h(s)$ and $k(s)$ collapse to a Dirac mass (as in [1, 4, 23], see also [9, 14] for models with memory). Below in the paper this limiting procedure will be frequently called a singular limit. We consider any fixed small parameters $0 < \sigma, \varepsilon \leq 1$ and

$$h_\sigma(s) = \frac{1}{\sigma} h\left(\frac{s}{\sigma}\right), \quad k_\varepsilon(s) = \frac{1}{\varepsilon} k\left(\frac{s}{\varepsilon}\right), \quad s \in \mathbb{R}^+, \tag{1.2}$$

where $\mathbb{R}^+ = (0, +\infty)$. Then we consider the system

$$\begin{cases} u_{tt} + (h_\sigma(0) + 1)\,\Delta^2 u + \int_0^\infty h'_\sigma(s)\Delta^2 u(t-s)ds \\ \qquad\qquad + \left(\Gamma - \int_\Omega |\nabla u|^2 \, d\mathbf{x}\right)\Delta u + \Delta v = p(\mathbf{x}), \\ v_t - \int_0^\infty k_\varepsilon(s)\Delta v(t-s)ds - \Delta u_t = 0 \end{cases} \tag{1.3}$$

instead of system (1.1). If we formally pass to the limit $\sigma, \varepsilon \to 0+$, the system above collapses into the following one:

$$\begin{cases} u_{tt} + \Delta^2 u_t + \Delta^2 u + \left(\Gamma - \int_\Omega |\nabla u|^2 \, d\mathbf{x}\right)\Delta u + \Delta v = p(\mathbf{x}), \\ v_t - \Delta v - \Delta u_t = 0. \end{cases} \tag{1.4}$$

System (1.4) preserves a viscous dissipation effect expressed by the term $\Delta^2 u_t$ which replaces $h_\sigma(0)\Delta^2 u + \int_0^\infty h'_\sigma(s)\Delta^2 u(t-s)ds$ in the first equation of system (1.3). The limiting heat process is described by usual heat equation. In this case the corresponding integral term $\int_0^\infty k_\varepsilon(s)\Delta v(t-s)ds$ in (1.3) is replaced by Δv in (1.4).

The boundary conditions for (1.3) are

$$\begin{cases} u(t) = 0, \ \ \mathbf{x} \in \partial\Omega, \ t \geq 0, \\ \Delta \left[(h_\sigma(0) + 1)\, u(t) + \int_0^\infty h'_\sigma(s) u(t-s) ds \right] = 0, \ \ \mathbf{x} \in \partial\Omega, \ t \geq 0, \\ v(t) = 0, \ \ \mathbf{x} \in \partial\Omega, \ t \geq 0 \end{cases} \tag{1.5}$$

and for (1.4),

$$\begin{cases} u(t) = \Delta \left[u_t(t) + u(t) \right] = 0, \ \mathbf{x} \in \partial\Omega, \ t \geq 0, \\ v(t) = 0, \ \ \mathbf{x} \in \partial\Omega, \ t \geq 0. \end{cases} \tag{1.6}$$

The boundary conditions on function u are widely used as simplified hinged (or edge-free) boundary conditions (e.g., the similar conditions were considered in [4, 5, 14, 17, 19, 22]).

As in a plenty of previous works on systems with memory (e.g., see [9, 13, 14, 20] and references therein), by following [10], we will introduce new auxiliary variables which replace the convolution integrals in original equation by a functional operator applied to one of the added variables. It makes possible to apply the asymptotic theory of semigroups.

The linear versions of the model considered and the related ones were studied in [13, 14, 17]. In [13], the model was linear and the memory effects were taken into account in thermal variable only (i.e., the kernel $h(s)$ satisfied $h(0) = h(\infty) = 0$). The convergence of solutions to zero point was obtained there. The linear version of the problem considered in this paper was studied in [14]. Uniform exponential decay to zero point with respect to the parameters σ and ε, which played the same role as in this paper, and the singular limit result were obtained in [14]. In this paper, the model has a nonlinear term, so additional question about the upper semicontinuity of attractors with respect to the parameters σ and ε is answered here. The similar result on the upper semicontinuity of attractors for nonlinear thermoviscoelastic Mindlin–Timoshenko model was obtained in [12].

Isothermal Berger model of oscillations of a plate without memory effects with emphasis on its asymptotic behaviour was studied in [4, 6]. The thermoelastic model with nonlinearity of Berger type and different boundary conditions was considered in [3, 15].

This work is a continuation of [22] where the model was considered with the fixed parameters σ and ε. The existence of the compact global attractor, its finite dimensionality and boundedness with respect to topology stronger than topology of the phase space were obtained. The main technique for treating the model is the so-called stabilizability inequality (in our paper this inequality is formulated in Th. 5.3). Similar inequalities were also obtained in various problems on the dissipative wave dynamics and have become an important part of the studying of the existence, smoothness and finite dimensionality of attractors (see [5–8] and references therein). One should notice that these estimates are not the consequences of some common abstract results but essentially depend on the peculiarities of the model under consideration. In works [5–8] the authors proposed how to use stabilizability estimates to obtain finite dimensionality and smoothness of the attractor and how to construct exponential attractors and determining functionals. In this paper we will show that the coefficients of stabilizability inequality obtained for our problem are uniform with respect to the parameters σ and ε. It will help to prove the upper semicontinuity of attractors (Th. 6.2).

There are two main results obtained in this paper. The first one is the closeness between the corresponding solutions of the model with memory and the memory-free limiting model on finite time intervals (singular limit). The second result is the closeness between the attractors of the model with memory and the memory-free limiting one. To get the result on the attractors we use a uniform stabilizability estimate. Besides, we provide analysis (well-posedness and existence of the compact global attractor) for limiting problem (1.4). Up to our knowledge, problem (1.4) has not been considered before.

For simplicity, we do not consider the limits for $\sigma \to 0$ with $\varepsilon = 0$ ($\varepsilon \to 0$ with $\sigma = 0$) separately, so below in the paper both parameters are either strictly larger than 0 or equal to 0 simultaneously.

We conclude the introduction with a brief plan of the paper.

In Section 2 we rewrite the system in abstract form. Besides, the assumptions to be used in the sequel are given. Section 3 collects the results obtained in [22] on the considered semigroup. The question about the singular limit on finite time intervals is answered in Section 4. Section 5 includes the assertions of the existence of global attractor and of the uniform stabilizability estimate for both (1.3) and (1.4). Section 6 contains the theorem on the upper semicontinuity of the family of attractors when $(\sigma, \varepsilon) \to (0, 0)$. The proofs are relegated to Section 7. Except the proofs of stabilizability estimates, Section 7 also contains the proof of the smoothness of the family of attractors which is used in Theorem 6.2.

For reader's convenience we note that the main results are formulated in Theorems 4.1 and 6.2.

2. Main Settings

2.1. Kernels

The conditions on the kernels $h(s)$ and $k(s)$ imposed below are similar to those in [9, 13, 14].

First, we assume that $h, k : [0, +\infty) \to \mathbb{R}^+$ are smooth, decreasing and summable functions. For the sake of simplicity, we assume that $h(0) = k(0) = 1$, moreover,

$$\int_0^\infty h(s)ds = \int_0^\infty k(s)ds = 1.$$

We set $\beta(s) = -h'(s)$ and $\mu(s) = -k'(s)$, where β and μ are supposed to satisfy

$$\beta, \mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+) \cap C[0,\infty),\ \beta(s) \geq 0,\ \ \mu(s) \geq 0, \tag{2.1}$$

and there exists a constant $\delta > 0$ such that

$$\beta'(s) + \delta\beta(s) \leq 0 \ \text{ and } \ \mu'(s) + \delta\mu(s) \leq 0. \tag{2.2}$$

Consider any $\sigma \in (0,1]$ and $\varepsilon \in (0,1]$. We set

$$\beta_\sigma(s) = \frac{1}{\sigma^2}\beta\left(\frac{s}{\sigma}\right) = -h'_\sigma(s) \ \text{ and } \ \mu_\varepsilon(s) = \frac{1}{\varepsilon^2}\mu\left(\frac{s}{\varepsilon}\right) = -k'_\varepsilon(s), \tag{2.3}$$

where $h_\sigma(s)$ and $k_\varepsilon(s)$ are defined in (1.2).

We note that the properties of (2.1) for $\beta_\sigma(s)$ and $\mu_\varepsilon(s)$ are preserved. Assumptions (2.2) turn into

$$\beta'_\sigma(s) + \frac{\delta}{\sigma}\beta_\sigma(s) \leq 0 \ \text{ and } \ \mu'_\varepsilon(s) + \frac{\delta}{\varepsilon}\mu_\varepsilon(s) \leq 0. \tag{2.4}$$

In the sequel we will frequently use the following equalities:

$$\int_0^\infty \beta_\sigma(s)ds = \frac{1}{\sigma},\ \int_0^\infty \mu_\varepsilon(s)ds = \frac{1}{\varepsilon},\ \int_0^\infty s\beta_\sigma(s)ds = \int_0^\infty s\mu_\varepsilon(s)ds = 1.$$

2.2. Spaces

We denote by H a separable Hilbert space with inner product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$. Let A be a selfadjoint positive linear operator defined on a domain $D(A) \subset H$. Assume that there exists an eigenbasis $\{e_k\}_{k=1}^\infty$ of the operator A such that

$$(e_k, e_j) = \delta_{kj},\ \ Ae_k = \lambda_k e_k,\ \ k,j = 1,2,\ldots,$$

and $0 < \lambda_1 \leq \lambda_2 \leq \ldots,\ \lim_{k\to\infty} \lambda_k = \infty$, where λ_k is a corresponding eigenvalue of the operator A.

We introduce the scale of Hilbert spaces $H^s = D(A^{s/2})$ with $s \in \mathbb{R}$ endowed with usual inner products $(v,w)_{2s} = (A^s v, A^s w)$.

We introduce the weighted Hilbert spaces $L^2_{\beta_\sigma}(\mathbb{R}^+; H^2)$ and $L^2_{\mu_\varepsilon}(\mathbb{R}^+; H^1)$ of measurable functions ξ with values in H^2 or H^1, respectively, such that

$$\|\xi\|^2_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)} \equiv \int_0^{+\infty} \beta_\sigma(s)\,\|\xi(s)\|_2^2\, ds < \infty$$

and

$$\|\xi\|^2_{L^2_{\mu_\varepsilon}(\mathbb{R}^+;H^1)} \equiv \int_0^{+\infty} \mu_\varepsilon(s)\,\|\xi(s)\|_1^2\,ds < \infty.$$

The following Cartesian product of Hilbert spaces will play the role of a phase space for the considered model ($m = 0$):

$$\mathcal{H}^m_{\sigma,\varepsilon} = \begin{cases} H^{m+2} \times H^m \times H^m \times L^2_{\beta_\sigma}(\mathbb{R}^+; H^{m+2}) \times L^2_{\mu_\varepsilon}(\mathbb{R}^+; H^{m+1}), & \text{if } \sigma, \varepsilon > 0, \\ H^{m+2} \times H^m \times H^m, & \text{if } \sigma = \varepsilon = 0. \end{cases}$$

We note also that if the index is equal to null, we may omit it (e.g., use $\mathcal{H}$ instead of $\mathcal{H}^0_{0,0}$).

