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Classification of $U_q(\mathfrak{sl}_2)$-Module Algebra Structures on the Quantum Plane

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A complete list of $U_q(\mathfrak{sl}_2)$-module algebra structures on the quantum plane is produced and the (uncountable family of) isomorphism classes of these structures are described. The composition series of representations in question are computed. The classical limits of the $U_q(\mathfrak{sl}_2)$-module algebra structures are discussed.

Key words: quantum universal enveloping algebra, Hopf algebra, Verma module, representation, composition series, weight.

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1. Introduction

The quantum plane [11] is known to be a starting point in studying modules over quantum universal enveloping algebras [3]. The structures existing on the quantum plane are widely used as a background to produce associated structures for more sophisticated quantum algebras [5, 4, 10]. There is one distinguished structure of $U_q(\mathfrak{sl}_2)$-module algebra on the quantum plane which was widely considered before (see, e.g., [8]). In addition, one could certainly mention the structure $h(\mathbf{v}) = \varepsilon(h)\mathbf{v}$, where $h \in U_q(\mathfrak{sl}_2)$, ε is the counit, $\mathbf{v}$ is a polynomial on the quantum plane. Normally it is disregarded because of its triviality. Nevertheless, it turns out that there exist more (in fact, an uncountable family of nonisomorphic) $U_q(\mathfrak{sl}_2)$-module algebra structures which are nontrivial and can be used in further development of the quantum group theory.

In this paper we suggest a complete description and classification of $U_q(\mathfrak{sl}_2)$-module algebra structures existing on the quantum plane. Specifically, in Section 3 we use a general form of the automorphism of quantum plane to render the notion of weight for $U_q(\mathfrak{sl}_2)$-actions considered here. In Section 4 we present our classification in terms of a pair of symbolic matrices, which relies upon considering the low dimensional (0-th and 1-st) homogeneous components of an action. In Section 5 we describe the composition series for the above structures viewed as representations in vector spaces.

2. Preliminaries

Let H be a Hopf algebra whose comultiplication is Δ, counit is ε, and antipode is S [1]. Also let A be a unital algebra with unit $\mathbf{1}$. We will also use the Sweedler notation $\Delta(h) = \sum_i h'_i \otimes h''_i$ [13].

Definition 2.1. *By a structure of H-module algebra on A we mean a homomorphism $\pi : H \to \mathrm{End}_{\mathbb{C}} A$ such that:*

(i) $\pi(h)(ab) = \sum_i \pi(h'_i)(a) \cdot \pi(h''_i)(b)$ for all $h \in H$, $a, b \in A$;

(ii) $\pi(h)(\mathbf{1}) = \varepsilon(h)\mathbf{1}$ for all $h \in H$.

The structures π_1, π_2 are said to be isomorphic if there exists an automorphism Ψ of the algebra A such that $\Psi\pi_1(h)\Psi^{-1} = \pi_2(h)$ for all $h \in H$.

Throughout the paper we assume that $q \in \mathbb{C} \setminus \{0\}$ is not a root of the unit ($q^n \neq 1$ for all nonzero integers n). Consider the quantum plane which is a unital algebra $\mathbb{C}_q[x, y]$ with two generators x, y and a single relation

$$yx = qxy. \tag{2.1}$$

The quantum universal enveloping algebra $U_q(\mathfrak{sl}_2)$ is a unital associative algebra determined by its (Chevalley) generators k, k^{-1}, e, f, and the relations

$$\mathsf{k}^{-1}\mathsf{k} = \mathbf{1}, \qquad \mathsf{k}\mathsf{k}^{-1} = \mathbf{1}, \tag{2.2}$$

$$\mathsf{k}\mathsf{e} = q^2\mathsf{e}\mathsf{k}, \tag{2.3}$$

$$\mathsf{k}\mathsf{f} = q^{-2}\mathsf{f}\mathsf{k}, \tag{2.4}$$

$$\mathsf{e}\mathsf{f} - \mathsf{f}\mathsf{e} = \frac{\mathsf{k} - \mathsf{k}^{-1}}{q - q^{-1}}. \tag{2.5}$$

The standard Hopf algebra structure on $U_q(\mathfrak{sl}_2)$ is determined by

$$\Delta(\mathsf{k}) = \mathsf{k} \otimes \mathsf{k}, \tag{2.6}$$

$$\Delta(\mathsf{e}) = \mathbf{1} \otimes \mathsf{e} + \mathsf{e} \otimes \mathsf{k}, \tag{2.7}$$

$$\Delta(\mathsf{f}) = \mathsf{f} \otimes \mathbf{1} + \mathsf{k}^{-1} \otimes \mathsf{f}, \tag{2.8}$$

$$\mathsf{S}(\mathsf{k}) = \mathsf{k}^{-1}, \qquad \mathsf{S}(\mathsf{e}) = -\mathsf{e}\mathsf{k}^{-1}, \qquad \mathsf{S}(\mathsf{f}) = -\mathsf{k}\mathsf{f},$$

$$\varepsilon(\mathsf{k}) = 1, \qquad \varepsilon(\mathsf{e}) = \varepsilon(\mathsf{f}) = 0.$$

3. Automorphisms of the Quantum Plane

Denote by $\mathbb{C}_q[x,y]_i$ the i-th homogeneous component of $\mathbb{C}_q[x,y]$, which is a linear span of the monomials $x^m y^n$ with $m+n=i$. Also, given a polynomial $p \in \mathbb{C}_q[x,y]$, denote by $(p)_i$ the i-th homogeneous component of p, that is the projection of p onto $\mathbb{C}_q[x,y]_i$ parallel to the direct sum of all other homogeneous components of $\mathbb{C}_q[x,y]$.

We rely upon a result by J. Alev and M. Chamarie which gives, in particular, a description of automorphisms of the algebra $\mathbb{C}_q[x,y]$ [2, Prop. 1.4.4(i)]. In fact, their claim is much more general, so in the special case we need here we present a quite elementary proof for the reader's convenience.

Proposition 3.1. *Let Ψ be an automorphism of $\mathbb{C}_q[x,y]$, then there exist nonzero constants α, β such that*

$$\Psi : x \mapsto \alpha x, \qquad y \mapsto \beta y. \tag{3.1}$$

First note that an automorphism as in (3.1) is well defined on the entire algebra, because the ideal of relations generated by (2.1) is Ψ-invariant. We split the proof into a series of lemmas.

Lemma 3.2. *One has $(\Psi(x))_0 = (\Psi(y))_0 = 0$.*

P r o o f. We start with proving $(\Psi(x))_0 = 0$. Suppose the contrary, that is $(\Psi(x))_0 \neq 0$. As $\Psi(y) \neq 0$, we choose the lowest i with $(\Psi(y))_i \neq 0$. Apply Ψ to the relation $yx = qxy$ and then project it to the i-th homogeneous component of $\mathbb{C}_q[x,y]$ (parallel to the direct sum of all other homogeneous components) to get $(\Psi(y)\Psi(x))_i = q(\Psi(x)\Psi(y))_i$. Clearly, $(\Psi(y)\Psi(x))_i$ is the lowest homogeneous component of $\Psi(y)\Psi(x)$, and $(\Psi(y)\Psi(x))_i = (\Psi(y))_i(\Psi(x))_0$. In a similar way $q(\Psi(x)\Psi(y))_i = q(\Psi(x))_0(\Psi(y))_i$. Because $(\Psi(x))_0$ is a constant, it commutes with $(\Psi(y))_i$, then $(\Psi(y))_i(\Psi(x))_0 = q(\Psi(y))_i(\Psi(x))_0$, and since $(\Psi(x))_0 \neq 0$, we also have $(\Psi(y))_i = q(\Psi(y))_i$. Recall that $q \neq 1$, hence $(\Psi(y))_i = 0$ which contradicts to our choice of i. Thus our claim is proved. The proof of another claim goes in a similar way. ■

Lemma 3.3. *One has $(\Psi(x))_1 \neq 0$, $(\Psi(y))_1 \neq 0$.*

P r o o f. Let us prove that $(\Psi(x))_1 \neq 0$. Suppose the contrary, which by virtue of Lemma 3.2 means that $\Psi(x) = \sum_i a_i x^{m_i} y^{n_i}$ with $m_i + n_i > 1$. The subsequent application of the inverse automorphism gives $\Psi^{-1}(\Psi(x))$ which is certainly x. On the other hand,

$$\Psi^{-1}(\Psi(x)) = \sum_i a_i (\Psi^{-1}(x))^{m_i} (\Psi^{-1}(y))^{n_i}.$$

By Lemma 3.2 every nonzero monomial in $\Psi^{-1}(x)$ and $\Psi^{-1}(y)$ has degree at least one, which implies that $\Psi^{-1}(\Psi(x))$ is a sum of monomials of degree at least 2. In particular, $\Psi^{-1}(\Psi(x))$ can not be x. This contradiction proves the claim. The rest of the statements can be proved in a similar way. ■

Lemma 3.4. *There exist nonzero constants $\alpha, \beta, \gamma, \delta$ such that $(\Psi(x))_1 = \alpha x$, $(\Psi(y))_1 = \beta y$.*

P r o o f. Let us apply Ψ to (2.1), then project it to $\mathbb{C}_q[x,y]_2$ to get $(\Psi(y)\Psi(x))_2 = q(\Psi(x)\Psi(y))_2$. It follows from Lemmas 3.2, 3.3 that $(\Psi(y)\Psi(x))_2 = (\Psi(y))_1(\Psi(x))_1$ and $(\Psi(x)\Psi(y))_2 = (\Psi(x))_1(\Psi(y))_1$. Let $(\Psi(x))_1 = \alpha x + \mu y$ and $(\Psi(y))_1 = \beta y + \nu x$, which leads to $(\beta y + \nu x)(\alpha x + \mu y) = q(\alpha x + \mu y)(\beta y + \nu x)$. This, together with (2.1) and Lemma 3.3, implies that $\mu = \nu = 0$, $\alpha \neq 0$, and $\beta \neq 0$. ■

Denote by $\mathbb{C}[x]$ and $\mathbb{C}[y]$ the linear spans of $\{x^n |\, n \geq 0\}$ and $\{y^n |\, n \geq 0\}$, respectively. Obviously, one has the direct sum decompositions

$$\mathbb{C}_q[x,y] = \mathbb{C}[x] \oplus y\mathbb{C}_q[x,y] = \mathbb{C}[y] \oplus x\mathbb{C}_q[x,y].$$

Given any polynomial $P \in \mathbb{C}_q[x,y]$, let $(P)_x$ be its projection to $\mathbb{C}[x]$ parallel to $y\mathbb{C}_q[x,y]$, and in a similar way define $(P)_y$. Obviously, $\mathbb{C}[x]$ and $\mathbb{C}[y]$ are commutative subalgebras.

Lemma 3.5. *One has $(\Psi(x))_y = (\Psi(y))_x = 0$.*

P r o o f. First we prove that $(\Psi(x))_y = 0$. Project $yx = qxy$ to $\mathbb{C}[y]$ to obtain $(\Psi(y))_y(\Psi(x))_y = q(\Psi(x))_y(\Psi(y))_y$. On the other hand, $(\Psi(y))_y(\Psi(x))_y = (\Psi(x))_y(\Psi(y))_y$, so that $(1-q)(\Psi(x))_y(\Psi(y))_y = 0$. Since $q \neq 1$, we deduce that $(\Psi(x))_y(\Psi(y))_y = 0$. It follows from Lemma 3.4 that $(\Psi(y))_y \neq 0$, and since $\mathbb{C}_q[x,y]$ is a domain [7], we finally obtain $(\Psi(x))_y = 0$. The proof of another claim goes in a similar way. ■

P r o o f of Proposition 3.1. It follows from Lemma 3.5 that $\Psi(x) = xP$ for some $P \in \mathbb{C}_q[x,y]$. An application of Ψ^{-1} gives $x = \Psi^{-1}(x)\Psi^{-1}(P)$. Since $\deg x = 1$, one should have either $\deg \Psi^{-1}(x) = 0$ or $\deg \Psi^{-1}(P) = 0$. Lemma 3.2 implies that $\deg \Psi^{-1}(x) \neq 0$, hence $\deg \Psi^{-1}(P) = 0$, that is $\Psi^{-1}(P)$ is a nonzero constant, and so $P = \Psi\Psi^{-1}(P)$ is the same constant (we denote it by α). The second claim can be proved in a similar way. ■

4. The Structures of $U_q(\mathfrak{sl}_2)$-Module Algebra on the Quantum Plane

We describe here the $U_q(\mathfrak{sl}_2)$-module algebra structures on $\mathbb{C}_q[x,y]$ and then classify them up to isomorphism.

For the sake of brevity, given a $U_q(\mathfrak{sl}_2)$-module algebra structure on $\mathbb{C}_q[x,y]$, we can associate a 2×3 matrix with entries from $\mathbb{C}_q[x,y]$

$$\mathsf{M} \stackrel{def}{=} \left\| \begin{array}{c} \mathsf{k} \\ \mathsf{e} \\ \mathsf{f} \end{array} \right\| \cdot \|x,y\| = \left\| \begin{array}{cc} \mathsf{k}(x) & \mathsf{k}(y) \\ \mathsf{e}(x) & \mathsf{e}(y) \\ \mathsf{f}(x) & \mathsf{f}(y) \end{array} \right\|, \tag{4.1}$$

where k, e, f are the generators of $U_q(\mathfrak{sl}_2)$ and x, y are the generators of $\mathbb{C}_q[x,y]$. We call M a *full action matrix*. Conversely, suppose we have a matrix M with entries from $\mathbb{C}_q[x,y]$ as in (4.1). To derive the associated $U_q(\mathfrak{sl}_2)$-module algebra structure on $\mathbb{C}_q[x,y]$ we set (using the Sweedler notation)

$$(\mathsf{ab})u \stackrel{def}{=} \mathsf{a}(\mathsf{b}u), \qquad \mathsf{a},\mathsf{b} \in U_q(\mathfrak{sl}_2), \qquad u \in \mathbb{C}_q[x,y], \tag{4.2}$$

$$\mathsf{a}(uv) \stackrel{def}{=} \Sigma_i(\mathsf{a}_i' u)\cdot(\mathsf{a}_i'' v), \qquad \mathsf{a} \in U_q(\mathfrak{sl}_2), \qquad u,v \in \mathbb{C}_q[x,y], \tag{4.3}$$

which determines a well-defined action of $U_q(\mathfrak{sl}_2)$ on $\mathbb{C}_q[x,y]$ iff the following properties hold. Firstly, an application (defined by (4.2)) of an element from the relation ideal of $U_q(\mathfrak{sl}_2)$ (2.2)–(2.5) to any $u \in \mathbb{C}_q[x,y]$ should produce zero. Secondly, a result of application (defined by (4.3)) of any $\mathsf{a} \in U_q(\mathfrak{sl}_2)$ to an element of the relation ideal of $\mathbb{C}_q[x,y]$ (2.1) vanishes. These conditions are to be verified in the specific cases considered below.

Note that, given a $U_q(\mathfrak{sl}_2)$-module algebra structure on the quantum plane, the action of the generator k determines an automorphism of $\mathbb{C}_q[x,y]$, which is a consequence of invertibility of k and $\Delta(\mathsf{k}) = \mathsf{k}\otimes\mathsf{k}$. In particular, it follows from (3.1) that k is determined completely by its action Ψ on the generators presented by a 1×2-matrix M_k as follows

$$\mathsf{M}_\mathsf{k} \stackrel{def}{=} \|\mathsf{k}(x), \mathsf{k}(y)\| = \|\alpha x, \beta y\| \tag{4.4}$$

for some $\alpha,\beta \in \mathbb{C}\setminus\{0\}$(which is certainly a minor of M (4.1)). Therefore every monomial $x^n y^m \in \mathbb{C}_q[x,y]$ is an eigenvector for k, and the associated eigenvalue $\alpha^n\beta^m$ will be referred to as a *weight* of this monomial, which will be written as $\mathbf{wt}(x^n y^m) = \alpha^n\beta^m$.

We will also need another minor of M as follows

$$\mathsf{M}_{\mathsf{ef}} \stackrel{def}{=} \left\| \begin{array}{cc} \mathsf{e}(x) & \mathsf{e}(y) \\ \mathsf{f}(x) & \mathsf{f}(y) \end{array} \right\|, \tag{4.5}$$

and we call M_k and M_{ef} an *action* k-*matrix* and an action ef-*matrix*, respectively.

It follows from (2.3)–(2.4) that each entry of M is a weight vector, in particular, all the nonzero monomials which constitute a specific entry should be of the same weight. Specifically, by some abuse of notation we can write

$$\mathbf{wt}(\mathsf{M}) \stackrel{def}{=} \begin{pmatrix} \mathbf{wt}(\mathsf{k}(x)) & \mathbf{wt}(\mathsf{k}(y)) \\ \mathbf{wt}(\mathsf{e}(x)) & \mathbf{wt}(\mathsf{e}(y)) \\ \mathbf{wt}(\mathsf{f}(x)) & \mathbf{wt}(\mathsf{f}(y)) \end{pmatrix}$$
$$\bowtie \begin{pmatrix} \mathbf{wt}(x) & \mathbf{wt}(y) \\ q^2\mathbf{wt}(x) & q^2\mathbf{wt}(y) \\ q^{-2}\mathbf{wt}(x) & q^{-2}\mathbf{wt}(y) \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ q^2\alpha & q^2\beta \\ q^{-2}\alpha & q^{-2}\beta \end{pmatrix},$$

where the relation $\bowtie$ between the two matrices $A = (a_{ij})$ and $B = (b_{ij})$ is defined as follows:

Notation. $A \bowtie B$ if for every pair of indices i, j such that both a_{ij} and b_{ij} are nonzero, one has $a_{ij} = b_{ij}$, e.g., $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \bowtie \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}$.

As an immediate consequence, we also have

Proposition 4.1. *Suppose that α/β is not a root of the unit. Then every homogeneous component $(\mathsf{e}(x))_n$, $(\mathsf{e}(y))_n$, $(\mathsf{f}(x))_n$, $(\mathsf{f}(y))_n$, $n \geq 0$, if nonzero, reduces to a monomial.*

P r o o f. Under our assumptions on α, β, the weights of the monomials $x^i y^{n-i}$, $0 \leq i \leq n$, of degree n are pairwise different. Since $\mathsf{e}(x)$, $\mathsf{e}(y)$, $\mathsf{f}(x)$, $\mathsf{f}(y)$ are weight vectors, our claim follows. ■

Our basic observation is that the $U_q(\mathfrak{sl}_2)$-actions in question are actually determined to a large extent by the projections of M to the lower homogeneous components of $\mathbb{C}_q[x, y]$.

Next, we denote by $(M)_i$ the i-th homogeneous component of M, whose elements are just the i-th homogeneous components of the corresponding entries of M. Thus every matrix element of M, if nonzero, admits a well-defined weight.

Let us introduce the constants $a_0, b_0, c_0, d_0 \in \mathbb{C}$ such that zero degree component of the full action matrix is

$$(\mathsf{M})_0 = \begin{pmatrix} 0 & 0 \\ a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}_0 . \tag{4.6}$$

Here we keep the subscript 0 to the matrix in the r.h.s. to emphasize the origin of this matrix as the 0-th homogeneous component of M. Note that the weights of nonzero projections of (weight) entries of M should have the same weight. Hence

$$\mathbf{wt}\left((\mathsf{M})_0\right) \bowtie \begin{pmatrix} 0 & 0 \\ q^2\alpha & q^2\beta \\ q^{-2}\alpha & q^{-2}\beta \end{pmatrix}_0 . \tag{4.7}$$

On the other hand, as all the entries of $(\mathsf{M})_0$ are constants (4.6), one also deduces

$$\mathbf{wt}\left((\mathsf{M})_0\right) \bowtie \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}_0 , \tag{4.8}$$

where the relation $\bowtie$ is understood as a set of elementwise equalities iff they are applicable, that is, when the corresponding entry of the projected matrix $(\mathsf{M})_0$ is nonzero. Therefore, it is not possible to have all nonzero entries in the 0-th homogeneous component of M simultaneously.

The classification of $U_q(\mathfrak{sl}_2)$-module algebra structures on the quantum plane we are about to suggest will be done in terms of a pair of symbolic matrices derived from the minor M_{ef} only. Now we use $(\mathsf{M}_{\mathsf{ef}})_i$ to construct a symbolic matrix $\left(\overset{\star}{\mathsf{M}}_{\mathsf{ef}}\right)_i$ whose entries are symbols $\mathbf{0}$ or $\star$ as follows: a nonzero entry of $(\mathsf{M}_{\mathsf{ef}})_i$ is replaced by $\star$, while a zero entry is replaced by the symbol $\mathbf{0}$.