To perform a comparison between a five-component vector from $\mathcal{H}^m_{\sigma,\varepsilon}$ and a three-component vector from $\mathcal{H}^m$, we have to introduce the following lifting and projection maps (we preserve the notations introduced formerly in [14]):

$$\begin{array}{rclrcl} \mathbb{L}_{\sigma,\varepsilon} & : & \mathcal{H}^m \to \mathcal{H}^m_{\sigma,\varepsilon}, & \mathbb{Q}_\sigma & : & \mathcal{H}^m_{\sigma,\varepsilon} \to L^2_{\beta_\sigma}(\mathbb{R}^+; H^{m+2}), \\ \mathbb{P} & : & \mathcal{H}^m_{\sigma,\varepsilon} \to \mathcal{H}^m, & \mathbb{Q}_\varepsilon & : & \mathcal{H}^m_{\sigma,\varepsilon} \to L^2_{\mu_\varepsilon}(\mathbb{R}^+; H^{m+1}) \end{array}$$

defined by

$$\mathbb{L}_{\sigma,\varepsilon}(u, u_t, v) = \begin{cases} (u, u_t, v, 0, 0), & \text{if } \sigma, \varepsilon > 0, \\ (u, u_t, v), & \text{if } \sigma = \varepsilon = 0, \end{cases}$$

$\mathbb{P}(u, u_t, v, \xi, \eta) = (u, u_t, v)$, $\mathbb{Q}_\sigma(u, u_t, v, \xi, \eta) = \xi$ and $\mathbb{Q}_\varepsilon(u, u_t, v, \xi, \eta) = \eta$, respectively.

2.3. Memory Variables

Following the ideas of Dafermos (see [9, 10, 14, 20] and references therein), we introduce additional variables, namely, the summed past history of u and v defined as

$$\xi^t(s) = u(t) - u(t-s), \quad \eta^t(s) = \int_0^s v(t-y)dy.$$

They formally satisfy the linear equations

$$\tfrac{\partial}{\partial t}\xi^t + \tfrac{\partial}{\partial s}\xi^t = u_t(t), \quad \tfrac{\partial}{\partial t}\eta^t + \tfrac{\partial}{\partial s}\eta^t = v(t)$$

and $\xi^t(0) = \eta^t(0) = 0$, whereas

$$\xi^0(s) = \xi_0(s) \equiv u_0(0) - u_0(s), \quad \eta^0(s) = \eta_0(s) \equiv \int_0^s v_0(y)dy.$$

Let T_σ, T_ε be the linear operators in $L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)$ and $L^2_{\mu_\varepsilon}(\mathbb{R}^+;H^1)$, respectively, with the domains

$$D(T_\sigma)=\left\{\xi\in L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)\,\Big|\,\xi_s\in L^2_{\beta_\sigma}(\mathbb{R}^+;H^2),\ \xi(0)=0\right\},$$
$$D(T_\varepsilon)=\left\{\eta\in L^2_{\mu_\varepsilon}(\mathbb{R}^+;H^1)\,\big|\,\eta_s\in L^2_{\mu_\varepsilon}(\mathbb{R}^+;H^1),\ \eta(0)=0\right\},$$

defined by $T_\sigma\xi=-\xi_s$ and $T_\varepsilon\eta=-\eta_s$ for all admissible ξ and η. Here η_s denotes the distributional derivative with respect to the variable s.

These operators satisfy the following inequalities (see, e.g., [20]):

$$(T_\sigma\xi,\xi)_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)}\le-\frac{\delta}{2\sigma}\left\|\xi\right\|^2_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)},\quad\forall\xi\in D(T_\sigma),\tag{2.5}$$

$$(T_\varepsilon\eta,\eta)_{L^2_{\mu_\varepsilon}(\mathbb{R}^+;H^1)}\le-\frac{\delta}{2\varepsilon}\left\|\eta\right\|^2_{L^2_{\mu_\varepsilon}(\mathbb{R}^+;H^1)},\quad\forall\eta\in D(T_\varepsilon).\tag{2.6}$$

2.4. Requirements on Nonlinearity

We impose conditions on function $M(\cdot)$

$$\begin{cases}\mathcal{M}(z)\equiv\int\limits_0^z M(\xi)d\xi\ge-az-b,\quad a\in(0,\lambda_1),\ b\in\mathbb{R},\\ M(z)\in C^2(\mathbb{R}^+).\end{cases}\tag{2.7}$$

We note that $M(z)=z-\Gamma$ satisfies (2.7). This $M(z)$ corresponds to the standard Berger nonlinearity.

2.5. Abstract Form of the Problem

In view of introduced settings, when $\sigma,\varepsilon>0$, original system (1.3) may be rewritten as follows:

$$\begin{cases}u_{tt}+A^2u+\int\limits_0^{+\infty}\beta_\sigma(s)A^2\xi^t(s)ds-Av=p-M\left(\left\|A^{1/2}u\right\|^2\right)Au,\\ v_t+\int\limits_0^{+\infty}\mu_\varepsilon(s)A\eta^t(s)ds+Au_t=0,\\ \xi^t_t=T_\sigma\xi^t+u_t(t),\quad \eta^t_t=T_\varepsilon\eta^t+v(t),\\ u|_{t=0}=u_0,\ \ u_t|_{t=0}=u_1,\ \ v|_{t=0}=v_0,\ \ \xi^t|_{t=0}=\xi_0,\ \ \eta^t|_{t=0}=\eta_0.\end{cases}\tag{2.8}$$

Limiting memory-free system (1.4) has the form

$$\begin{cases}u_{tt}+A^2u+A^2u_t-Av=p-M\left(\left\|A^{1/2}u\right\|^2\right)Au,\\ v_t+Av+Au_t=0,\\ u|_{t=0}=u_0,\ \ u_t|_{t=0}=u_1,\ \ v|_{t=0}=v_0.\end{cases}\tag{2.9}$$

To formulate the existence and uniqueness results, we have to define various types of solutions following the theory of semigroups of linear operators (see [21]). For this, we write the linear part of equation (2.8) as the linear operator $\mathcal{L}^{\sigma,\varepsilon} : D(\mathcal{L}^{\sigma,\varepsilon}) \subset \mathcal{H}_{\sigma,\varepsilon} \to \mathcal{H}_{\sigma,\varepsilon}$ given by

$$\mathcal{L}^{\sigma,\varepsilon}U = \begin{pmatrix} w \\ -A^2u - \int\limits_0^\infty \beta_\sigma(s)A^2\xi(s)ds + Av \\ -\int\limits_0^\infty \mu_\varepsilon(s)A\eta(s)ds - Au_t \\ T_\sigma\xi + w \\ T_\varepsilon\eta + v \end{pmatrix}, \quad U = \begin{pmatrix} u \\ w \\ v \\ \xi \\ \eta \end{pmatrix} \in \mathcal{H}_{\sigma,\varepsilon}$$

and equipped with the domain

$$D(\mathcal{L}^{\sigma,\varepsilon}) = \left\{ U = \begin{pmatrix} u \\ w \\ v \\ \xi \\ \eta \end{pmatrix} \in \mathcal{H}_{\sigma,\varepsilon} \;\middle|\; \begin{array}{l} \xi \in D(T_\sigma),\ \eta \in D(T_\varepsilon), \\ w \in H^2, v \in H^1, \\ A^2u + \int\limits_0^\infty \beta_\sigma(s)A^2\xi(s)ds - Av \in H, \\ \int\limits_0^\infty \mu_\varepsilon(s)A\eta(s)ds \in H. \end{array} \right\}.$$

For problem (2.9), the linear part is represented by $\mathcal{L}^{0,0} : D(\mathcal{L}^{0,0}) \subset \mathcal{H} \to \mathcal{H}$ given as follows

$$\mathcal{L}^{0,0} = \begin{pmatrix} w \\ -A^2\left[w + u\right] + Av \\ -Av - Aw \end{pmatrix}, \quad U = \begin{pmatrix} u \\ w \\ v \end{pmatrix} \in \mathcal{H}$$

and equipped with the domain

$$D(\mathcal{L}^{0,0}) = \left\{ U = \begin{pmatrix} u \\ w \\ v \end{pmatrix} \in \mathcal{H} \;\middle|\; \begin{array}{l} u + w \in H^4, \\ w, v \in H^2 \end{array} \right\}.$$

Using the standard method, it is possible to prove that $\mathcal{L}^{\sigma,\varepsilon}$ is an infinitesimal operator of a strongly continuous linear semigroup. For the case $\sigma, \varepsilon > 0$ we refer to [13, 22]. To prove the same statement for the operator $\mathcal{L}^{0,0}$ it is sufficient to note that (i) the point $0 \in \mathbb{C}$ belongs to the resolvent set of the operator $\mathcal{L}^{0,0}$, and (ii) the operator $\mathcal{L}^{0,0}$ is dissipative, i.e., for all $U \in D(\mathcal{L}^{0,0}) : (\mathcal{L}^{0,0}U, U)_{\mathcal{H}} \leq 0$. Therefore, Theorem 1.4.3 in [21] (the Lumer–Phillips Theorem) is applicable, and $\mathcal{L}^{0,0}$ is also an infinitesimal operator of a strongly continuous linear semigroup.

After having made final notations for the nonlinear term, namely,

$$f(U) = (0, -M\left(\left\|A^{1/2}u\right\|^2\right)Au + p, 0, 0, 0)^{\mathrm{T}},$$

we may rewrite the nonlinear problem (2.8) (or (2.9)) as a first order problem of the form

$$\begin{cases} \dot{U}(t) = \mathcal{L}^{\sigma,\varepsilon}U(t) + f(U(t)) \\ U(0) = U_0 \in \mathcal{H}_{\sigma,\varepsilon}. \end{cases} \tag{2.10}$$

We recall that according to [21], $U(t)$ is a *mild solution* of (2.10) if $U(t)$ satisfies the following equality:

$$U(t) = e^{t\mathcal{L}^{\sigma,\varepsilon}}U_0 + \int_0^t e^{(t-\tau)\mathcal{L}^{\sigma,\varepsilon}} f(U(\tau))d\tau,$$

where $e^{t\mathcal{L}^{\sigma,\varepsilon}}$ is the linear semigroup on $\mathcal{H}_{\sigma,\varepsilon}$ whose infinitesimal operator is $\mathcal{L}^{\sigma,\varepsilon}$. We say that $U(t)$ is a *strong solution* on the interval $[0,T)$ if it is continuously differentiable, its values lie in $D(\mathcal{L}^{\sigma,\varepsilon})$ and it satisfies (2.10).

3. Nonlinear Semigroup

We collect here some results on the generation of nonlinear semigroup and some of its properties. All results (Th. 3.1, Props. 3.2, 3.4, 3.5) are shown in [22].

3.1. Well-Posedness

We recall that $S_{\sigma,\varepsilon}(t) : \mathcal{H}_{\sigma,\varepsilon} \to \mathcal{H}_{\sigma,\varepsilon}$ is called a continuous semigroup of operators if (i) $S_{\sigma,\varepsilon}(0) = I$, $S_{\sigma,\varepsilon}(t+\tau) = S_{\sigma,\varepsilon}(t)S_{\sigma,\varepsilon}(\tau)$, $t, \tau \geq 0$, (ii) the mapping $t \to S_{\sigma,\varepsilon}(t)U_0$ is continuous for any $U_0 \in \mathcal{H}_{\sigma,\varepsilon}$, and (iii) the mapping $U \to S_{\sigma,\varepsilon}(t)U$ is continuous for any $t \geq 0$. Also the couple $(\mathcal{H}_{\sigma,\varepsilon}, S_{\sigma,\varepsilon})$ is called a dynamical system on the phase space $\mathcal{H}_{\sigma,\varepsilon}$.

Theorem 3.1. *Let assumptions* (2.1), (2.4) *and* (2.7) *hold true and* $\sigma, \varepsilon > 0$ *or* $\sigma = \varepsilon = 0$. *Assume also that* $p \in H$. *Then for all* $U_0 \in \mathcal{H}_{\sigma,\varepsilon}$ *and* $T > 0$ *there exists a unique mild solution* $U(t) \in C(0,T;\mathcal{H}_{\sigma,\varepsilon})$. *Therefore, we may set* $S_{\sigma,\varepsilon}(t)U_0 = U(t)$ *as a nonlinear semigroup corresponding to problem* (2.8) $(\sigma, \varepsilon > 0)$ *or to problem* (2.9) $(\sigma = \varepsilon = 0)$.

Besides, if $U_1, U_2 \in \mathcal{H}_{\sigma,\varepsilon}$ *and* $\|U_i\|_{\mathcal{H}_{\sigma,\varepsilon}} \leq R$, *then there exists a positive constant* $C_{R,T}$ *such that*

$$\|S_{\sigma,\varepsilon}(t)U_1 - S_{\sigma,\varepsilon}(t)U_2\|_{\mathcal{H}_{\sigma,\varepsilon}} \leq C_{R,T}\,\|U_1 - U_2\|_{\mathcal{H}_{\sigma,\varepsilon}}, \quad t \in [0,T]. \tag{3.1}$$

Therefore, $S_{\sigma,\varepsilon}(t)$ *is a continuous semigroup of operators. And if* $U_0 \in D(\mathcal{L}^{\sigma,\varepsilon})$, *then the corresponding mild solution* $U(t)$ *is a strong solution.*

The proof is based on the abstract result from [21] on the perturbation of s.c. linear semigroup. For details we refer to [22, Th. 2.1].

Further analysis requires additional information about the considered semigroup.

Consider the functional

$$\Phi(U) = \frac{1}{2}\|U\|^2_{\mathcal{H}_{\sigma,\varepsilon}} + \frac{1}{2}\mathcal{M}\left(\left\|A^{1/2}u\right\|^2\right) - (p,u),$$

which we call the Lyapunov function after having substituted a mild solution $U(t)$ instead of U.

It is shown in [22, Subsect. 2.5] that the Lyapunov function possesses the properties that can be formulated as follows (if compared with [22], here we add the obvious uniformity with respect to σ and ε of constants C and α)

Proposition 3.2. *There holds (i) if $\|U\|_{\mathcal{H}_{\sigma,\varepsilon}} \le R$, then*

$$\alpha\|U\|^2_{\mathcal{H}_{\sigma,\varepsilon}} - C \le \Phi(U) \le \max_{s\in[0,R^2]} |\mathcal{M}(s)| + \|U\|^2_{\mathcal{H}_{\sigma,\varepsilon}} + C, \tag{3.2}$$

where α and C are strictly positive and do not depend on σ, ε and R, (ii) for each mild solution $U(t)$ of problem (2.8) *the function $\Phi(U(t))$ is nonincreasing with respect to t, and (iii) if $\Phi(S_{\sigma,\varepsilon}(t)U_0) = \Phi(U_0)$ for any $t > 0$, then $S_{\sigma,\varepsilon}(t)U_0 = U_0$ for any $t > 0$.*

The lemma below will be needed in Section 4. Besides, it asserts that spaces $\mathcal{H}^m_{\sigma,\varepsilon}$ are positively invariant (i.e., $S_{\sigma,\varepsilon}(t)\mathcal{H}^m_{\sigma,\varepsilon} \subseteq \mathcal{H}^m_{\sigma,\varepsilon}$).

Lemma 3.3. *Let $m \ge 0$ and $U_0 \in B_{\mathcal{H}^m_{\sigma,\varepsilon}}(R)$. Then there exist positive constants C_R, L_R which do not depend on the parameters σ and ε such that*

$$\|U(t)\|^2_{\mathcal{H}^m_{\sigma,\varepsilon}} \le C_R e^{L_R t},$$

where $U(t) = (u(t), u_t(t), v(t), \xi^t, \eta^t)$ is a solution to (2.8) *(or to* (2.9) *if $\sigma = 0$ and $\varepsilon = 0$) with $U(0) = U_0$.*

P r o o f. The case $m = 0$ is obtained, e.g., if we apply the properties of Φ ((i) and (ii) from Prop. 3.2). Namely, for all $U_0 \in B_{\mathcal{H}^m_{\sigma,\varepsilon}}(R)$ there exists C_R uniform with respect to σ and ε such that $\|S_{\sigma,\varepsilon}(t)U_0\|^2_{\mathcal{H}_{\sigma,\varepsilon}} \le C_R$.

Now we turn to $m > 0$. Here we need to introduce the projector

$$P_N \ : \ H \longrightarrow \mathcal{L}in\{e_1, \dots, e_N\},$$

where each e_k is an eigenvector of the operator A.

Multiplying the equations in (2.8), or (2.9)), by $A^m P_N u_t$ in H, $A^m P_N v$ in H, $P_N \xi^t$ in $L^2_{\beta_\sigma}(\mathbb{R}^+; H^{m+2})$, $P_N \eta^t$ in $L^2_{\mu_\varepsilon}(\mathbb{R}^+; H^{m+1})$, we obtain (for similar calculations we refer to [4, Ch. 4])

$$\frac{1}{2}\frac{d}{dt}\|P_N U(t)\|^2_{\mathcal{H}^m_{\varepsilon,\sigma}} \le \left(M\left(\left\|A^{1/2}u\right\|^2\right)Au - p, P_N u_t\right)_m \le C^*_R(1+\|P_N U(t)\|^2_{\mathcal{H}^m_{\varepsilon,\sigma}}).$$

We used above the desired estimate for $m = 0$, and $P_N U(t)$ denoted the vector $(P_N u(t), P_N u_t(t), P_N v(t), P_N \xi^t, P_N \eta^t)$. The Gronwall Lemma implies

$$\|P_N U(t)\|^2_{\mathcal{H}^m_{\varepsilon,\sigma}} \le C^{**}_R (1 + \|P_N U(0)\|^2_{\mathcal{H}^m_{\varepsilon,\sigma}}) e^{L_R t}.$$

Now we may extend this estimate to the case of any $m \ge 0$ by passing to the limit with respect to N what is possible due to the fact that initial data lies in $\mathcal{H}^m_{\sigma,\varepsilon}$. ■

3.2. Set of Stationary Points

We define

$$\mathcal{N} = \{U_0 \in \mathcal{H}_{\sigma,\varepsilon} \,|S_{\sigma,\varepsilon}(t)U_0 = U_0 \text{ for all } t > 0\},$$

which is called the set of stationary points to problem (2.8). It turns out that this set depends neither on σ nor on ε, i.e., the following proposition holds.

Proposition 3.4. *The set $\mathcal{N}$ can be represented as follows:*

$$\mathcal{N} = \left\{ U = (u; 0; 0; 0; 0) \ : \ A^2 u + M\left(\left\|A^{1/2} u\right\|^2\right) Au = p \right\},$$

and there exists such $R > 0$ that $\|u\|_2 \le R$, $\forall u \in H^2 : (u; 0; 0; 0; 0) \in \mathcal{N}$ and R apparently depends neither on σ nor on ε.

3.3. Explicit Representation Formulas

In the sequel we need the typical for equations with infinite memory explicit representation formulas (similar to those considered in [9, 13, 14, 22]).

Proposition 3.5. *Let $U(t) = (u(t); w(t); v(t); \xi^t; \eta^t)$ be a mild solution of (2.10) with initial data $U_0 = (u_0; w_0; v_0; \xi_0; \eta_0)$. Then the following representations hold true:*

$$\xi^t(s) = \begin{cases} u(t) - u(t-s), & t > s > 0, \\ \xi_0(s-t) + u(t) - u(0), & t \le s, \end{cases} \tag{3.3}$$

and

$$\eta^t(s) = \begin{cases} \int_0^s v(t-y)dy, & t > s > 0, \\ \eta_0(s-t) + \int_0^t v(t-y)dy, & t \le s. \end{cases} \tag{3.4}$$

4. A Singular Limit on Finite Time Intervals

In this section we prove our first main result — the closeness between the solutions of (2.8) and (2.9) as ε and σ tend to zero on finite time intervals and with sufficiently smooth initial data. In the proof we rely on the ideas applied in [9, 14]. We note here that the model in [14] is linear and exponential stable, so the analysis there is simpler and allows to state the singular limit result on the infinite time interval. The work [9] deals only with the thermal convolution integral collapsing into $-\Delta v$ and with another type of nonlinearity. Traditionally, C will denote a generic positive constant. In further proof C is allowed to depend on R and T, positive parameters that will be introduced in theorem below.

Theorem 4.1. *Let $R > 0$ and $T > 0$. Assume also that $U_0 \in B_{\mathcal{H}^2_{\sigma,\varepsilon}}(R)$. Then for all $t \in [0,T]$ there hold*

$$\|\mathbb{P}S_{\sigma,\varepsilon}(t)U_0 - S_{0,0}(t)\mathbb{P}U_0\|_{\mathcal{H}} \le C(\sqrt[8]{\sigma} + \sqrt[8]{\varepsilon}), \tag{4.1}$$

$$\|\mathbb{Q}_\sigma S_{\sigma,\varepsilon}(t)U_0\|_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)} + \|\mathbb{Q}_\varepsilon S_{\sigma,\varepsilon}(t)U_0\|_{L^2_{\mu_\varepsilon}(\mathbb{R}^+;H^1)}$$
$$\le e^{-\frac{\delta t}{4\sigma}}\|\xi_0\|_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)} + e^{-\frac{\delta t}{4\varepsilon}}\|\eta_0\|_{L^2_{\mu_\varepsilon}(\mathbb{R}^+;H^1)} + C(\sqrt{\sigma} + \sqrt{\varepsilon}). \tag{4.2}$$

R e m a r k. If we collect both above estimates, we may formulate the statement of the theorem as follows:
For all $R > 0$ and $T > \tau > 0$ there holds

$$\lim_{\substack{\varepsilon \to 0+ \\ \sigma \to 0+}} \sup_{U_0 \in B_{\mathcal{H}^2_{\sigma,\varepsilon}}(R)} \sup_{t \in [\tau,T]} \|S_{\sigma,\varepsilon}(t)U_0 - \mathbb{L}_{\sigma,\varepsilon}S_{0,0}(t)\mathbb{P}U_0\|_{\mathcal{H}_{\varepsilon,\sigma}} = 0. \tag{4.3}$$

The necessity of introducing $\tau > 0$ in (4.3) is caused by the presence of the terms with exponents of the form $e^{-\frac{\delta t}{4\sigma}}$ multiplied by the norm of initial memory in (4.2).

P r o o f. Let us assume that $(\hat{u}(t), \hat{u}_t(t), \hat{v}(t), \hat{\xi}^t, \hat{\eta}^t)$ is a solution to (2.8) with initial data $U_0 = (u_0, u_1, v_0, \xi_0, \eta_0)$, and first three components of the time-dependent vector $(u(t), u_t(t), v(t), \xi(t); \eta(t))$ stand for the solution to (2.9) with initial data (u_0, u_1, v_0). Two other components satisfy the following two Cauchy problems in $L^2_{\beta_\sigma}(\mathbb{R}^+; H^2)$ and $L^2_{\mu_\varepsilon}(\mathbb{R}^+; H^1)$:

$$\begin{cases} \xi^t_t = T_\sigma \xi^t + u_t(t), & t > 0, \\ \xi^0 = \xi_0, \end{cases} \qquad \begin{cases} \eta^t_t = T_\sigma \eta^t + v(t), & t > 0, \\ \eta^0 = \eta_0. \end{cases}$$

Besides, we introduce the functions $\overline{u}(t) = \hat{u}(t) - u(t)$, $\overline{u}_t(t) = \hat{u}_t(t) - u_t(t)$, $\overline{v}(t) = \hat{v}(t) - v(t)$, $\overline{\xi}^t = \hat{\xi}^t - \xi^t$ and $\overline{\eta}^t = \hat{\eta}^t - \eta^t$, which satisfy the system

$$\begin{cases} \overline{u}_{tt} + A^2\overline{u} + \int\limits_0^\infty \beta_\sigma(s)A^2\hat{\xi}^t(s)ds - A^2u_t - A\overline{v} \\ = M\left(\left\|A^{1/2}u\right\|^2\right)Au - M\left(\left\|A^{1/2}\hat{u}\right\|^2\right)A\hat{u}, \\ \overline{v}_t + \int\limits_0^\infty \mu_\varepsilon(s)A\hat{\eta}^t(s)ds - Av + Au_t = 0, \\ \overline{\xi}^t_t = T_\sigma\overline{\xi}^t + \overline{u}_t(t), \quad \overline{\eta}^t_t = T_\varepsilon\overline{\eta} + \overline{v}(t) \end{cases} \tag{4.4}$$

with the null initial data.