In the case of 0-th components the specific elementwise relations involved in (4.7) imply that each column of $\left(\overset{\star}{\mathsf{M}}_{\mathsf{ef}}\right)_0$ should contain at least one $\mathbf{0}$, and so $\left(\overset{\star}{\mathsf{M}}_{\mathsf{ef}}\right)_0$ can be either of the following 9 matrices:

$$\begin{gathered}
\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0 , \\
\begin{pmatrix} \star & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0 , \begin{pmatrix} \mathbf{0} & \star \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0 , \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \star & \mathbf{0} \end{pmatrix}_0 , \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \star \end{pmatrix}_0 , \\
\begin{pmatrix} \star & \star \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0 , \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \star & \star \end{pmatrix}_0 , \begin{pmatrix} \star & \mathbf{0} \\ \mathbf{0} & \star \end{pmatrix}_0 , \begin{pmatrix} \mathbf{0} & \star \\ \star & \mathbf{0} \end{pmatrix}_0 .
\end{gathered} \tag{4.9}$$

An application of e and f to (2.1) by using (4.4) gives

$$y\mathsf{e}(x) - q\beta\mathsf{e}(x)y = qx\mathsf{e}(y) - \alpha\mathsf{e}(y)x, \tag{4.10}$$

$$\mathsf{f}(x)y - q^{-1}\beta^{-1}y\mathsf{f}(x) = q^{-1}\mathsf{f}(y)x - \alpha^{-1}x\mathsf{f}(y). \tag{4.11}$$

After projecting (4.10)–(4.11) to $\mathbb{C}_q[x,y]_1$ we obtain

$$a_0(1-q\beta)y = b_0(q-\alpha)x,$$
$$d_0\left(1-q\alpha^{-1}\right)x = c_0\left(q-\beta^{-1}\right)y,$$

which certainly implies

$$a_0(1-q\beta) = b_0(q-\alpha) = d_0\left(1-q\alpha^{-1}\right) = c_0\left(q-\beta^{-1}\right) = 0.$$

This determines the weight constants α and β as follows:

$$a_0 \neq 0 \Longrightarrow \beta = q^{-1}, \tag{4.12}$$
$$b_0 \neq 0 \Longrightarrow \alpha = q, \tag{4.13}$$
$$c_0 \neq 0 \Longrightarrow \beta = q^{-1}, \tag{4.14}$$
$$d_0 \neq 0 \Longrightarrow \alpha = q. \tag{4.15}$$

The deduction compared to (4.7), (4.8) implies that the symbolic matrices from (4.9) containing two $\star$'s should be excluded. Also, using (4.7) and (4.12)–(4.15) we conclude that the position of $\star$ in the remaining symbolic matrices completely determines the associated weight constants by

$$\begin{pmatrix} \star & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0 \quad\Longrightarrow\quad \alpha = q^{-2}, \quad \beta = q^{-1}, \tag{4.16}$$
$$\begin{pmatrix} \mathbf{0} & \star \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0 \quad\Longrightarrow\quad \alpha = q, \quad \beta = q^{-2}, \tag{4.17}$$
$$\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \star & \mathbf{0} \end{pmatrix}_0 \quad\Longrightarrow\quad \alpha = q^{2}, \quad \beta = q^{-1}, \tag{4.18}$$
$$\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \star \end{pmatrix}_0 \quad\Longrightarrow\quad \alpha = q, \quad \beta = q^{2}. \tag{4.19}$$

As for the matrix $\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0$, it does not determine the weight constants at all.

Next, for the 1-st homogeneous component, one has $\mathbf{wt}(\mathsf{e}(x)) = q^2\mathbf{wt}(x) \neq \mathbf{wt}(x)$ (because $q^2 \neq 1$), which implies $(\mathsf{e}(x))_1 = a_1 y$, and in a similar way we have

$$(\mathsf{M}_{\mathsf{ef}})_1 = \begin{pmatrix} a_1 y & b_1 x \\ c_1 y & d_1 x \end{pmatrix}_1$$

with $a_1, b_1, c_1, d_1 \in \mathbb{C}$. This allows us to introduce a symbolic matrix $\left(\overset{\star}{\mathsf{M}}_{\mathsf{ef}}\right)_1$ as above. Using the relations between the weights similar to (4.7), we obtain

$$\mathbf{wt}((\mathsf{M}_{\mathsf{ef}})_1) \bowtie \begin{pmatrix} q^2\alpha & q^2\beta \\ q^{-2}\alpha & q^{-2}\beta \end{pmatrix}_1 \bowtie \begin{pmatrix} \beta & \alpha \\ \beta & \alpha \end{pmatrix}_1, \tag{4.20}$$

here $\bowtie$ is implicit for a set of the elementwise equalities applicable iff the respective entry of the projected matrix $(\mathsf{M})_1$ is nonvanishing.

This means that every row and every column of $\left(\overset{\star}{\mathsf{M}}_{\mathsf{ef}}\right)_1$ may contain at least one $\mathbf{0}$. Now project (4.10)–(4.11) to $\mathbb{C}_q[x,y]_2$ to obtain

$$a_1(1 - q\beta)y^2 = b_1(q - \alpha)x^2,$$
$$d_1\left(1 - q\alpha^{-1}\right)x^2 = c_1\left(q - \beta^{-1}\right)y^2,$$

whence $a_1(1-q\beta) = b_1(q-\alpha) = d_1\left(1-q\alpha^{-1}\right) = c_1\left(q-\beta^{-1}\right) = 0$. As a consequence we have

$$a_1 \neq 0 \quad\Longrightarrow\quad \beta = q^{-1}, \tag{4.21}$$
$$b_1 \neq 0 \quad\Longrightarrow\quad \alpha = q, \tag{4.22}$$
$$c_1 \neq 0 \quad\Longrightarrow\quad \beta = q^{-1}, \tag{4.23}$$
$$d_1 \neq 0 \quad\Longrightarrow\quad \alpha = q. \tag{4.24}$$

A comparison of (4.20) with (4.21)–(4.24) allows one to discard the symbolic matrix $\begin{pmatrix} \star & \mathbf{0} \\ \mathbf{0} & \star \end{pmatrix}_1$ from the list of symbolic matrices with at least one $\mathbf{0}$ at every row or column. As for other symbolic matrices with the above property, we get

$$\begin{pmatrix} \star & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1 \Longrightarrow \alpha = q^{-3}, \quad \beta = q^{-1}, \tag{4.25}$$
$$\begin{pmatrix} \mathbf{0} & \star \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1 \Longrightarrow \alpha = q, \quad \beta = q^{-1}, \tag{4.26}$$
$$\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \star & \mathbf{0} \end{pmatrix}_1 \Longrightarrow \alpha = q, \quad \beta = q^{-1}, \tag{4.27}$$
$$\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \star \end{pmatrix}_1 \Longrightarrow \alpha = q, \quad \beta = q^{3}, \tag{4.28}$$
$$\begin{pmatrix} \mathbf{0} & \star \\ \star & \mathbf{0} \end{pmatrix}_1 \Longrightarrow \alpha = q, \quad \beta = q^{-1}. \tag{4.29}$$

The matrix $\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1$ does not determine the weight constants in the way described above.

In view of the above observations we see that in most cases a pair of symbolic matrices corresponding to 0-th and 1-st homogeneous components determines completely the weight constants of the conjectured associated actions. It will be clear from the subsequent arguments that the higher homogeneous components are redundant within the presented classification. Therefore, we introduce the table of families of $U_q(\mathfrak{sl}_2)$-module algebra structures, each family is labelled by two symbolic matrices $\left(\overset{\star}{\mathsf{M}}_{\mathsf{ef}}\right)_0$, $\left(\overset{\star}{\mathsf{M}}_{\mathsf{ef}}\right)_1$, and we call such a family a $\left[\left(\overset{\star}{\mathsf{M}}_{\mathsf{ef}}\right)_0 ; \left(\overset{\star}{\mathsf{M}}_{\mathsf{ef}}\right)_1\right]$-series. Note that the series labelled with pairs of nonzero symbolic matrices at both positions are empty, because each of the matrices determines a pair of specific weight constants α and β (4.16)–(4.19) which fails to coincide to any pair of such constants associated to the set of nonzero symbolic matrices at the second position (4.25)–(4.29). Also, the series with zero symbolic matrix at the first position and symbolic matrices containing only one $\star$ at the second position are empty.

For instance, show that $\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0 ; \begin{pmatrix} \star & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1\right]$-series is empty. If we suppose the contrary, then it follows from (2.5) that within this series we have

$$e(f(x)) - f(e(x)) = -(1 + q^2 + q^{-2})x.$$

We claim that the projection of the l.h.s. to $\mathbb{C}_q[x, y]_1$ is zero. Start with observing that, if the first symbolic matrix consists of $\mathbf{0}$'s only, one cannot reduce a degree of any monomial by applying e or f. On the other hand, within this series $f(x)$ is a sum of the monomials whose degree is at least 2. Therefore, the term $e(f(x))$ has zero projection to $\mathbb{C}_q[x, y]_1$. Similarly, $f(e(x))$ has also zero projection to $\mathbb{C}_q[x, y]_1$. The contradiction we get proves our claim.

In a similar way, one can prove that all other series with zero symbolic matrix at the first position and symbolic matrices containing only one $\star$ at the second position are empty.

In the framework of our classification we obtained 24 "empty" $\left[\left(\overset{\star}{\mathsf{M}}_{\mathsf{ef}}\right)_0 ; \left(\overset{\star}{\mathsf{M}}_{\mathsf{ef}}\right)_1\right]$-series. Next turn to "nonempty" series. We start with the simplest case in which the action ef-matrix is zero, while the full action matrix is

$$\mathsf{M} = \left\| \begin{matrix} \alpha x & \beta y \\ 0 & 0 \\ 0 & 0 \end{matrix} \right\|.$$

Theorem 4.2. *The* $\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0 ; \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1\right]$*-series consists of 4* $U_q(\mathfrak{sl}_2)$*-module algebra structures on the quantum plane given by*

$$\mathsf{k}(x) = \pm x, \qquad \mathsf{k}(y) = \pm y, \tag{4.30}$$

$$\mathsf{e}(x) = \mathsf{e}(y) = \mathsf{f}(x) = \mathsf{f}(y) = 0, \tag{4.31}$$

which are pairwise nonisomorphic.