For further analysis one more lemma is required.

Lemma 4.2. *Assume that the conditions of Theorem 4.1 hold. Then for any $t \in [0,T]$ there holds*

$$\max\left\{\left\|\hat{\xi}^t\right\|^2_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)}, \left\|\xi^t\right\|^2_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)}\right\} \le \|\xi_0\|^2_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)} e^{-\frac{\delta t}{2\sigma}} + C\sigma, \tag{4.5}$$

and

$$\max\left\{\left\|\hat{\eta}^t\right\|^2_{L^2_{\mu_\varepsilon}(\mathbb{R}^+;H^1)}, \left\|\eta^t\right\|^2_{L^2_{\mu_\varepsilon}(\mathbb{R}^+;H^1)}\right\} \le \|\eta_0\|^2_{L^2_{\mu_\varepsilon}(\mathbb{R}^+;H^1)} e^{-\frac{\delta t}{2\varepsilon}} + C\varepsilon. \tag{4.6}$$

P r o o f. The result can be obtained by multiplying the equation of ξ (or η) by ξ (or η) in $L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)$ (or $L^2_{\mu_\varepsilon}(\mathbb{R}^+;H^1)$). Taking (2.5), (2.6) and Lemma 3.3, we have

$$\begin{aligned} \tfrac{d}{dt}\|\xi\|^2_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)} + \tfrac{\delta}{\sigma}\|\xi\|^2_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)} & \\ \le C\int\limits_0^\infty \beta_\sigma(s)\|A\xi(s)\|\,ds & \\ \le C\left(\int\limits_0^\infty \beta_\sigma(s)ds\right)^{1/2}\left(\int\limits_0^\infty \beta_\sigma(s)\|A\xi(s)\|^2 ds\right)^{1/2} & \\ \le \tfrac{C}{\sqrt{\sigma}}\|\xi\|_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)} & \\ \le \tfrac{\delta}{2\sigma}\|\xi\|^2_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)} + C. & \end{aligned}$$

And the Gronwall Lemma leads to inequality (4.5). Inequality (4.6) is obtained in the same manner. ∎

We proceed the proof of Theorem 4.1 by considering system (4.4). Multiplying the first equation by $\overline{u}_t$ in H, the second, by $\overline{v}$ in H, the third, by $\overline{\xi}$ in

$L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)$, the fourth, by $\overline{\eta}$ in $L^2_{\mu_\varepsilon}(\mathbb{R}^+;H^1)$, and summerizing the four obtained equalities we get

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\left(\|A\overline{u}\|^2+\|\overline{u}_t\|^2+\|\overline{v}\|^2+\left\|\overline{\xi}\right\|^2_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)}+\|\overline{\eta}\|^2_{L^2_{\mu_\varepsilon}(\mathbb{R}^+;H^1)}\right)\\&\le J_\sigma(t)+I_\varepsilon(t)+(M\left(\left\|A^{1/2}u\right\|^2\right)Au-M\left(\left\|A^{1/2}\hat{u}\right\|^2\right)A\hat{u},\overline{u}_t),\end{aligned}$$

where

$$\begin{aligned}J_\sigma(t)&=-\int\limits_0^\infty\beta_\sigma(s)(A\xi^t(s),A\overline{u}_t(t))ds+(Au_t(t),A\overline{u}_t),\\I_\varepsilon(t)&=-\int\limits_0^\infty\mu_\varepsilon(s)(A^{1/2}\eta^t(s),A^{1/2}\overline{v}(t))ds+(A^{1/2}v(t),A^{1/2}\overline{v}(t)).\end{aligned}$$

We write each of introduced terms, J_σ and I_ε, as follows:

$$J_\sigma=\sum_{j=1}^5 J_j(t),\quad I_\varepsilon=\sum_{j=1}^5 I_j(t),$$

where the terms J_j are defined by

$$\begin{aligned}J_1(t)&=\textstyle\int_{\sqrt{\sigma}}^\infty s\beta_\sigma(s)(Au_t(t),A\overline{u}_t(t))ds,\\J_2(t)&=-\textstyle\int_{\sqrt{\sigma}}^\infty\beta_\sigma(s)(A\xi^t(s),A\overline{u}_t(t))ds,\\J_3(t)&=-\textstyle\int_{\min\{\sqrt{\sigma},t\}}^{\sqrt{\sigma}}\beta_\sigma(s)(A\xi_0(s-t),A\overline{u}_t(t))ds,\\J_4(t)&=\textstyle\int_{\min\{\sqrt{\sigma},t\}}^{\sqrt{\sigma}}(s-t)\beta_\sigma(s)(Au_t,A\overline{u}_t(t))ds,\\J_5(t)&=\textstyle\int_0^{\sqrt{\sigma}}\beta_\sigma(s)\left[\int_0^{\min\{s,t\}}(A(u_t(t)-u_t(t-y)),A\overline{u}_t(t))dy\right]ds,\end{aligned}$$

and the terms I_i are defined by

$$\begin{aligned}I_1(t)&=\textstyle\int_{\sqrt{\varepsilon}}^\infty s\mu_\varepsilon(s)(A^{1/2}v(t),A^{1/2}\overline{v}(t))ds,\\I_2(t)&=-\textstyle\int_{\sqrt{\varepsilon}}^\infty\mu_\varepsilon(s)(A^{1/2}\eta^t(s),A^{1/2}\overline{v}(t))ds,\\I_3(t)&=-\textstyle\int_{\min\{\sqrt{\varepsilon},t\}}^{\sqrt{\varepsilon}}\mu_\varepsilon(s)(A^{1/2}\eta_0(s-t),A^{1/2}\overline{v}(t))ds,\\I_4(t)&=\textstyle\int_{\min\{\sqrt{\varepsilon},t\}}^{\sqrt{\varepsilon}}(s-t)\mu_\varepsilon(s)(A^{1/2}v,A^{1/2}\overline{v}(t))ds,\\I_5(t)&=\textstyle\int_0^{\sqrt{\varepsilon}}\mu_\varepsilon(s)\left[\int_0^{\min\{s,t\}}(A^{1/2}(v(t)-v(t-y)),A^{1/2}\overline{v}(t))dy\right]ds.\end{aligned}$$

We also note that the inequalities below hold

$$\int\limits_{\sqrt{\sigma}}^\infty s\beta_\sigma(s)ds\le C\sigma,\quad\int\limits_{\sqrt{\sigma}}^\infty\beta_\sigma(s)ds\le C\sqrt{\sigma}.\tag{4.7}$$

The first relation (for $\sigma \in (0,1]$) can be obtained as follows:

$$\int_{\sqrt{\sigma}}^{\infty} s\beta_\sigma(s)ds = \int_{1/\sqrt{\sigma}}^{\infty} s\beta(s)ds \le \frac{e^{\delta}\beta(1)}{\delta^2}\left(1+\frac{\delta}{\sqrt{\sigma}}\right)e^{-\frac{\delta}{\sqrt{\sigma}}} \le C_\alpha \sigma^\alpha, \ \ \forall \alpha \in \mathbb{R},$$

where (2.4) is used. The second one holds due to

$$\int_{\sqrt{\sigma}}^{\infty} \beta_\sigma(s)ds \le \frac{1}{\sqrt{\sigma}}\int_{\sqrt{\sigma}}^{\infty} s\beta_\sigma(s)ds \le C_\alpha \sigma^\alpha, \ \ \forall \alpha \in \mathbb{R}.$$

It is apparent that the same inequalities take place for μ_ε

$$\int_{\sqrt{\varepsilon}}^{\infty} s\mu_\varepsilon(s)ds \le C\varepsilon, \ \ \int_{\sqrt{\varepsilon}}^{\infty} \mu_\varepsilon(s)ds \le C\sqrt{\varepsilon}. \tag{4.8}$$

Now we will estimate J_j and I_j by using

$$\left\|A^2\hat{u}(t)\right\|^2 + \|A\hat{u}_t(t)\|^2 + \left\|A^{1/2}\hat{v}(t)\right\|^2 \le C, \tag{4.9}$$

$$\left\|A^2 u(t)\right\|^2 + \int_0^t \left\|A^2 u_t(\tau)\right\|^2 d\tau + \int_0^t \|Av(\tau)\|^2 d\tau$$
$$+ \|Au_t(t)\|^2 + \left\|A^{1/2}v(t)\right\|^2 \le C, \tag{4.10}$$

which follow from Lemma 3.3 and the condition that the initial data U_0 is taken from a closed ball in $\mathcal{H}^4_{\sigma,\varepsilon}$ of radius R.

1. Using (4.9), (4.10) and the properties of the kernels (4.7) and (4.8), we immediately have

$$J_1(t) = \textstyle\int_{\sqrt{\sigma}}^{\infty} s\beta_\sigma(s)(Au_t(t), A\overline{u}_t(t))ds \le C\sigma,$$
$$I_1(t) = \textstyle\int_{\sqrt{\varepsilon}}^{\infty} s\mu_\varepsilon(s)(A^{1/2}v(t), A^{1/2}\overline{v}(t))ds \le C\varepsilon.$$

2. To estimate $J_2(t)$ (and $I_2(t)$) we apply Lemma 4.2

$$J_2(t) = -\int_{\sqrt{\sigma}}^{\infty} \beta_\sigma(s)(A\xi^t(s), A\overline{u}_t(t))ds \le C\int_{\sqrt{\sigma}}^{\infty} \beta_\sigma(s)\left\|A\xi^t(s)\right\| ds$$
$$\le C\int_{\sqrt{\sigma}}^{\infty} \beta_\sigma(s)\left\|A\xi^t(s)\right\|^2 ds + C\int_{\sqrt{\sigma}}^{\infty} \beta_\sigma(s)ds.$$

And application of Lemma 4.2 finally gives $J_2(t) \le Ce^{-\frac{\delta t}{2\sigma}} + C\sigma$. The inequality $I_2(t) \le Ce^{-\frac{\delta t}{2\varepsilon}} + C\varepsilon$ can be obtained in a similar way.

3. If $t \geq \sqrt{\sigma}$, then $J_3(t)$ vanishes. If we consider the case $t < \sqrt{\sigma}$, then

$$J_3(t) = -\int_t^{\sqrt{\sigma}} \beta_\sigma(s)(A\xi_0(s-t), A\overline{u}_t(t))ds \leq C \int_0^{\sqrt{\sigma}-t} \beta_\sigma(y+t) \|A\xi_0(y)\| \, dy$$
$$\leq Ce^{-\frac{\delta t}{\sigma}} \int_0^\infty \beta_\sigma(s) \|A\xi_0(s)\| \, ds \leq Ce^{-\frac{\delta t}{\sigma}} \left(\int_0^\infty \beta_\sigma(s) ds \right)^{1/2} = \frac{C}{\sqrt{\sigma}} e^{-\frac{\delta t}{\sigma}},$$

and, similarly, $I_3(t) \leq \frac{C}{\sqrt{\varepsilon}} e^{-\frac{\delta t}{\varepsilon}}$.

4. $J_4(t)$ is treated in the same way as $J_3(t)$, namely,

$$J_4(t) = \int_t^{\sqrt{\sigma}} (s-t)\beta_\sigma(s)(Au_t(t), A\overline{u}_t(t))ds \leq C \int_0^{\sqrt{\sigma}-t} s\beta_\sigma(s+t)ds \leq Ce^{-\frac{\delta t}{\sigma}},$$

and, similarly, $I_4(t) \leq Ce^{-\frac{\delta t}{\sigma}}$.

5. Before estimating $J_5(t)$, we note that the first equation in (2.9) gives

$$\int_{t-y}^{t} \|u_{tt}(\varsigma)\|^2 \, d\varsigma \leq 5 \left[\int_{t-y}^{t} \left\|A^2 u_t(\varsigma)\right\|^2 d\varsigma + \int_{t-y}^{t} \left\|A^2 u(\varsigma)\right\|^2 d\varsigma \right.$$
$$\left. + \int_{t-y}^{t} (M\left(\left\|A^{1/2}u\right\|^2\right) \|Au\|)^2 d\varsigma + \int_{t-y}^{t} \|Av(\varsigma)\|^2 \, d\varsigma + \int_{t-y}^{t} \|p\|^2 \, d\varsigma \right] \leq C,$$

which implies

$$\|u_t(t) - u_t(t-y)\| \leq \int_{t-y}^{t} \|u_{tt}(\varsigma)\| \, d\varsigma \leq \sqrt{y} \left(\int_{t-y}^{t} \|u_{tt}(\varsigma)\|^2 \, d\varsigma \right)^{1/2} \leq C\sqrt{y}.$$

Moreover,

$$\|Au_t(t) - Au_t(t-y)\| \leq \|u_t(t) - u_t(t-y)\|^{1/2} \left\|A^2 u_t(t) - A^2 u_t(t-y)\right\|^{1/2}$$
$$\leq \|u_t(t) - u_t(t-y)\|^{1/2} \left(\left\|A^2 u_t(t)\right\|^{1/2} + \left\|A^2 u_t(t-y)\right\|^{1/2} \right)$$
$$\leq C \sqrt[4]{y} \left(\left\|A^2 u_t(t)\right\|^{1/2} + \left\|A^2 u_t(t-y)\right\|^{1/2} \right).$$

Hence,

$$J_5(t) = \int_0^{\sqrt{\sigma}} \beta_\sigma(s) \left[\int_0^{\min\{s,t\}} (Au_t(t) - Au_t(t-y), A\overline{u}_t(t))dy \right] ds$$
$$\leq C \int_0^{\sqrt{\sigma}} \beta_\sigma(s) \left[\int_0^{\min\{s,t\}} \sqrt[4]{y} \left(\left\|A^2 u_t(t)\right\|^{1/2} + \left\|A^2 u_t(t-y)\right\|^{1/2} \right) dy \right] ds$$
$$\leq C \sqrt[8]{\sigma} \int_0^{\sqrt{\sigma}} \beta_\sigma(s) \left[\int_0^{\min\{s,t\}} \left(\left\|A^2 u_t(t)\right\|^{1/2} + \left\|A^2 u_t(t-y)\right\|^{1/2} \right) dy \right] ds \equiv C \sqrt[8]{\sigma} g_{2,\sigma}(t),$$

and, similarly, $I_5(t) \leq C\sqrt[8]{\varepsilon} g_{2,\varepsilon}$ with $g_{2,\varepsilon}$ defined by

$$g_{2,\varepsilon}(t) = \int_0^{\sqrt{\varepsilon}} \mu_\varepsilon(s) \left[\int_0^{\min\{s,t\}} \left(\|Av(t)\|^{1/2} + \|Av(t-y)\|^{1/2} \right) dy \right] ds.$$

Functions $g_{2,\sigma}$ and $g_{2,\varepsilon}$ possess the property

$$\int_0^T [g_{2,\sigma}(t) + g_{2,\varepsilon}(t)]\, dt \leq C. \tag{4.11}$$

Collecting all above estimates, including

$$(M\left(\left\|A^{1/2}u\right\|^2\right) Au - M\left(\left\|A^{1/2}\hat{u}\right\|^2\right) A\hat{u}, \overline{u}_t) \leq C\left(\|A\overline{u}\|^2 + \|\overline{u}_t\|^2\right),$$

we finally obtain

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\left(\|A\overline{u}\|^2 + \|\overline{u}_t\|^2 + \|\overline{v}\|^2 + \left\|\overline{\xi}\right\|^2_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)} + \|\overline{\eta}\|^2_{L^2_{\mu_\varepsilon}(\mathbb{R}^+;H^1)} \right) \\ &\leq g_{1,\sigma} + g_{1,\varepsilon} + C\sqrt[8]{\sigma} g_{2,\sigma}(t) + C\sqrt[8]{\varepsilon} g_{2,\varepsilon}(t) + C\left(\|A\overline{u}\|^2 + \|\overline{u}_t\|^2\right), \end{aligned}$$

where $g_{i,\varsigma} = C\left(\sqrt{\varsigma} + \frac{1}{\sqrt{\varsigma}} e^{-\frac{\delta t}{2\varsigma}}\right)$, $i = 1, 2$, for which the following inequality holds:

$$\int_0^t g_{i,\varsigma}(y) dy \leq C\sqrt{\varsigma}, \quad t \in [0, T].$$

Applying the Gronwall Lemma and taking into account (4.11), we get the result

$$\tfrac{1}{2}\left(\|A\overline{u}\|^2 + \|\overline{u}_t\|^2 + \|\overline{v}\|^2 + \left\|\overline{\xi}\right\|^2_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)} + \|\overline{\eta}\|^2_{L^2_{\mu_\varepsilon}(\mathbb{R}^+;H^1)} \right) \leq C(\sqrt[8]{\sigma} + \sqrt[8]{\varepsilon}).$$

■

5. Attractors and Their Properties

First, we recall some definitions.

Definition 5.1. *$\mathcal{A}^{\sigma,\varepsilon} \subset \mathcal{H}_{\sigma,\varepsilon}$ is called a global attractor if (i) $\mathcal{A}^{\sigma,\varepsilon}$ is a closed bounded strictly invariant set ($S_{\sigma,\varepsilon}(t)\mathcal{A}^{\sigma,\varepsilon} = \mathcal{A}^{\sigma,\varepsilon}\ \forall t \geq 0$) and (ii) $\mathcal{A}^{\sigma,\varepsilon}$ possesses the uniform attraction property, i.e., for any bounded set $B \subset \mathcal{H}_{\sigma,\varepsilon}$ the following equality holds true:*

$$\lim_{t\to+\infty} \sup_{U\in B} \operatorname{dist}_{\mathcal{H}_{\sigma,\varepsilon}} (S_{\sigma,\varepsilon}(t)U, \mathcal{A}^{\sigma,\varepsilon}) = 0.$$

We also define (for details we refer to [4, 25]) the unstable manifold $\mathcal{M}^u(\mathcal{N})$ emanating from $\mathcal{N}$ as a set of all $U \in \mathcal{H}_{\sigma,\varepsilon}$ such that there exists a full trajectory $\gamma = \{U(t) \,:\, t \in \mathbb{R}\}$ with the properties

$$U(0) = U \quad \text{and} \quad \lim_{t\to-\infty} \operatorname{dist}_{\mathcal{H}_{\sigma,\varepsilon}}(U(t), \mathcal{N}) = 0.$$

5.1. Case $\sigma, \varepsilon > 0$

The result [22, Th. 3.9] obtained on the attractors of the case $\sigma, \varepsilon > 0$ and some of their properties is stated as follows

Theorem 5.2. *Assume that all conditions of Theorem 3.1 hold. Then for any positive values of parameters σ and ε the dynamical system $(\mathcal{H}_{\sigma,\varepsilon}, S_{\sigma,\varepsilon}(t))$ possesses a compact global attractor $\mathcal{A}^{\sigma,\varepsilon}$ of finite fractal dimension.*

Moreover, the attractor $\mathcal{A}^{\sigma,\varepsilon}$ consists of full trajectories belonging to the domain $D(\mathcal{L}^{\sigma,\varepsilon})$ and, moreover, $\mathcal{A}^{\sigma,\varepsilon} = \mathcal{M}^u(\mathcal{N})$, where $\mathcal{N}$ denotes the set of stationary points.

To provide the standard procedure of the proof of upper semicontinuity of attractors in Section 6 (see [1, 23]) we need the two theorems below. We refer to [22, Th. 3.11 and estimate (4.2)], where these theorems are stated and proved with fixed $\sigma > 0$ and $\varepsilon > 0$. Here we need a uniform (with respect to σ and ε) variant of these theorems. Section 7 is devoted to the verification that all constants appeared in Theorems 5.3 and 5.4 are uniform with respect to σ and ε.

The first auxiliary result is the stabilizability estimate. For the similar estimates and their application to asymptotic analysis we refer to [3, 5–8].

Theorem 5.3. *Assume $\sigma, \varepsilon > 0$. Let $(u^1(t); v^1(t); \xi^{1,t}; \eta^{1,t})$ and $(u^2(t); v^2(t); \xi^{2,t}; \eta^{2,t})$ be two solutions of problem (2.10) with initial data $U^i = (u_0^i; u_1^i; v_0^i; \xi_0^i; \eta_0^i)$, $i = 1, 2$. Assume that*

$$\left\|Au^i(t)\right\|^2 + \left\|u_t^i(t)\right\|^2 + \left\|v^i(t)\right\|^2 + \left\|\xi^{i,t}\right\|^2_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)} + \left\|\eta^{i,t}\right\|^2_{L^2_{\mu_\varepsilon}(\mathbb{R}^+;H^1)} \le R^2$$

for all $t \ge 0$. Let

$$Z(t) = \left(u^1(t) - u^2(t); u_t^1(t) - u_t^2(t); v^1(t) - v^2(t); \xi^{1,t} - \xi^{2,t}; \eta^{1,t} - \eta^{2,t}\right)$$

and $z(t) = u^1(t) - u^2(t)$.

Then there exist positive constants C_R and γ depending neither on σ nor on ε such that

$$\|Z(t)\|^2_{\mathcal{H}_{\sigma,\varepsilon}} \le C_R \|Z(0)\|^2_{\mathcal{H}_{\sigma,\varepsilon}} e^{-\gamma t} + C_R \sup_{0\le\tau\le t} \|z(\tau)\|^2 . \tag{5.1}$$

We note that this theorem also implies that the fractal dimension of $\mathcal{A}^{\sigma,\varepsilon}$ can be estimated by constant uniform with respect to σ and ε (we refer to the similar uniform results of [3, 5, 6]).

The second auxiliary result is the theorem stating that the attractor is a bounded subset of some "smoother" space.

Theorem 5.4. *There exists a positive constant C not depending on t, σ and ε such that for any trajectory $U(t) = (u(t); u_t(t); v(t); \xi^t; \eta^t)$ lying in the attractor we have*

$$\|u_{tt}(t)\|^2 + \|Au_t(t)\|^2 + \|v_t(t)\|^2 + \left\|\xi_t^t\right\|^2_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)} + \left\|\eta_t^t\right\|^2_{L^2_{\mu_\varepsilon}(\mathbb{R}^+;H^1)}$$
$$+ \left\|A^{3/2}u(t)\right\|^2 + \left\|A^{1/2}v(t)\right\|^2 + \left\|T_\sigma \xi^t\right\|^2_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)} + \left\|T_\varepsilon \eta^t\right\|^2_{L^2_{\mu_\varepsilon}(\mathbb{R}^+;H^1)} \leq C^2 \quad (5.2)$$

for all $t \in \mathbb{R}$.

5.2. Case $\sigma = \varepsilon = 0$

Up to our knowledge, the following theorem is new. Its statement follows from stabilizability estimate (5.3) in the same way as in [3, 5–8] for the gradient dynamical systems possessing stabilizability estimates.

Theorem 5.5. *Assume that condition* (2.7) *holds and $p \in H$. Then the dynamical system $(\mathcal{H}, S_{0,0}(t))$ possesses a compact global attractor $\mathcal{A}^{0,0}$ of finite fractal dimension.*

And the result on the stabilizability estimate to problem (2.9) is formulated as follows

Theorem 5.6. *Let $(u^1(t); v^1(t))$ and $(u^2(t); v^2(t))$ be two solutions of problem* (2.9) *with initial data $U^i = (u_0^i; u_1^i; v_0^i)$, $i = 1, 2$. Assume that $\left\|Au^i(t)\right\|^2 + \left\|u_t^i(t)\right\|^2 + \left\|v^i(t)\right\|^2 \leq R^2$ for all $t \geq 0$. Let*

$$Z(t) = \left(u^1(t) - u^2(t); u_t^1(t) - u_t^2(t); v^1(t) - v^2(t)\right)$$

and $z(t) = u^1(t) - u^2(t)$.