P r o o f. It is evident that (4.30)–(4.31) determine a well-defined $U_q(\mathfrak{sl}_2)$-action consistent with the multiplication in $U_q(\mathfrak{sl}_2)$ and in the quantum plane, as well as with comultiplication in $U_q(\mathfrak{sl}_2)$. Prove that there are no other $U_q(\mathfrak{sl}_2)$-actions here. Note that an application of the l.h.s. of (2.5) to x or y has zero projection to $\mathbb{C}_q[x, y]_1$, because in this series e and f send any monomial to a sum of the monomials of higher degree. Therefore, $\left(\mathsf{k} - \mathsf{k}^{-1}\right)(x) = \left(\mathsf{k} - \mathsf{k}^{-1}\right)(y) = 0$, and hence $\alpha - \alpha^{-1} = \beta - \beta^{-1} = 0$, which leads to $\alpha, \beta \in \{1, -1\}$. To prove (4.31), note that $\mathbf{wt}(\mathsf{e}(x)) = q^2 \mathbf{wt}(x) = \pm q^2 \neq \pm 1$. On the other hand, the weight of any nonzero weight vector in this series is ± 1. This and similar arguments which involve e, f, x, y imply (4.31).

To see that the $U_q(\mathfrak{sl}_2)$-module algebra structures are pairwise nonisomorphic, observe that all the automorphisms of the quantum plane commute with the action of k (see Sect. 3). ∎

The action we reproduce in the next theorem is well known [9, 12], and here is the place for it in our classification.

Theorem 4.3. *The* $\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0 ; \begin{pmatrix} \mathbf{0} & \star \\ \star & \mathbf{0} \end{pmatrix}_1\right]$*-series consists of a one-parameter ($\tau \in \mathbb{C} \setminus \{0\}$) family of $U_q(\mathfrak{sl}_2)$-module algebra structures on the quantum plane*

$$\mathsf{k}(x) = qx, \qquad \mathsf{k}(y) = q^{-1}y, \tag{4.32}$$

$$\mathsf{e}(x) = 0, \qquad \mathsf{e}(y) = \tau x, \tag{4.33}$$

$$\mathsf{f}(x) = \tau^{-1}y, \qquad \mathsf{f}(y) = 0. \tag{4.34}$$

All these structures are isomorphic, in particular, to the action as above with $\tau = 1$.

The full action matrix related to (4.32)–(4.34) is

$$\mathsf{M} = \left\| \begin{matrix} qx & q^{-1}y \\ 0 & x \\ y & 0 \end{matrix} \right\|.$$

P r o o f. It is easy to check that (4.32)–(4.34) are compatible to all the relations in $U_q(\mathfrak{sl}_2)$ and $\mathbb{C}_q[x, y]$, hence determine a well-defined $U_q(\mathfrak{sl}_2)$-module algebra structure on the quantum plane [12].

Prove that the $\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0 ; \begin{pmatrix} \mathbf{0} & \star \\ \star & \mathbf{0} \end{pmatrix}_1\right]$-series contains no other actions except those given by (4.32)–(4.34). Let us first prove that the matrix elements of M_{ef} (4.5) contain no terms of degree higher than one, i.e. $(\mathsf{M}_{\mathsf{ef}})_n = 0$ for $n \geq 2$. A general form for $\mathsf{e}(x)$ and $\mathsf{e}(y)$ here is

$$\mathsf{e}(x) = \sum_{m+n \geq 2} \bar{\rho}_{mn} x^m y^n, \qquad \mathsf{e}(y) = \tau_{\mathsf{e}} x + \sum_{m+n \geq 2} \bar{\sigma}_{mn} x^m y^n, \tag{4.35}$$

where $\tau_{\mathsf{e}}, \bar{\rho}_{mn}, \bar{\sigma}_{mn} \in \mathbb{C}$, $\tau_{\mathsf{e}} \neq 0$. Note that in this series

$$\mathbf{wt}\,(\mathsf{M}_{\mathsf{ef}}) = \begin{pmatrix} q^3 & q \\ q^{-1} & q^{-3} \end{pmatrix}.$$

In particular, $\mathbf{wt}(\mathsf{e}(x)) = q^3$ and $\mathbf{wt}(\mathsf{e}(y)) = q$, which reduces the general form (4.35) to a sum of terms with each one having the same fixed weight

$$\mathsf{e}(x) = \sum_{m\geq 0} \rho_m x^{m+3} y^m, \tag{4.36}$$

$$\mathsf{e}(y) = \tau_{\mathsf{e}} x + \sum_{m\geq 0} \sigma_m x^{m+2} y^{m+1}. \tag{4.37}$$

Substitute (4.36)–(4.37) to (4.10) and then project it to the one-dimensional subspace $\mathbb{C}x^{m+3}y^{m+1}$ (for every $m \geq 0$) to obtain

$$\frac{\rho_m}{\sigma_m} = -q\frac{1-q^{m+1}}{1-q^{m+3}}.$$

In a similar way, the relations $\mathbf{wt}(\mathsf{f}(x)) = q^{-1}$ and $\mathbf{wt}(\mathsf{f}(y)) = q^{-3}$ imply that

$$\mathsf{f}(x) = \tau_f y + \sum_{n\geq 0} \rho'_n x^{n+1} y^{n+2}, \tag{4.38}$$

$$\mathsf{f}(y) = \sum_{n\geq 0} \sigma'_n x^n y^{n+3}, \tag{4.39}$$

where $\tau_f \in \mathbb{C} \setminus \{0\}$. An application of (4.38)–(4.39) and (4.11) with subsequent projection to $\mathbb{C}x^{n+1}y^{n+3}$ (for every $n \geq 0$) allows one to get

$$\frac{\rho'_n}{\sigma'_n} = -q^{-1}\frac{1-q^{n+3}}{1-q^{n+1}}.$$

Thus we have

$$\mathsf{M}_{\mathsf{ef}} = \begin{pmatrix} 0 & \tau_e x \\ \tau_f y & 0 \end{pmatrix} + \sum_{n\geq 0} \begin{pmatrix} -\mu_n q(1-q^{n+1})x^{n+3}y^n & \mu_n(1-q^{n+3})x^{n+2}y^{n+1} \\ \nu_n(1-q^{n+3})x^{n+1}y^{n+2} & -\nu_n q(1-q^{n+1})x^n y^{n+3} \end{pmatrix},$$

where $\mu_n, \nu_n \in \mathbb{C}$. We intend to prove that the second matrix in this sum is zero. Assume the contrary. In the case there exist both nonzero μ_n's and ν_n's, and since the sums here are finite, for the first row choose the largest index n_e with $\mu_{n_e} \neq 0$ and for second row, the largest index n_f with $\nu_{n_f} \neq 0$. Then using (2.7)–(2.8), we deduce that the highest degree of the monomials in $(ef - fe)(x)$ is $2n_e + 2n_f + 5$. This monomial appears to be unique, and its precise computation gives $\mu_{n_e}\nu_{n_f} q^{n_e n_f - 1}(1-q^{n_2+n_f+4})(1-q^{2n_e+2n_f+6})x^{n_e+n_f+3}y^{n_e+n_f+2}$. Therefore, $(ef - fe)(x)$ has a nonzero projection onto the one dimensional subspace spanned by the monomial $x^{n_e+n_f+3}y^{n_e+n_f+2}$, the latter being of degree higher than 1. This contradicts to (2.5) whose r.h.s. applied to x has degree 1.

In the case when all ν_n's are zero and some μ_n's are nonvanishing we have that the highest degree monomial of $(ef - fe)(x)$ is of the form

$$\tau_f \mu_{n_e} \frac{(1-q^{n_e+3})(1-q^{2n_e+4})}{q^{n_e+1}(1-q^2)} x^{n_e+2} y^{n_e+1},$$

which is nonzero under our assumptions on q. This again produces the same contradiction as above. In the opposite case when all μ_n's are zero and some ν_n's are nonvanishing, a similar computation works, which also leads to a contradiction. Therefore, all μ_n's and ν_n's are zero.

Finally, an application of (2.5) to x yields $\tau_e \tau_f = 1$ so that $\tau_e = \tau$ and $\tau_f = \tau^{-1}$ for some $\tau \in \mathbb{C} \setminus \{0\}$.

We claim that all the actions corresponding to nonzero τ are isomorphic to the specific action with $\tau = 1$. The desired isomorphism is given by the automorphism $\Phi_\tau : x \mapsto x$, $y \mapsto \tau y$. In particular, $\left(\Phi_\tau \mathsf{e}_\tau \Phi_\tau^{-1}\right)(y) = \tau^{-1}\Phi_\tau(\tau x) = x = \mathsf{e}_1(y)$, where $\mathsf{e}_\tau(y)$ denotes the action from (4.33) with an arbitrary $\tau \neq 0$. ■

Now we consider the actions whose symbolic matrix $\left(\overset{\star}{\mathsf{M}}_{\mathsf{ef}}\right)_0$ contains one $\star$. Seemingly, the corresponding actions described below never appeared in the literature before, so we present more detailed computations.

Theorem 4.4. *The* $\left[\begin{pmatrix} \mathbf{0} & \star \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0 ; \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1\right]$*-series consists of a one-parameter ($b_0 \in \mathbb{C} \setminus \{0\}$) family of $U_q(\mathfrak{sl}_2)$-module algebra structures on the quantum plane*

$$\mathsf{k}(x) = qx, \qquad \mathsf{k}(y) = q^{-2}y, \tag{4.40}$$

$$\mathsf{e}(x) = 0, \qquad \mathsf{e}(y) = b_0, \tag{4.41}$$

$$\mathsf{f}(x) = b_0^{-1}xy, \qquad \mathsf{f}(y) = -qb_0^{-1}y^2. \tag{4.42}$$

All these structures are isomorphic, in particular to the action as above with $b_0 = 1$.