Then there exist positive constants C_R and γ such that

$$\|Z(t)\|^2_{\mathcal{H}} \leq C_R \|Z(0)\|^2_{\mathcal{H}} e^{-\gamma t} + C_R \sup_{0 \leq \tau \leq t} \|z(\tau)\|^2. \quad (5.3)$$

The proof of Theorem 5.6 is relegated to Subsection 7.2.

6. Upper Semicontinuous Family of Attractors

Our second main result is the closeness between $\mathcal{A}^{\sigma,\varepsilon}$ and $\mathcal{A}^{0,0}$. To state the corresponding theorem we need the following definition.

Definition 6.1. *Let X be a Banach space, and $\mathcal{B}_1, \mathcal{B}_2 \subset X$. We denote by*

$$\mathrm{dist}_X(\mathcal{B}_1, \mathcal{B}_2) = \sup_{z_1 \in \mathcal{B}_1} \inf_{z_2 \in \mathcal{B}_2} \|z_1 - z_2\|_X$$

the Hausdorff semidistance in X from $\mathcal{B}_1$ to $\mathcal{B}_2$.

Theorem 6.2. *Let $\mathcal{A}^{\sigma,\varepsilon}$ be an attractor of the dynamical system $(\mathcal{H}_{\sigma,\varepsilon}, S_{\sigma,\varepsilon}(t))$. $\mathcal{A}^{0,0}$ denotes an attractor of $(\mathcal{H}, S_{0,0}(t))$. Then*

$$\lim_{\substack{\sigma \to 0+ \\ \varepsilon \to 0+}} \Big[\mathrm{dist}_{\mathcal{H}}\left(\mathbb{P}\mathcal{A}^{\sigma,\varepsilon}, \mathcal{A}^{0,0}\right) + \sup_{U \in \mathcal{A}^{\sigma,\varepsilon}} \Big(\|\mathbb{Q}_\sigma U\|_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)} + \|\mathbb{Q}_\varepsilon U\|_{L^2_{\mu_\varepsilon}(\mathbb{R}^+;H^1)}\Big)\Big] = 0. \quad (6.1)$$

P r o o f. In this proof we need Theorems 5.3, 5.4 and 5.6. We should notice that their proofs do not depend on the arguments used in this section.

For further arguments we note that Theorem 5.2 implies that $\mathcal{A}^{\sigma,\varepsilon}$ consists of full trajectories $\{U(t) = S_{\sigma,\varepsilon}(t)U_0\}_{t\in\mathbb{R}}$. Moreover, each of these trajectories corresponds to strong solutions to problem (2.10) (it is also stated in Th. 5.2).

We split our arguments in steps.

Step I. We consider any $\{U(t) = S_{\sigma,\varepsilon}(t)U_0\}_{t\in\mathbb{R}} \subset \mathcal{A}^{\sigma,\varepsilon}$. The following estimate holds true:

$$\|U_t(t)\|_{\mathcal{H}_{\sigma,\varepsilon}} + \|\mathbb{P}U(t)\|_{\mathcal{H}^1} \le C, \quad (6.2)$$

where constant C from (5.2) is independent of σ, ε.

In particular, it means that the set

$$\left\{\mathbb{P}U \in C(\mathbb{R}; \mathcal{H}^1) |\ U(t) \in \mathcal{A}^{\sigma,\varepsilon},\ \sigma, \varepsilon \in (0, 1]\right\}$$

is bounded in $C(0, T; \mathcal{H}^1)$. Analogously, the set

$$\{\mathbb{P}U_t \in C(\mathbb{R}; \mathcal{H}) |\ U(t) \in \mathcal{A}^{\sigma,\varepsilon},\ \sigma, \varepsilon \in (0, 1]\}$$

is bounded in $C(0, T; \mathcal{H})$.

Step II. Assume by contradiction that there exists $\rho > 0$, sequences $\sigma_n \to 0+$, $\varepsilon_n \to 0+$ and the corresponding elements $U_n \in \mathcal{A}^{\sigma_n,\varepsilon_n}$ such that

$$\|U_n - \mathbb{L}_{\sigma_n,\varepsilon_n} U_0\|_{\mathcal{H}_{\sigma_n,\varepsilon_n}} \ge \rho > 0 \ \ \forall U_0 \in \mathcal{A}^0.$$

Let $U_n(t) = (u_n(t)u_{t,n}(t), v_n(t), \xi^{n,t}, \eta^{n,t}) \subset \mathcal{H}_{\sigma_n,\varepsilon_n}$ be a full trajectory in attractor $\mathcal{A}^{\sigma_n,\varepsilon_n}$ with initial data equal to U_n, i.e., $U_n(t) = S_{\sigma_n,\varepsilon_n}(t)U_n$. Then

$$\begin{array}{lcl} \{\mathbb{P}U_n\}_{n=1}^{\infty} & - & \text{bounded in } C([-T,T];\mathcal{H}^1), \\ \{\mathbb{P}U_{t,n}\}_{n=1}^{\infty} & - & \text{bounded in } C([-T,T];\mathcal{H}) \end{array}$$

for arbitrary $T > 0$. That implies the existence of a triple

$$U^*(t) = (u^*(t), u_t^*(t), v^*(t)) \in C([-T,T];\mathcal{H})$$

such that

$$\lim_{n\to\infty} \max_{t\in[-T,T]} \left[\|u_n(t) - u_n^*(t)\|_2 + \left\|u_{t,n}(t) - u_{t,n}^*(t)\right\| + \|v_n(t) - v_n^*(t)\|\right] = 0$$

and, moreover,

$$\sup_{t\in\mathbb{R}} \|U^*(t)\|_{\mathcal{H}} \le C. \tag{6.3}$$

Now we will show that $\xi^{n,t}$ and $\eta^{n,t}$ vanish as $n \to +\infty$. For this we apply Lemma 4.2 application of which is justified because of

$$\|Au_{t,n}(t)\| + \left\|A^{1/2}v_n(t)\right\| \le C,$$

where the constant C is taken from (5.2). Then the convergences

$$\lim_{n\to\infty} \left\|\xi_n^0\right\|_{L^2_{\beta_{\sigma_n}}(\mathbb{R}^+;H^2)} = \lim_{n\to\infty} \left\|\eta_n^0\right\|_{L^2_{\mu_{\varepsilon_n}}(\mathbb{R}^+;H^1)} = 0$$

follow from the invariance of attractor. (We can take $U_n(-\tau)$, where $\tau > 0$ as the initial data in (4.5) and (4.6), and then pass to the limit $n \to +\infty$.)

We meet a contradiction if we show that $U^*(0) \in \mathcal{A}^{0,0}$ that occurs if and only if $U^*(t)$ is a full bounded trajectory of $S_{0,0}(t)$. The boundedness follows from (6.3). The fact that $U^*(t)$ is a solution to (2.9) will be shown in the next step.

Step III. Functions $U_n(t)$ satisfy (2.10), i.e.,

$$\begin{cases} \dot{U}_n(t) = \mathcal{L}^{\sigma_n,\varepsilon_n}U_n(t) + f(U_n(t)), \\ U_n(0) = U_n. \end{cases}$$

We need to show that

$$\begin{cases} \dot{U}^*(t) = \mathcal{L}^{0,0}U^*(t) + f(U^*(t)), \\ U^*(0) = U_0^*. \end{cases} \tag{6.4}$$

For this we note that

$$\int_0^\infty \beta_{\sigma_n}(s)\xi^{n,t}(s)ds \to u_t^*(t) \text{ in } C([-T,T];H) \tag{6.5}$$

and, similarly,

$$\int_0^\infty \mu_{\varepsilon_n}(s)\eta^{n,t}(s)ds \to v^*(t) \text{ in } C([-T,T];H). \tag{6.6}$$

To obtain (6.5), we rewrite the memory variable ξ^n in terms of u_t^n

$$\int_0^\infty \beta_{\sigma_n}(s)\xi^{n,t}(s)ds = \int_0^\infty h_{\sigma_n}(s)u_t^n(t-s)ds.$$

Then

$$\left\|\int_0^\infty h_{\sigma_n}(s)u_t^n(t-s)ds - u_t^n(t)\right\| = \left\|\int_0^\infty h_{\sigma_n}(s)\left[u_t^n(t-s) - u_t^n(t)\right]ds\right\|$$
$$= \left\|\int_0^\infty h_{\sigma_n}(s)\int_0^s u_{tt}^n(t-y)dyds\right\| \le C\sigma_n,$$

which implies (6.5), since

$$\int_0^\infty h_{\sigma_n}(s)u_t^n(t-s)ds - u_t^*(t) = \int_0^\infty h_{\sigma_n}(s)u_t^n(t-s)ds - u_t^n(t) + u_t^n(t) - u_t^*(t).$$

Convergence (6.6) is obtained in a similar way.

Thus,

$$\mathbb{P}\left(\mathcal{L}^{\sigma_n,\varepsilon_n}U_n(t) + f(U_n(t))\right) \to \mathcal{L}^{0,0}U^*(t) + f(U^*(t)) \text{ in } C([-T,T];\mathcal{H}^{-4}),$$

and in view that

$$\|\mathbb{P}\left(\mathcal{L}^{\sigma_n,\varepsilon_n}U_n(t) + f(U_n(t))\right)\|_{C([-T,T];\mathcal{H})} \le C,$$

the function $W(t) \equiv \mathcal{L}^{0,0}U^*(t) + f(U^*(t))$ belongs to $C([-T,T];\mathcal{H})$. If we pass to the limit in the relation

$$U_n(t) - U_n(0) = \int_0^t U_{n,t}(\tau)d\tau,$$

we obtain that $W(t) = \frac{d}{dt}U^*(t)$ and this ends the proof of Theorem 6.2. ■

7. Proofs

In what follows we check whether the arguments provided in [22] allow to obtain the necessary estimates uniform with respect to σ and ε. In further proofs all generic constants C_R and C are supposed to be uniform with respect to σ and ε. Also, we consider the case $\sigma = \varepsilon = 0$ since it was not considered in [22].

7.1. Proof of Theorem 5.3

Let $(u^1, v^1, \xi^1, \eta^1)$ and $(u^2, v^2, \xi^2, \eta^2)$ be two strong solutions of problem (2.10) with initial data $U^i = (u_0^i, u_1^i, v_0^i, \xi_0^i, \eta_0^i)$, $i = 1, 2$, and assume that

$$\left\|Au^i(t)\right\|^2 + \left\|u_t^i(t)\right\|^2 + \left\|v^i(t)\right\|^2 + \left\|\xi^{i,t}\right\|^2_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)} + \left\|\eta^{i,t}\right\|^2_{L^2_{\mu_\varepsilon}(\mathbb{R}^+;H^1)} \le R^2 \quad (7.1)$$

for $\forall t \ge 0$.

The components of $Z(t)$ ($Z(t)$ was introduced in the statement of Th. 5.3) satisfy the following equation

$$\begin{cases} z_{tt} + A^2 z + \int\limits_0^\infty \beta_\sigma(s) A^2 \xi^t(s) ds - A\vartheta = F(t), \\ \vartheta_t + \int\limits_0^\infty \mu_\varepsilon(s) A \eta^t(s) ds + A z_t = 0, \\ \xi_t^t + \xi_s^t = z_t, \quad \eta_t^t + \eta_s^t = \vartheta, \end{cases} \quad (7.2)$$

where $F(t) = M\left(\left\|A^{1/2}u^2\right\|^2\right) Au^2 - M\left(\left\|A^{1/2}u^1\right\|^2\right) Au^1$.

To obtain an appropriate form of energy relation from (7.2), we first transform the term $(F(t), z_t)$.