The full action matrix of an action within this isomorphism class is of the form

$$\mathsf{M} = \begin{Vmatrix} qx & q^{-2}y \\ 0 & 1 \\ xy & -qy^2 \end{Vmatrix}.$$

P r o o f. First we demonstrate that an extension of (4.40)–(4.42) to the entire action of $U_q(\mathfrak{sl}_2)$ on $\mathbb{C}_q[x, y]$ passes through all the relations. It is clear that (4.40) is compatible with the relation $\mathsf{k}\mathsf{k}^{-1} = \mathsf{k}^{-1}\mathsf{k} = \mathbf{1}$. Then we apply the relations (2.3)–(2.5) to the quantum plane generators

$$\begin{aligned}
(\mathsf{ke} - q^2\mathsf{ek})(x) &= \mathsf{k}(0) - q^3\mathsf{e}(x) = 0,\\
(\mathsf{ke} - q^2\mathsf{ek})(y) &= \mathsf{k}(b_0) - \mathsf{e}(y) = b_0 - b_0 = 0,\\
(\mathsf{kf} - q^{-2}\mathsf{fk})(x) &= \mathsf{k}\left(b_0^{-1}xy\right) - q^{-1}\mathsf{f}(x)\\
&= b_0^{-1}q^{-1}xy - q^{-1}b_0^{-1}xy = 0,\\
(\mathsf{kf} - q^{-2}\mathsf{fk})(y) &= \mathsf{k}\left(-qb_0^{-1}y^2\right) - q^{-4}\mathsf{f}(y)\\
&= -qb_0^{-1}q^{-4}y^2 + q^{-4}\left(qb_0^{-1}y^2\right) = 0,\\
\left(\mathsf{ef} - \mathsf{fe} - \frac{\mathsf{k} - \mathsf{k}^{-1}}{q - q^{-1}}\right)(x) &= \mathsf{e}\left(b_0^{-1}xy\right) - \mathsf{f}(0) - x = b_0^{-1}\mathsf{e}(xy) - x\\
&= b_0^{-1}x\mathsf{e}(y) + b_0^{-1}\mathsf{e}(x)\mathsf{k}(y) - x = 0,\\
\left(\mathsf{ef} - \mathsf{fe} - \frac{\mathsf{k} - \mathsf{k}^{-1}}{q - q^{-1}}\right)(y) &= -qb_0^{-1}\mathsf{e}\left(y^2\right) - \mathsf{f}(b_0) - \frac{q^{-2} - q^2}{q - q^{-1}}y\\
&= -qb_0^{-1}\mathsf{e}\left(y^2\right) + \left(q + q^{-1}\right)y\\
&= -qb_0^{-1}y\mathsf{e}(y) - qb_0^{-1}\mathsf{e}(y)\mathsf{k}(y) + \left(q + q^{-1}\right)y\\
&= -qy - q^{-1}y + \left(q + q^{-1}\right)y = 0.
\end{aligned}$$

Now apply the generators of $U_2\left(\mathfrak{sl}_2\right)$ to (2.1) and get

$$\begin{aligned}
\mathsf{k}\left(yx - qxy\right) &= q^{-2}y \cdot qx - qqx \cdot q^{-2}y = 0,\\
\mathsf{e}\left(yx - qxy\right) &= y\mathsf{e}\left(x\right) + \mathsf{e}\left(y\right)\mathsf{k}\left(x\right) - qx\mathsf{e}\left(y\right) - q\mathsf{e}\left(x\right)\mathsf{k}\left(y\right)\\
&= 0 + b_0qx - qxb_0 - 0 = 0,\\
\mathsf{f}\left(yx - qxy\right) &= \mathsf{f}\left(y\right)x + \mathsf{k}^{-1}\left(y\right)\mathsf{f}\left(x\right) - q\mathsf{f}\left(x\right)y - q\mathsf{k}^{-1}\left(x\right)\mathsf{f}\left(y\right)\\
&= -qb_0^{-1}y^2x + q^2yb_0^{-1}xy - qb_0^{-1}xy \cdot y + qq^{-1}x \cdot qb_0^{-1}y^2\\
&= -q^3b_0^{-1}xy^2 + q^3b_0^{-1}xy^2 - qb_0^{-1}xy^2 + qb_0^{-1}xy^2 = 0.
\end{aligned}$$

Next prove that $\left[\begin{pmatrix} \mathbf{0} & \star \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0 ; \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1\right]$-series contains no actions except (4.40)–(4.42). Show that the matrix elements of M_{ef} (4.5) have no terms of degree higher than two, viz. $(\mathsf{M}_{\mathsf{ef}})_n = 0$ for $n \geq 3$. Now a general form for $\mathsf{e}(x)$, $\mathsf{e}(y)$, $\mathsf{f}(x)$, $\mathsf{f}(y)$ is

$$\mathsf{e}(x) = \sum_{m+n \geq 0} \bar{\rho}_{mn}x^my^n, \qquad \mathsf{e}(y) = \sum_{m+n \geq 0} \bar{\sigma}_{mn}x^my^n, \tag{4.43}$$

$$\mathsf{f}(x) = \sum_{m+n \geq 0} \bar{\rho}'_{mn}x^my^n, \qquad \mathsf{f}(y) = \sum_{m+n \geq 0} \bar{\sigma}'_{mn}x^my^n \tag{4.44}$$

where $\bar{\rho}_{mn}, \bar{\sigma}_{mn}, \bar{\rho}'_{mn}, \bar{\sigma}'_{mn} \in \mathbb{C}$. Within this series one has the matrix of weights

$$\mathbf{wt}(\mathsf{M}_{\mathsf{ef}}) = \begin{pmatrix} q^3 & 1 \\ q^{-1} & q^{-4} \end{pmatrix}.$$

In view of this, the general form (4.43)–(4.44) should be a sum of terms of the same weight

$$\mathsf{e}(x) = \sum_{m \geq 0} \rho_m x^{2m+3} y^m, \tag{4.45}$$

$$\mathsf{e}(y) = b' + \sum_{m \geq 0} \sigma_m x^{2m+2} y^{m+1}, \tag{4.46}$$

$$\mathsf{f}(x) = b'' xy + \sum_{n \geq 0} \rho'_n x^{2n+3} y^{n+2}, \tag{4.47}$$

$$\mathsf{f}(y) = b''' y^2 + \sum_{n \geq 0} \sigma'_n x^{2n+2} y^{n+3}. \tag{4.48}$$

Now we combine (4.45)–(4.46), (4.47)–(4.48)) with (4.10), (4.11), respectively, then project the resulting relation to the one-dimensional subspace $\mathbb{C}x^{2m+3}y^{m+2}$ (resp. $\mathbb{C}x^{2n+3}y^{n+3}$) (for every $m \geq 0$, resp. $n \geq 0$) to obtain

$$\frac{\rho_m}{\sigma_m} = -q^2 \frac{1 - q^{m+1}}{1 - q^{2m+4}},$$
$$\frac{\rho'_n}{\sigma'_n} = -q^{-1} \frac{1 - q^{n+3}}{1 - q^{2n+4}}.$$

Thus we get

$$\mathsf{M}_{\mathsf{ef}} = \begin{pmatrix} 0 & b' \\ b'' xy & b''' y^2 \end{pmatrix} + \sum_{n \geq 0} \begin{pmatrix} \mu_n q^2 (1 - q^{n+1}) x^{2n+3} y^n & -\mu_n (1 - q^{2n+4}) x^{2n+2} y^{n+1} \\ -\nu_n (1 - q^{n+3}) x^{2n+3} y^{n+2} & \nu_n q (1 - q^{2n+4}) x^{2n+2} y^{n+3} \end{pmatrix}, \tag{4.49}$$

where $\mu_n, \nu_n \in \mathbb{C}$. To prove that the second matrix vanishes, assume the contrary. First consider the case when there exist both nonzero μ_n's and ν_n's. As the sums here are finite, for the first row choose the largest index n_e with $\mu_{n_e} \neq 0$ and for the second row, the largest index n_f with $\nu_{n_f} \neq 0$. After applying (2.7)–(2.8) one concludes that the highest degree of monomials in $(ef - fe)(x)$ is $3n_e + 3n_f + 7$. This monomial is unique, and its computation gives

$$\mu_{n_e} \nu_{n_f} q^{2n_e n_f + 2n_e} (1 - q^{n_e + n_f + 4})(1 - q^{2n_e + 2n_f + 6}) x^{2n_e + 2n_f + 5} y^{n_e + n_f + 2}. \tag{4.50}$$

Under our assumptions on q, since $n_e \geq 0$, $n_f \geq 0$, $\mu_{n_e}\nu_{n_f} \neq 0$, it becomes clear that (4.50) is a nonzero monomial of degree higher than 1. This breaks (2.5) whose r.h.s. applied to x has degree 1. An application of (2.5) to x and y together with (4.49) leads to (up to terms of degree higher than 1)

$$\left(\mathsf{ef} - \mathsf{fe} - \frac{\mathsf{k} - \mathsf{k}^{-1}}{q - q^{-1}}\right)(x) = 0 = b'b''x - x,$$

$$\left(\mathsf{ef} - \mathsf{fe} - \frac{\mathsf{k} - \mathsf{k}^{-1}}{q - q^{-1}}\right)(y) = 0 = b'b'''(1 + q^{-2})y + \left(q + q^{-1}\right)y,$$

which yields

$$b' = b_0, \qquad b'' = b_0^{-1}, \qquad b''' = -qb_0^{-1}$$

for some $b_0 \neq 0$.

A similar, but simpler computation also shows that in the case when all ν_n's are zero and some μ_n's are nonzero we have the highest degree monomial of $(ef - fe)(x)$ of the form

$$b_0^{-1}\mu_{n_e}\frac{(1 - q^{n_e+3})(q^{2n_e+4} - 1)}{1 - q^2}x^{2n_e+3}y^{n_e+1}.$$

This monomial is nonzero due to our assumption on q, which gives the same contradiction as above. The opposite case, when all μ_n's are zero and some ν_n's are nonvanishing, can be treated similarly and also leads to a contradiction. Therefore, all μ_n's and ν_n's are zero. This gives the desired relations (4.40)–(4.42).

Finally we show that the actions (4.40)–(4.42) with nonzero b_0 are isomorphic to the specific action with $b_0 = 1$. The desired isomorphism is as follows Φ_{b_0} : $x \mapsto x$, $y \mapsto b_0 y$. In fact,

$$\left(\Phi_{b_0}\mathsf{e}_{b_0}\Phi_{b_0}^{-1}\right)(y) = \Phi_{b_0}\mathsf{e}_{b_0}\left(b_0^{-1}y\right) = b_0^{-1}\Phi_{b_0}(b_0) = \Phi_{b_0}(1) = 1 = \mathsf{e}_1(y),$$

$$\left(\Phi_{b_0}\mathsf{f}_{b_0}\Phi_{b_0}^{-1}\right)(x) = \Phi_{b_0}\mathsf{f}_{b_0}(x) = b_0^{-1}\Phi_{b_0}(xy) = b_0^{-1}b_0xy = xy = \mathsf{f}_1(x),$$

$$\left(\Phi_{b_0}\mathsf{f}_{b_0}\Phi_{b_0}^{-1}\right)(y) = \Phi_{b_0}\mathsf{f}_{b_0}\left(b_0^{-1}y\right) = b_0^{-1}\Phi_{b_0}\left(-qb_0^{-1}y^2\right) = -qb_0^{-2}b_0^2y^2 =$$
$$= -qy^2 = \mathsf{f}_1(y).$$

The theorem is proved. ■

Theorem 4.5. *The* $\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \star & \mathbf{0} \end{pmatrix}_0 ; \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1\right]$*-series consists of a one-parameter ($c_0 \in \mathbb{C} \setminus \{0\}$) family of $U_q(\mathfrak{sl}_2)$-module algebra structures on the quantum plane*

$$\mathsf{k}(x) = q^2x, \qquad \mathsf{k}(y) = q^{-1}y, \tag{4.51}$$

$$\mathsf{e}(x) = -qc_0^{-1}x^2, \qquad \mathsf{e}(y) = c_0^{-1}xy, \tag{4.52}$$

$$\mathsf{f}(x) = c_0, \qquad \mathsf{f}(y) = 0. \tag{4.53}$$

All these structures are isomorphic, in particular to the action as above with $c_0 = 1$.