Lemma 7.1. *Let $(u^1(t), v^1(t), \xi^{1,t}, \eta^{1,t})$ and $(u^2(t), v^2(t), \xi^{2,t}, \eta^{2,t})$ be strong solutions to problem* (2.10) *satisfying* (7.1). *Then the following representation holds:*

$$(F(t), z_t) = \frac{d}{dt} Q(t) + P(t), \quad (7.3)$$

where the functions $Q(t) \in C^1(\mathbb{R}^+)$ and $P(t) \in C(\mathbb{R}^+)$ satisfy the relations:

$$|Q(t)| \le C_R \|Az\| \, \|z\|, \quad (7.4)$$

$$|P(t)| \le C_R \left| \left(T_\sigma \xi^{2,t}, \xi^{2,t}\right)_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)} \right|^{1/2} \left(\|Az\|^2 + \|z_t\|^2\right) \quad (7.5)$$

with the constant C_R chosen to be independent of σ and ε.

P r o o f. In [22], it is proved that the following representation holds:

$$(F(t), z_t) = \frac{d}{dt} Q(t) + P(t),$$

where

$$Q(t) = Q_0(t) + \sigma Q_1(t), \quad P(t) = \sigma \left[P_1(t) - P_2(t)\right],$$

and

$$Q_0(t) = -\int_0^1 \mathrm{I}_1[u^2 + \lambda z, u^2] d\lambda \left(Au^2, z\right) - \int_0^1 \lambda M \left(\left\|A^{1/2}(u^2 + \lambda z)\right\|^2\right) d\lambda \left(Az, z\right),$$

$$Q_1(t) = \int_0^\infty \beta_\sigma(s) \left(\xi^{2,t}(s), \mathrm{I}_2 \cdot Au^2 + \mathrm{I}_1[u^1, u^2] \cdot Az\right) ds,$$

$$P_1(t) = \int_0^\infty \beta_\sigma(s) \left(\xi_s^{2,t}(s), \mathrm{I}_2 \cdot Au^2 + \mathrm{I}_1[u^1, u^2] \cdot Az\right) ds,$$

$$P_2(t) = \int_0^\infty \beta_\sigma(s) \left(\xi^{2,t}(s), \mathrm{I}_4 \cdot Au^2 + \mathrm{I}_2 \cdot Au_t^2 + 2\mathrm{I}_3 \cdot Az + \mathrm{I}_1[u^1, u^2] \cdot Az_t\right) ds.$$

Terms I_i admit the following estimates:

$$|\mathrm{I}_1 [u_1, u_2]| \le C_R \|u_1 - u_2\|, \quad |\mathrm{I}_2| \le C_R \|Az\| \|z\|,$$
$$|\mathrm{I}_3| \le C_R \left(\|Az\| + \|z_t\|\right), \quad |\mathrm{I}_4| \le C_R \left(\|Az\|^2 + \|z_t\|^2\right).$$

A generic constant C_R in the estimates above and in further arguments is independent of σ and ε.

We estimate $Q_i(t)$ and $P_i(t)$ as follows:

1. For $Q(t) = Q_0(t) + \sigma Q_1(t)$ we have $|Q_0(t)| \le C_R \|Az\| \|z\|$ and

$$|Q_1(t)| \le C_R \int_0^\infty \beta_\sigma(s) \left\|\xi^{2,t}(s)\right\| ds \|Az\| \|z\| \le \frac{C_R}{\sigma^{1/2}} \|Az\| \|z\|.$$

Therefore, $|Q(t)| \le C_R(1 + \sigma^{1/2}) \|Az\| \|z\|$.

2. For $P(t) = \sigma \left[P_1(t) - P_2(t)\right]$ we have

$$|P_1(t)| \le C_R \int_0^\infty (-\beta_\sigma'(s)) \left\|\xi^{2,t}(s)\right\|_2 ds \|Az\|^2$$
$$\le C_R \left(\int_0^\infty (-\beta_\sigma'(s)) ds\right)^{1/2} \sqrt{-\left(T_\sigma \xi^{2,t}, \xi^{2,t}\right)_{L^2_\sigma(\mathbb{R}^+; H^2)}} \|Az\|^2$$
$$\le \frac{C_R}{\sigma} \sqrt{-\left(T_\sigma \xi^{2,t}, \xi^{2,t}\right)_{L^2_\sigma(\mathbb{R}^+; H^2)}} \|Az\|^2.$$

To estimate $P_2(t)$ we apply (2.5)

$$|P_2(t)| \le C_R \int_0^\infty \beta_\sigma(s) \left\|\xi^{2,t}(s)\right\|_2 ds \left(\|Az\|^2 + \|z_t\|^2\right)$$
$$\le \frac{C_R}{\sigma^{1/2}} \left\|\xi^{2,t}\right\|_{L^2_{\beta_\sigma}(\mathbb{R}^+; H^2)} \left(\|Az\|^2 + \|z_t\|^2\right)$$
$$\le C_R \sqrt{-\left(T_\sigma \xi^{2,t}, \xi^{2,t}\right)_{L^2_\sigma(\mathbb{R}^+; H^2)}} \left(\|Az\|^2 + \|z_t\|^2\right).$$

And therefore, $|P(t)| \le C_R(1+\sigma)\sqrt{-(T_\sigma \xi^{2,t}, \xi^{2,t})_{L^2_\sigma(\mathbb{R}^+;H^2)}}\left(\|Az\|^2 + \|z_t\|^2\right)$.
Thus, all necessary estimates are obtained and this concludes the proof. ■

Now we return to the proof of Theorem 5.3.
By (7.3), for these solutions we have the energy relation

$$\frac{d}{dt}\mathcal{E}^0(t) = \left(T_\sigma \xi^t, \xi^t\right)_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)} + \left(T\eta^t, \eta^t\right)_{L^2_{\mu_\varepsilon}(\mathbb{R}^+;H^1)} + P(t), \qquad (7.6)$$

where

$$\mathcal{E}^0(t) = \tfrac{1}{2}\Big[\|z_t(t)\|^2 + \|Az(t)\|^2 + \|\vartheta(t)\|^2 + \left\|\xi^t\right\|^2_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)} + \left\|\eta^t\right\|^2_{L^2_{\mu_\varepsilon}(\mathbb{R}^+;H^1)} - 2Q(t)\Big].$$

It follows from (7.4) that

$$\frac{3}{8}\|Z(t)\|^2_{\mathcal{H}} - C_R\|z(t)\|^2 \le \mathcal{E}^0(t) \le \frac{5}{8}\|Z(t)\|^2_{\mathcal{H}} + C_R\|z(t)\|^2. \qquad (7.7)$$

Now we consider $V(t) \equiv \mathcal{E}^0(t) + \omega\sum_{i=1}^{3}\Phi_i(t)$, where we set $\Phi_1(t) = (z_t, z)$, $\Phi_2(t) = -\sigma(A^{-2}z_t, \xi^t)_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)}$ and $\Phi_3(t) = -\varepsilon(z + A^{-1}\vartheta, \eta^t)_{L^2_{\mu_\varepsilon}(\mathbb{R}^+;H^1)}$.

A positive constant ω will be chosen in the sequel and it will be independent of σ and ε. For $V(t)$ we have an estimate similar to (7.7)

$$\frac{1}{4}\|Z(t)\|^2_{\mathcal{H}} - C_R\|z(t)\|^2 \le V(t) \le \|Z(t)\|^2_{\mathcal{H}} + C_R\|z(t)\|^2 \qquad (7.8)$$

as soon as ω is sufficiently small.

Now we compute the derivatives of $\Phi_i(t)$

$$\begin{aligned}
\tfrac{d}{dt}\Phi_1(t) &= -\|Az\|^2 - \int_0^\infty \beta_\sigma(s)(\xi^t(s), z)_1 ds + (\vartheta, Az) + (F(t), z) + \|z_t\|^2, \\
\tfrac{d}{dt}\Phi_2(t) &= \sigma\int_0^\infty \beta_\sigma(s)\left(A^2 z + \int_0^\infty \beta_\sigma(\tau)A^2\xi^t(\tau)d\tau - A\vartheta - F(t), \xi^t(s)\right)ds \\
&\quad +\sigma\int_0^\infty \beta_\sigma(s)(z_t, \xi^t_s)ds - \|z_t\|^2. \\
\tfrac{d}{dt}\Phi_3(t) &= \varepsilon\left(\int_0^\infty \mu_\varepsilon(\tau)\eta^t(\tau)d\tau, \eta^t\right)_{L^2_{\mu_\varepsilon}(\mathbb{R}^+;H^1)} - (Az, \vartheta) - \|\vartheta\|^2 \\
&\quad +\varepsilon(z + A^{-1}\vartheta, \eta^t_s)_{L^2_{\mu_\varepsilon}(\mathbb{R}^+;H^1)}.
\end{aligned}$$

Using the auxiliary inequalities below, we will be able to estimate Φ_i in appropriate way.

First,

$$\left\|\int_0^\infty \beta_\sigma(s)\xi^t(s)ds\right\|^2 \le \left(\int_0^\infty \beta_\sigma(s)\left\|\xi^t(s)\right\| ds\right)^2$$
$$\le \frac{1}{\sigma}\left\|\xi\right\|^2_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)} \le C\left|\left(T_\sigma\xi^t,\xi^t\right)_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)}\right|.$$

Second,

$$\int_0^\infty \beta_\sigma(s)(\xi^t,z)_1 ds \le \int_0^\infty \beta_\sigma(s)\left(\frac{1}{\sigma}\left\|\xi^t\right\|_1^2 + \sigma\left\|z\right\|_1^2\right)ds$$
$$= \frac{1}{\sigma}\left\|\xi^t\right\|^2_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)} + \left\|z\right\|_1^2 \le C\left|\left(T_\sigma\xi^t,\xi^t\right)_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)}\right| + \left\|z\right\|_1^2.$$

Third,

$$\int_0^\infty \beta_\sigma(s)(z_t,\xi^t_s(s))ds = \int_0^\infty(-\beta'_\sigma(s))(z_t,\xi^t(s))ds$$
$$\le \int_0^\infty(-\beta'_\sigma(s))\left\|\xi^t(s)\right\|ds\left\|z_t\right\| \le \frac{C}{\sigma}\left(\int_0^\infty(-\beta'_\sigma(s))\left\|\xi^t(s)\right\|ds\right)^{1/2}\left\|z_t\right\|$$
$$\le \frac{C}{\sigma}\left(\left|\left(T_\sigma\xi^t,\xi^t\right)_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)}\right| + \left\|z_t\right\|^2\right).$$

We are in position to write the estimates for $\frac{d}{dt}\sum_{i=1}^3\Phi_i$:

$$\frac{d}{dt}\sum_{i=1}^3\Phi_i(t) = C_R\left|\left(T_\sigma\xi^t,\xi^t\right)_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)}\right| + C_R\left|\left(T_\varepsilon\eta^t,\eta^t\right)_{L^2_{\mu_\varepsilon}(\mathbb{R}^+;H^1)}\right|$$
$$-\frac{1}{2}\left[\left\|Az\right\|^2 + \left\|z_t\right\|^2 + \left\|\vartheta\right\|^2\right] + C_R\left\|z\right\|^2.$$

Choosing sufficiently small ω, we obtain that for some positive α^*

$$\frac{d}{dt}V(t) + \alpha^*\left\|Z(t)\right\|_{\mathcal{H}_{\varepsilon,\sigma}} \le P(t) + C_R\left\|z(t)\right\|^2,$$

and then the application of (7.5) and (7.8) implies the existence of $\gamma > 0$ such that

$$\frac{d}{dt}V(t) + \gamma V(t) \le C_R\left\|z(t)\right\|^2 + C_R\left|\left(T_\sigma\xi^{2,t},\xi^{2,t}\right)_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)}\right|\left(\left\|Az\right\|^2 + \left\|z_t\right\|^2\right).$$

Using the Gronwall Lemma, we obtain

$$\left\|Z(t)\right\|^2_{\mathcal{H}_{\sigma,\varepsilon}} \le C_R\left\|Z(0)\right\|^2_{\mathcal{H}_{\sigma,\varepsilon}}e^{-\gamma t} + C_R\max_{\tau\in[0,t]}\left\|z(t)\right\|^2$$
$$+C_R\int_0^t e^{-\gamma(t-\tau)}\left|\left(T_\sigma\xi^{2,\tau},\xi^{2,\tau}\right)_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)}\right|\left\|Z(\tau)\right\|^2_{\mathcal{H}_{\sigma,\varepsilon}}d\tau.$$

Now using the fact that $\int\limits_0^{+\infty} \left| \left(T_\sigma \xi^{2,t}, \xi^{2,t}\right)_{L^2_{\beta_\sigma}(\mathbb{R}^+;H^2)} \right| dt \leq C_R$, which follows from the energy relation and inequality (7.1), we obtain the stabilizability estimate with uniform (with respect to parameters σ and ε) coefficients.

7.2. Proof of Theorem 5.6

We use the same procedure as in the previous subsection. So, we need to replace (7.1) by

$$\left\|Au^i(t)\right\|^2 + \left\|u^i_t(t)\right\|^2 + \left\|v^i(t)\right\|^2 \leq R^2, \ \ t \geq 0, \tag{7.9}$$

and (7.2) by

$$\begin{cases} z_{tt} + A^2 z_t + A^2 z - A\vartheta = F(t), \\ \vartheta_t + A\vartheta + Az_t = 0, \end{cases} \tag{7.10}$$

where $F(t)$ is the same as in (7.2). Next lemma holds

Lemma 7.2. *Let $(u^1(t), v^1(t))$ and $(u^2(t), v^2(t))$ be strong solutions to problem (2.9) satisfying (7.9). Then the representation $(F(t), z_t) = \frac{d}{dt}Q(t) + P(t)$ holds, where the functions $Q(t) \in C^1(\mathbb{R}^+)$ and $P(t) \in C(\mathbb{R}^+)$ satisfy the relations*

$$|Q(t)| \leq C_R \left\|Az\right\| \left\|z\right\|, \ \ |P(t)| \leq C_R \left\|u^2_t\right\| \left(\left\|Az\right\|^2 + \left\|z_t\right\|^2\right).$$

This lemma can be proved in the same way as Lemma 7.1 (see also [3]).

Following the same procedure as in the previous subsection, we consider the auxiliary function $V(t)$ given by $V(t) = \mathcal{E}_0(t) + \omega\Phi(t)$, where $\Phi(t) = (z_t, z)$. We do not repeat all arguments, since the proof is similar to [5, Th. 5.6], [22, Th. 3.11] and Subsection 7.1, and give just a sketch of the proof.

As before

$$\frac{1}{4}\left\|Z(t)\right\|^2_{\mathcal{H}} - C\left\|z(t)\right\|^2 \leq V(t) \leq \left\|Z(t)\right\|^2_{\mathcal{H}} + C\left\|z(t)\right\|^2.$$

Then

$$\begin{aligned} \tfrac{d}{dt}\Phi(t) \ \ &= \ \ -(Az_t, Az) - \left\|Az\right\|^2 + (Az, \vartheta) + (F(t), z) + \left\|z_t\right\|^2 \\ &\leq C\left\|Az_t\right\|^2 - \tfrac{1}{2}\left\|Az\right\|^2 + C\left\|A^{1/2}\vartheta\right\|^2 + C\left\|z\right\|^2. \end{aligned}$$

Taking ω sufficiently small, the following inequality holds with some strictly positive constant γ

$$V(t) \leq V(0)e^{-\gamma t} + C\max_{\tau\in[0,t]}\left\|z(\tau)\right\|^2 + C\int\limits_0^t e^{-\gamma(t-\tau)}\left\|u^2_t(\tau)\right\|^2_2 \left\|Z(\tau)\right\|^2_{\mathcal{H}} d\tau.$$

And we get the desired conclusion in the same way as in the previous case (Subsect. 7.1) by using the estimate

$$\int_0^\infty \left\|u_t^2(t)\right\|_2^2 dt < C.$$

7.3. Proof of Theorem 5.4

Step I. Theorem 5.2 states that $\mathcal{A}^{\sigma,\varepsilon} = \mathcal{M}^u(\mathcal{N})$.

We also know that the functional $\Phi(U)$ introduced in Subsection 3.2 decreases on nonstationary trajectories. Since $\mathcal{N} \subset \mathcal{A}^{\sigma,\varepsilon}$, it implies that

$$\max\left\{\Phi(U) : \ U \in \mathcal{A}^{\sigma,\varepsilon}\right\} = \max\left\{\Phi(U) : \ U \in \mathcal{N}\right\},$$

where the right-hand side does not depend on σ and ε because of the form of $\mathcal{N}$ (see Prop. 3.4). If in addition we notice that (see Prop. 3.2)

$$\alpha \left\|U\right\|^2_{\mathcal{H}_{\sigma,\varepsilon}} - C \le \Phi(U),$$

we may assert that there exists positive R independent of σ, ε such that

$$\left\|U\right\|_{\mathcal{H}_{\sigma,\varepsilon}} \le R \quad \forall \sigma \ \ \forall \varepsilon \ \ \forall U \in \mathcal{A}^{\sigma,\varepsilon}. \tag{7.11}$$

In the steps below we will emphasize the dependence of generic constants on R. But this dependence is formal since R is independent of σ, ε.

Step II. Current step follows the arguments of Step II from [22, Th. 4.1]. We also note that the main idea of this step is borrowed from [5].

Due to the uniform (with respect to σ,ε) feature of stabilizability inequality (5.1) all constants below are uniform with respect to σ and ε.

Let $\left\{U(t) \equiv (u(t); u_t(t); v(t); \xi^t; \eta^t)\right\} \subset \mathcal{H}_{\sigma,\varepsilon}$ be a full trajectory from the attractor $\mathcal{A}^{\sigma,\varepsilon}$. Let $|\omega| < 1$. Applying Theorem 5.3 with $U^1 = U(s+\omega)$, $U^2 = U(s)$ (and, accordingly, the interval $[s,t]$ in place of $[0,t]$), we have

$$\begin{aligned} \left\|U(t+\omega) - U(t)\right\|^2_{\mathcal{H}_{\sigma,\varepsilon}} \le C_1 e^{-\gamma(t-s)} \left\|U(s+\omega) - U(s)\right\|^2_{\mathcal{H}_{\sigma,\varepsilon}} \\ + C_2 \max_{\tau \in [s,t]} \left\|u(\tau+\omega) - u(\tau)\right\|^2 \end{aligned} \tag{7.12}$$

for any $t, s \in \mathbb{R}$ such that $s \le t$ and for any ω with $|\omega| < 1$. Taking the limit $s \to -\infty$, (7.12) gives

$$\left\|U(t+\omega) - U(t)\right\|^2_{\mathcal{H}_{\sigma,\varepsilon}} \le C_2 \max_{\tau \in (-\infty,t]} \left\|u(\tau+\omega) - u(\tau)\right\|^2$$

for any $t \in \mathbb{R}$ and $|\omega| < 1$. We obviously have

$$\frac{1}{\omega}\left\|u(\tau+\omega) - u(\tau)\right\| \le \frac{1}{\omega}\int_0^\omega \left\|u_t(\tau+t)\right\| dt, \quad \tau \in \mathbb{R}.$$

Therefore, by (7.11) we obtain

$$\max_{\tau\in\mathbb{R}}\left\|\frac{U(\tau+\omega)-U(\tau)}{\omega}\right\|_{\mathcal{H}_{\sigma,\varepsilon}}\le C_R \ \text{ for } \ |\omega|<1.$$

The last estimate implies that the function $U(t)$ is absolutely continuous and thus possesses a derivative almost everywhere which satisfies $\|U_t(t)\|_{\mathcal{H}_{\sigma,\varepsilon}}\le C_R$.

And as it is stated in Theorem 5.2, $U(t)$ is a strong solution to (2.10) and, besides,

$$\|\mathcal{L}^{\sigma,\varepsilon}U\|_{\mathcal{H}_{\sigma,\varepsilon}}\le C_R. \tag{7.13}$$

Step III. Now we verify that $\left\|A^{1/2}v\right\|\le C_R$.

It follows from the second and the fourth equalities of (2.8) and estimate (7.13) that

$$\int_0^\infty \mu_\varepsilon(s)A\eta(s)ds=-v_t-Au_t\equiv v^*,\ \ \|v^*\|\le C_R,$$
$$\eta_s-v=-\eta_t\equiv\eta^*,\ \ \|\eta^*\|_{L^2_{\mu_\varepsilon}(\mathbb{R}^+;H^1)}\le C_R.$$

Since $\eta\in D(T_\varepsilon)$, the second equality gives

$$\eta(s)=sv+\int_0^s\eta^*(y)dy.$$

Now we substitute this to the first equality

$$Av=-\int_0^\infty\mu_\varepsilon(s)\int_0^s A\eta^*(y)dyds+v^*,$$

where the right-hand side is estimated by generic constant C_R in space H^{-1} since

$$\left\|\int_0^\infty\mu_\varepsilon(s)\int_0^s A\eta^*(y)dyds\right\|_{-1}\le\int_0^\infty\mu_\varepsilon(s)\int_0^s\left\|A^{1/2}\eta^*(y)\right\|dyds$$
$$=\int_0^\infty\left\|A^{1/2}\eta^*(y)\right\|\int_y^\infty\mu_\varepsilon(s)dsdy\le\frac{\varepsilon}{\delta}\int_0^\infty\mu_\varepsilon(y)\left\|A^{1/2}\eta^*(y)\right\|dy$$
$$\le C_R\sqrt{\varepsilon}\int_0^\infty\mu_\varepsilon(y)\left\|\eta^*(y)\right\|_1^2dy\le C_R\sqrt{\varepsilon}.$$

Step IV. Our purpose is to obtain $\|u(t)\|_3\le C_R$.

Let us consider the following Volterra equation:

$$u(t)-\int_{-\infty}^t\frac{\beta_\sigma(t-y)}{1+\frac{1}{\sigma}}u(y)dy=\frac{h(t)}{1+\frac{1}{\sigma}},$$

where

$$h(t) \equiv -A^{-2}u_{tt} - A^{-2}M\left(\left\|A^{1/2}u\right\|^2\right)Au + A^{-2}p + A^{-1}v, \quad \|h(t)\|_3 \le C_R.$$

We will use the standard iteration method. Namely, $w_0(t) \equiv 0$ and

$$w_n(t) = \frac{h(t)}{1+\frac{1}{\sigma}} + \int\limits_{-\infty}^{t} \frac{\beta_\sigma(t-y)}{1+\frac{1}{\sigma}} w_{n-1}(y)dy, \quad n = 1, 2, 3, \ldots.$$

The sequence introduced converges to $u(t)$ in $C([-T,T]; H^3)$ for arbitrary $T > 0$ in view that

$$\begin{aligned}\sup_{t\in\mathbb{R}} \|w_{n+1}(t) - w_n(t)\|_3 &\le \frac{1}{1+\sigma} \sup_{t\in\mathbb{R}} \|w_n(t) - w_{n-1}(t)\|_3 \\ &\le \left(\frac{1}{1+\sigma}\right)^n \frac{1}{1+\frac{1}{\sigma}} \sup_{t\in\mathbb{R}} \|h(t)\|_3 .\end{aligned}$$

And, in particular,

$$\sup_{t\in[-T,T]} \|w_{n+1}(t) - w_n(t)\|_3 \le C_R \left(\frac{1}{1+\sigma}\right)^n \frac{1}{1+\frac{1}{\sigma}}.$$

To prove the final estimate, we need to observe

$$\sup_{t\in\mathbb{R}} \|u(t)\|_3 \le \sum_{n=0}^{\infty} \sup_{t\in\mathbb{R}} \|w_{n+1}(t) - w_n(t)\|_3 \le \sup_{t\in\mathbb{R}} \|h(t)\|_3 \le C_R.$$

This concludes the proof.

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