The full action matrix for this isomorphism class (with $c_0 = 1$) is

$$\mathsf{M} = \left\| \begin{array}{cc} q^2 x & q^{-1} y \\ -q x^2 & xy \\ 1 & 0 \end{array} \right\|.$$

P r o o f. Quite literally repeats that of the previous theorem. ■

Theorem 4.6. *The* $\left[\begin{pmatrix} \star & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0 ; \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1 \right]$*-series consists of a three-parameter* ($a_0 \in \mathbb{C} \setminus \{0\}$, $s, t \in \mathbb{C}$) *family of* $U_q(\mathfrak{sl}_2)$*-actions on the quantum plane*

$$\mathsf{k}(x) = q^{-2} x, \qquad \mathsf{k}(y) = q^{-1} y, \tag{4.54}$$

$$\mathsf{e}(x) = a_0, \qquad \mathsf{e}(y) = 0, \tag{4.55}$$

$$\mathsf{f}(x) = -q a_0^{-1} x^2 + t y^4, \qquad \mathsf{f}(y) = -q a_0^{-1} xy + s y^3. \tag{4.56}$$

The generic domain $\{(a_0, s, t) | \ s \neq 0, \ t \neq 0\}$ *with respect to the parameters splits into uncountably many disjoint subsets* $\{(a_0, s, t) | \ s \neq 0, \ t \neq 0, \ \varphi = \mathrm{const}\}$, *where* $\varphi = \dfrac{t}{a_0 s^2}$. *Each of those subsets corresponds to an isomorphism class of* $U_q(\mathfrak{sl}_2)$*-module algebra structures. Additionally, there exist three more isomorphism classes corresponding to the subsets*

$$\{(a_0, s, t) | \ s \neq 0, t = 0\}, \quad \{(a_0, s, t) | s = 0, t \neq 0\}, \quad \{(a_0, s, t) | \ s = 0, t = 0\}.$$

P r o o f. A routine verification demonstrates that (4.54)–(4.56) pass through all the relations as before, hence admit an extension to a well-defined series of $U_q(\mathfrak{sl}_2)$-actions on the quantum plane.

Now check that $\left[\begin{pmatrix} \star & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0 ; \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1 \right]$-series contains no other actions except (4.54)–(4.56). First consider the polynomial $\mathsf{e}(x)$. Since its weight is $q^2 \mathbf{wt}(x) = 1$, and the weight of any monomial other than constant is a negative degree of q (within the series under consideration), hence not 1, one gets $\mathsf{e}(x) = a_0$. In a similar way, the only possibility for $\mathsf{e}(y)$ is zero, because if not, $\mathbf{wt}(\mathsf{e}(y)) = q^2 \mathbf{wt}(y) = q$, which is impossible in view of the above observations.

Turn to $\mathsf{f}(x)$ and observe that $\mathbf{wt}(\mathsf{f}(x)) = q^{-4}$. It is easy to see that all the monomials with this weight are x^2, xy^2, y^4, that is $\mathsf{f}(x) = ux^2 + vxy^2 + wy^4$. In a similar way $\mathbf{wt}(\mathsf{f}(y)) = q^{-3}$ and so $\mathsf{f}(y) = zxy + sy^3$. A substitution to (2.5) yields $\left(1 + q^{-2}\right) u a_0 = -\left(q + q^{-1}\right)$, $v = 0$, $z a_0 q^{-1} = -1$. Note that (4.11) gives no new relations for u, v, z and provides no restriction on w and s at all. This leads to (4.56).

To distinguish the isomorphism classes of the structures within this series, we use Theorem 3.1 in writing down the general form of an automorphism of $\mathbb{C}_q[x,y]$ as $\Phi_{\theta,\omega}: x \mapsto \theta x,\ y \mapsto \omega y$. Certainly, this commutes with the action of k. For other generators we get

$$\begin{aligned}
\left(\Phi_{\theta,\omega}\mathsf{e}_{a_0,s,t}\Phi_{\theta,\omega}^{-1}\right)(x) &= \Phi_{\theta,\omega}\mathsf{e}_{a_0,s,t}\left(\theta^{-1}x\right) = \theta^{-1}a_0,\\
\left(\Phi_{\theta,\omega}\mathsf{e}_{a_0,s,t}\Phi_{\theta,\omega}^{-1}\right)(y) &= \Phi_{\theta,\omega}\mathsf{e}_{a_0,s,t}\left(\omega^{-1}y\right) = \omega^{-1}\Phi_{\theta,\omega}\mathsf{e}_{a_0,s,t}(y) = 0,\\
\left(\Phi_{\theta,\omega}\mathsf{f}_{a_0,s,t}\Phi_{\theta,\omega}^{-1}\right)(x) &= \Phi_{\theta,\omega}\mathsf{f}_{a_0,s,t}\left(\theta^{-1}x\right) = \theta^{-1}\Phi_{\theta,\omega}\left(-qa_0^{-1}x^2 + ty^4\right)\\
&= -qa_0^{-1}\theta x^2 + \theta^{-1}t\omega^4 y^4,\\
\left(\Phi_{\theta,\omega}\mathsf{f}_{a_0,s,t}\Phi_{\theta,\omega}^{-1}\right)(y) &= \Phi_{\theta,\omega}\mathsf{f}_{a_0,s,t}\left(\omega^{-1}y\right) = \omega^{-1}\Phi_{\theta,\omega}\left(-qa_0^{-1}xy + sy^3\right)\\
&= -q\theta a_0^{-1}xy + s\omega^2 y^3.
\end{aligned}$$

That is, the automorphism $\Phi_{\theta,\omega}$ transforms the parameters of actions (4.55)–(4.56) as follows:

$$a_0 \mapsto \theta^{-1}a_0, \qquad s \mapsto \omega^2 s, \qquad t \mapsto \theta^{-1}\omega^4 t.$$

In particular, this means that within the domain $\{s \neq 0,\ t \neq 0\}$ one obtains an invariant $\varphi = \dfrac{t}{a_0 s^2}$ of the isomorphism class. Obviously, the complement to this domain further splits into three distinct subsets $\{s \neq 0,\ t = 0\}$, $\{s = 0,\ t \neq 0\}$, $\{s = 0,\ t = 0\}$ corresponding to the isomorphism classes listed in the formulation, and our result follows. ■

Note that up to isomorphism of $U_q(\mathfrak{sl}_2)$-module algebra structure, the full action matrix corresponding to (4.54)–(4.56) is of the form

$$\mathsf{M} = \left\|\begin{matrix} q^{-2}x & q^{-1}y \\ 1 & 0 \\ -qx^2 + ty^4 & -qxy + sy^3 \end{matrix}\right\|.$$

Theorem 4.7. *The* $\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \star \end{pmatrix}_0 ; \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1\right]$*-series consists of three-parameter ($d_0 \in \mathbb{C}\setminus\{0\}$, $s,t \in \mathbb{C}$) family of $U_q(\mathfrak{sl}_2)$-actions on the quantum plane*

$$\mathsf{k}(x) = qx, \qquad \mathsf{k}(y) = q^2 y, \tag{4.57}$$

$$\mathsf{e}(x) = -qd_0^{-1}xy + sx^3, \qquad \mathsf{e}(y) = -qd_0^{-1}y^2 + tx^4, \tag{4.58}$$

$$\mathsf{f}(x) = 0, \qquad \mathsf{f}(y) = d_0. \tag{4.59}$$

Here we have the domain $\{(d_0, s, t)|\ s \neq 0,\ t \neq 0\}$ *which splits into the disjoint subsets* $\{(d_0, s, t)|\ s \neq 0,\ t \neq 0,\ \varphi = \text{const}\}$ *with* $\varphi = \dfrac{t}{d_0 s^2}$. *This uncountable family of subsets is in one-to-one correspondence to the isomorphism classes of* $U_q(\mathfrak{sl}_2)$*-module algebra structures. Aside of those, one also has three more isomorphism classes labelled by the subsets* $\{(d_0, s, t)|\ s \neq 0,\ t = 0\}$, $\{(d_0, s, t)|\ s = 0,\ t \neq 0\}$, $\{(d_0, s, t)|\ s = 0,\ t = 0\}$.

P r o o f. Is the same as that of the previous theorem. ■

Here, also up to isomorphism of $U_q(\mathfrak{sl}_2)$-module algebra structures, the full action matrix is

$$\mathsf{M} = \left\| \begin{array}{cc} qx & q^2 y \\ -qxy + sx^3 & -qy^2 + tx^4 \\ 0 & 1 \end{array} \right\|.$$

R e m a r k 4.8. There could be no isomorphisms between the $U_q(\mathfrak{sl}_2)$-module algebra structures on $\mathbb{C}_q[x, y]$ picked from different series. This is because every automorphism of the quantum plane commutes with the action of k, hence, the restrictions of isomorphic actions to k are always the same. On the other hand, the actions of k in different series are different.

R e m a r k 4.9. The list of $U_q(\mathfrak{sl}_2)$-module algebra structures on $\mathbb{C}_q[x, y]$ presented in the theorems of this section is complete. This is because the assumptions of those theorems exhaust all admissible forms for the components $(\mathsf{M}_{\mathsf{ef}})_0$, $(\mathsf{M}_{\mathsf{ef}})_1$ of the action ef-matrix.

R e m a r k 4.10. In all series of $U_q(\mathfrak{sl}_2)$-module algebra structures listed in Theorems 4.2–4.7, except the series $\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0; \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1\right]$, the weight constants α and β satisfy the assumptions of Proposition 4.1. So the claim of this proposition is well visible in a rather simple structure of nonzero homogeneous components of $\mathsf{e}(x)$, $\mathsf{e}(y)$, $\mathsf{f}(x)$, $\mathsf{f}(y)$, which everywhere reduce to monomials.

5. Composition Series

Let us view the $U_q(\mathfrak{sl}_2)$-module algebra structures on $\mathbb{C}_q[x, y]$ listed in the theorems of the previous section merely as representations of $U_q(\mathfrak{sl}_2)$ in the *vector space* $\mathbb{C}_q[x, y]$. Our immediate intention is to describe the composition series for these representations.

Proposition 5.1. *The representations corresponding to* $\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0; \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1\right]$*-series described in* (4.30)–(4.31) *split into the direct sum* $\mathbb{C}_q[x, y] = \oplus_{m=0}^{\infty} \oplus_{n=0}^{\infty}$

$\mathbb{C}x^m y^n$ *of (irreducible) one-dimensional subrepresentations. These subrepresentations may belong to two isomorphism classes, depending on the weight of a specific monomial* $x^m y^n$ *which can be* ± 1 *(see Th. 4.2).*

P r o o f. Since e and f are represented by zero operators and the monomials $x^m y^n$ are eigenvectors for k, then every direct summand is $U_q(\mathfrak{sl}_2)$-invariant. ■

Now turn to nontrivial $U_q(\mathfrak{sl}_2)$-module algebra structures and start with the well-known case [8, 12].

Proposition 5.2. *The representations corresponding to* $\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0 ; \begin{pmatrix} \mathbf{0} & \star \\ \star & \mathbf{0} \end{pmatrix}_1\right]$*-series described in* (4.32)–(4.34) *split into the direct sum* $\mathbb{C}_q[x,y] = \oplus_{n=0}^{\infty} \mathbb{C}_q[x,y]_n$ *of irreducible finite-dimensional subrepresentations, where* $\mathbb{C}_q[x,y]_n$ *is the* n*-th homogeneous component (introduced in Sect. 3) with* $\dim \mathbb{C}_q[x,y]_n = n+1$ *and the isomorphism class of this subrepresentation is* $\mathcal{V}_{1,n}$ *[8, Ch. VI].*

P r o o f. Is that of Theorem VII.3.3 (b) from [8]. ■

In the subsequent observations we encounter a split picture which does not reduce to a collection of purely finite-dimensional sub- or quotient modules. We recall the definition of the Verma modules in our specific case of $U_q(\mathfrak{sl}_2)$.

Definition 5.3. *A Verma module* $\mathcal{V}(\lambda)$ $(\lambda \in \mathbb{C} \setminus \{0\})$ *is a vector space with a basis* $\{v_i,\ i \geq 0\}$*, where the* $U_q(\mathfrak{sl}_2)$ *action is given by*

$$\mathsf{k}v_i = \lambda q^{-2i} v_i, \qquad \mathsf{k}^{-1} v_i = \lambda^{-1} q^{2i} v_i,$$
$$\mathsf{e}v_0 = 0, \qquad \mathsf{e}v_{i+1} = \frac{\lambda q^{-i} - \lambda^{-1} q^i}{q - q^{-1}} v_i, \qquad \mathsf{f}v_i = \frac{q^{i+1} - q^{-i-1}}{q - q^{-1}} v_{i+1}.$$

Note that the Verma module $\mathcal{V}(\lambda)$ is generated by the highest weight vector v_0 whose weight is λ (for details see, e.g., [8]).

Proposition 5.4. *The representations corresponding to* $\left[\begin{pmatrix} \mathbf{0} & \star \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0 ; \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1\right]$*-series described in* (2.2)–(4.42) *split into the direct sum of subrepresentations* $\mathbb{C}_q[x,y] = \oplus_{n=0}^{\infty} \mathcal{V}_n$*, where* $\mathcal{V}_n = x^n \mathbb{C}[y]$*. Each* $\mathcal{V}_n$ *admits a composition series of the form* $0 \subset \mathcal{J}_n \subset \mathcal{V}_n$*. The simple submodule* $\mathcal{J}_n$ *of dimension* $n+1$ *is the linear span of* $x^n, x^n y, \ldots, x^n y^{n-1}, x^n y^n$*, whose isomorphism class is* $\mathcal{V}_{1,n}$ *and* $\mathcal{J}_n$ *is not a direct summand in the category of* $U_q(\mathfrak{sl}_2)$*-modules (there exist no submodule* $\mathcal{W}$ *such that* $\mathcal{V}_n = \mathcal{J}_n \oplus \mathcal{W}$*). The quotient module* $\mathcal{V}_n \diagup \mathcal{J}_n = \mathcal{Z}_n$ *is isomorphic to the (simple) Verma module* $\mathcal{V}\left(q^{-n-2}\right)$*.*

P r o o f. Due to the isomorphism statement of Theorem 4.4, it suffices to set the parameter of the series $b_0 = 1$ in (2.2)–(4.42). An application of e and f

to the basis elements of $\mathbb{C}_q[x,y]$ gives

$$\mathsf{e}(x^n y^p) = q^{1-p}\frac{q^p - q^{-p}}{q-q^{-1}} x^n y^{p-1} \neq 0, \qquad \forall p > 0, \tag{5.1}$$

$$\mathsf{e}(x^n) = 0, \tag{5.2}$$

$$\mathsf{f}(x^n y^p) = q^{-n}\frac{q^{2n} - q^{2p}}{q-q^{-1}} x^n y^{p+1}, \qquad \forall p \geq 0, \tag{5.3}$$

which already implies that each $\mathcal{V}_n$ is $U_q(\mathfrak{sl}_2)$-invariant. Also $\mathcal{J}_n$ is a submodule of $\mathcal{V}_n$ generated by the highest weight vector x^n, as the sequence of weight vectors $\mathsf{f}(x^n y^p)$ terminates because $\mathsf{f}(x^n y^n) = 0$. The highest weight of $\mathcal{J}_n$ is q^n, hence by Theorem VI.3.5 of [8], the submodule $\mathcal{J}_n$ is simple and its isomorphism class is $\mathcal{V}_{1,n}$.

Now assume the contrary to our claim, that is $\mathcal{V}_n = \mathcal{J}_n \oplus \mathcal{W}$ for some submodule $\mathcal{W}$ of $\mathcal{V}_n$, and $\mathcal{V}_n \ni x^n y^{n+1} = u + w$, $u \in \mathcal{J}_n$, $w \in \mathcal{W}$ is the associated decomposition. In view of (5.1)–(5.2), an application of e^{n+1} gives $A(q)x^n = \mathsf{e}^{n+1}(w)$ for some nonzero constant $A(q)$, because $\mathsf{e}^{n+1}|_{\mathcal{J}_n} = 0$. This is a contradiction, because $\mathcal{J}_n \cap \mathcal{W} = \{0\}$, thus there exist no submodule $\mathcal{W}$ as above.

The quotient module $\mathcal{Z}_n$ is spanned by its basis vectors $z_{n+1}, z_{n+2}, \ldots$ which are the projections of $x^n y^{n+1}, x^n y^{n+2}, \ldots$, respectively, to $\mathcal{V}_n / \mathcal{J}_n$. It follows from (5.1), that z_{n+1} is the highest weight vector whose weight is q^{-n-2}, and it generates $\mathcal{Z}_n$ by (5.3). Now the universality property of the Verma modules (see, e.g., [8, Prop. VI.3.7]) implies that there exists a surjective morphism of modules $\Pi : \mathcal{V}\left(q^{-n-2}\right) \to \mathcal{Z}_n$. It follows from Proposition 2.5 of [7] that $\ker \Pi = 0$, hence Π is an isomorphism. ■

The next series, unlike the previous one, involves the lowest weight Verma modules. In all other respects the proof of the following proposition is the same (we also set here $d_0 = 1$).

Proposition 5.5. *The representations corresponding to* $\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \star & \mathbf{0} \end{pmatrix}_0 ; \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1\right]$*-series described in* (4.51)–(4.53) *split into the direct sum of subrepresentations* $\mathbb{C}_q[x,y] = \oplus_{n=0}^{\infty} \mathcal{V}_n$*, where* $\mathcal{V}_n = \mathbb{C}[x]y^n$*. Each* $\mathcal{V}_n$ *admits a composition series of the form* $0 \subset \mathcal{J}_n \subset \mathcal{V}_n$*. The simple submodule* $\mathcal{J}_n$ *of dimension* $n+1$ *is the linear span of* $y^n, xy^n, \ldots, x^{n-1}y^n, x^n y^n$*. This is a finite-dimensional* $U_q(\mathfrak{sl}_2)$*-module whose lowest weight vector is* y^n *with weight* q^{-n}*, and its isomorphism class is* $\mathcal{V}_{1,n}$*. Now the submodule* $\mathcal{J}_n$ *is not a direct summand in the category of* $U_q(\mathfrak{sl}_2)$*-modules (there exists no submodule* $\mathcal{W}$ *such that* $\mathcal{V}_n = \mathcal{J}_n \oplus \mathcal{W}$*). The quotient module* $\mathcal{V}_n / \mathcal{J}_n = \mathcal{Z}_n$ *is isomorphic to the (simple) Verma module with lowest weight* q^{n+2}*.*

Now turn to considering the three parameter series as in Theorems 4.6, 4.7. Despite we have now three parameters, the entire series has the same split picture.

Proposition 5.6. *The representations corresponding to* $\left[\begin{pmatrix} \star & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0; \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1\right]$*-series described in* (4.54)–(4.56) *split into the direct sum of subrepresentations* $\mathbb{C}_q[x,y] = \oplus_{n=0}^{\infty} \mathcal{V}_n$*, where* $\mathcal{V}_n$ *is a submodule generated by its highest weight vector* y^n*. Each* $\mathcal{V}_n$ *with* $n \geq 1$ *is isomorphic to a simple highest weight Verma module* $\mathcal{V}(q^{-n})$*. The submodule* $\mathcal{V}_0$ *admits a composition series of the form* $0 \subset \mathcal{J}_0 \subset \mathcal{V}_0$*, where* $\mathcal{J}_0 = \mathbb{C}\mathbf{1}$*. The submodule* $\mathcal{J}_0$ *is not a direct summand in the category of* $U_q(\mathfrak{sl}_2)$*-modules (there exists no submodule* $\mathcal{W}$ *such that* $\mathcal{V}_0 = \mathcal{J}_0 \oplus \mathcal{W}$*). The quotient module* $\mathcal{V}_0 / \mathcal{J}_0$ *is isomorphic to the (simple) Verma module* $\mathcal{V}\left(q^{-2}\right)$*.*

P r o o f. First, let us consider the special case of (4.55), (4.56) in which $s = t = 0$ and $a_0 = 1$. Then $\mathcal{V}_n = \mathbb{C}[x]y^n$ are $U_q(\mathfrak{sl}_2)$-invariant, and we calculate

$$\begin{aligned} \mathsf{e}(x^p y^n) &= q^{-n-p+1}\frac{q^p - q^{-p}}{q - q^{-1}} x^{p-1} y^n \neq 0, \qquad \forall p > 0, \\ \mathsf{e}(y^n) &= 0, \\ \mathsf{f}(x^p y^n) &= q^{n+p}\frac{q^{p+n} - q^{-p-n}}{q - q^{-1}} x^{p+1} y^n, \qquad \forall p \geq 0. \end{aligned} \tag{5.4}$$

Note that $\mathsf{f}(x^p y^n) = 0$ only when $p = n = 0$. Therefore $\mathcal{V}_n$ admits a generating highest weight vector y^n whose weight is q^{-n}. As in the proof of Proposition 5.4 we deduce that each $\mathcal{V}_n$ with $n \geq 1$ is isomorphic to the (highest weight simple) Verma module $\mathcal{V}(q^{-n})$. In the case $n = 0$, it is clear that $\mathcal{V}_0$ contains an obvious submodule $\mathbb{C}\mathbf{1}$ which is not a direct summand by an argument in the proof of Proposition 5.4.

Turn to the general case when the three parameters are unrestricted. The formulas (4.54)–(4.56) imply the existence of a descending sequence of submodules

$$\ldots \subset \mathcal{F}_{n+1} \subset \mathcal{F}_n \subset \mathcal{F}_{n-1} \subset \ldots \subset \mathcal{F}_2 \subset \mathcal{F}_1 \subset \mathcal{F}_0 = \mathbb{C}_q[x,y],$$

where $\mathcal{F}_n = \cup_{k=n}^{\infty} \mathbb{C}[x]y^k$, because operators of the action, being applied to a monomial, can only increase its degree in y. Note that the quotient module $\mathcal{F}_n / \mathcal{F}_{n+1}$ with unrestricted parameters is isomorphic to the module $\mathbb{C}[x]y^n \cong \mathcal{V}(q^{-n})$, just as in the case $s = t = 0$.

Now we claim that $\mathcal{F}_{n+1}$ is a direct summand in $\mathcal{F}_n$, namely $\mathcal{F}_n = \mathcal{V}_n \oplus \mathcal{F}_{n+1}$, $n \geq 0$, with $\mathcal{V}_n = U_q(\mathfrak{sl}_2)\, y^n$ for $n \geq 1$ and $\mathcal{V}_0 = U_q(\mathfrak{sl}_2)\, x$.

First consider the case $n \geq 1$. By virtue of (4.54)–(4.56), y^n is a generating highest weight vector of the submodule $\mathcal{V}_n = U_q(\mathfrak{sl}_2)y^n$, whose weight is q^{-n}. Another application of the argument in the proof of Proposition 5.4 establishes an isomorphism $\mathcal{V}_n \cong \mathcal{V}(q^{-n})$; in particular, $\mathcal{V}_n$ is a simple module by Proposition 2.5 of [7]. Hence $\mathcal{V}_n \cap \mathcal{F}_{n+1}$ can not be a proper submodule of $\mathcal{V}_n$. Since $\mathcal{V}_n$ is not contained in $\mathcal{F}_{n+1}$ (as $y^n \notin \mathcal{F}_{n+1}$), the latter intersection is zero, and the sum

$\mathcal{V}_n + \mathcal{F}_{n+1}$ is direct. On the other hand, a comparison of (4.56) and (5.4) allows one to deduce that $\mathcal{V}_n + \mathcal{F}_{n+1}$ contains all the monomials $x^p y^m$, $m \geq n$, $p \geq 0$. This already proves $\mathcal{F}_n = \mathcal{V}_n \oplus \mathcal{F}_{n+1}$.

Turn to the case $n = 0$. The composition series $0 \subset \mathbb{C}\mathbf{1} \subset \mathcal{V}_0 = U_q(\mathfrak{sl}_2)x$ is treated in the same way as that for $\mathcal{V}_0$ in Proposition 5.4; in particular, the quotient module $\mathcal{V}_0/\mathbb{C}\mathbf{1}$ is isomorphic to the simple Verma module $\mathcal{V}\left(q^{-2}\right)$. Let $\pi : \mathcal{V}_0 \to \mathcal{V}_0/\mathbb{C}\mathbf{1}$ be the natural projection map. Obviously, $\mathcal{F}_1$ does not contain $\mathbb{C}\mathbf{1}$, hence the restriction of π to $\mathcal{V}_0 \cap \mathcal{F}_1$ is one-to-one. Thus, to prove that the latter intersection is zero, it suffices to verify that $\pi(\mathcal{V}_0 \cap \mathcal{F}_1)$ is zero. As the module $\mathcal{V}_0/\mathbb{C}\mathbf{1}$ is simple, the only alternative to $\pi(\mathcal{V}_0 \cap \mathcal{F}_1) = \{0\}$ could be $\pi(\mathcal{V}_0 \cap \mathcal{F}_1) = \mathcal{V}_0/\mathbb{C}\mathbf{1}$. Under the latter assumption, there should exist some element of $\mathcal{V}_0 \cap \mathcal{F}_1$, which is certainly of the form Py for some $P \in \mathbb{C}_q[x,y]$, and such that $\pi(x) = \pi(Py)$. This relation is equivalent to $x - Py = \gamma$ for some constant γ, which is impossible, because the monomials that form Py, together with x and $\mathbf{1}$, are linearly independent. The contradiction we get this way proves that $\mathcal{V}_0 \cap \mathcal{F}_1 = \{0\}$, hence the sum $\mathcal{V}_0 + \mathcal{F}_1$ is direct. On the other hand, a comparison of (4.56) and (5.4) allows one to deduce that $\mathcal{V}_0 + \mathcal{F}_1$ contains all the monomials $x^p y^m$, with $m, p \geq 0$. Thus the relation $\mathcal{F}_n = \mathcal{V}_n \oplus \mathcal{F}_{n+1}$ is now proved for all $n \geq 0$. This, together with $\cap_{i=0}^{\infty} \mathcal{F}_i = \{0\}$, implies that

$$\mathbb{C}_q[x,y] = (\oplus_{n=1}^{\infty} U_q(\mathfrak{sl}_2) y^n) \oplus U_q(\mathfrak{sl}_2)x,$$

which was to be proved. ■

In a similar way we obtain the following

Proposition 5.7. *The representations corresponding to* $\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \star \end{pmatrix}_0 ; \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1\right]$*-series described in* (4.57)–(4.59) *split into the direct sum of subrepresentations* $\mathbb{C}_q[x,y] = \oplus_{n=0}^{\infty} \mathcal{V}_n$*, where* $\mathcal{V}_n$ *is a submodule generated by its lowest weight vector* x^n*. Each* $\mathcal{V}_n$ *with* $n \geq 1$ *is isomorphic to a simple lowest weight Verma module whose lowest weight is* q^n*. The submodule* $\mathcal{V}_0$ *admits a composition series of the form* $0 \subset \mathcal{J}_0 \subset \mathcal{V}_0$*, where* $\mathcal{J}_0 = \mathbb{C}\mathbf{1}$*. The submodule* $\mathcal{J}_0$ *is not a direct summand in the category of* $U_q(\mathfrak{sl}_2)$*-modules (there exists no submodule* $\mathcal{W}$ *such that* $\mathcal{V}_0 = \mathcal{J}_0 \oplus \mathcal{W}$*). The quotient module* $\mathcal{V}_0 \diagup \mathcal{J}_0$ *is isomorphic to the (simple) lowest weight Verma module whose lowest weight is* q^2*.*

The associated classical limit actions of the Lie algebra $\mathfrak{sl}_2$ (here it is the Lie algebra generated by e, f, h subject to the relations $[h,e] = 2e$, $[h,f] = -2f$, $[e,f] = h$) on $\mathbb{C}[x,y]$ by differentiations is derived from the quantum action via substituting $k = q^h$ with subsequent formal passage to the limit as $q \to 1$.

In this way we present all quantum and classical actions in Table 1. It should be noted that there exist more $\mathfrak{sl}_2$-actions on $\mathbb{C}[x,y]$ by differentiations (see, e.g.,

[6]) than one can see in Table 1. It follows from our results that the rest of the classical actions admit no quantum counterparts. On the other hand, among the quantum actions listed in the first row of Table 1, the only one to which the above classical limit procedure is applicable, is the action with $k(x) = x$, $k(y) = y$. The rest three actions of this series admit no classical limit in the above sense.

Table 1.

Symbolic matrices	$\mathbf{U_q(\mathfrak{sl}_2)}$ **– symmetries**	**Classical limit** $\mathfrak{sl}_2$ **– actions by differentiations**
$\left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_0 ; \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_1\right]$	$k(x) = \pm x,\ k(y) = \pm y,$ $e(x) = e(y) = 0,$ $f(x) = f(y) = 0,$	$h(x) = 0, \quad h(y) = 0,$ $e(x) = e(y) = 0,$ $f(x) = f(y) = 0,$
$\left[\begin{pmatrix} 0 & \star \\ 0 & 0 \end{pmatrix}_0 ; \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_1\right]$	$k(x) = qx,$ $k(y) = q^{-2}y,$ $e(x) = 0, \quad e(y) = b_0,$ $f(x) = b_0^{-1}xy,$ $f(y) = -qb_0^{-1}y^2$	$h(x) = x,$ $h(y) = -2y,$ $e(x) = 0, \quad e(y) = b_0,$ $f(x) = b_0^{-1}xy,$ $f(y) = -b_0^{-1}y^2$
$\left[\begin{pmatrix} 0 & 0 \\ \star & 0 \end{pmatrix}_0 ; \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_1\right]$	$k(x) = q^2x,$ $k(y) = q^{-1}y,$ $e(x) = -qc_0^{-1}x^2,$ $e(y) = c_0^{-1}xy,$ $f(x) = c_0, \quad f(y) = 0$	$h(x) = 2x,$ $h(y) = -y,$ $e(x) = -c_0^{-1}x^2,$ $e(y) = c_0^{-1}xy,$ $f(x) = c_0, \quad f(y) = 0$
$\left[\begin{pmatrix} \star & 0 \\ 0 & 0 \end{pmatrix}_0 ; \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_1\right]$	$k(x) = q^{-2}x,$ $k(y) = q^{-1}y,$ $e(x) = a_0, \quad e(y) = 0,$ $f(x) = -qa_0^{-1}x^2 + ty^4,$ $f(y) = -qa_0^{-1}xy + sy^3$	$h(x) = -2x,$ $h(y) = -y,$ $e(x) = a_0, \quad e(y) = 0,$ $f(x) = -a_0^{-1}x^2 + ty^4,$ $f(y) = -a_0^{-1}xy + sy^3$
$\left[\begin{pmatrix} 0 & 0 \\ 0 & \star \end{pmatrix}_0 ; \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_1\right]$	$k(x) = qx,\ k(y) = q^2y,$ $e(x) = -qd_0^{-1}xy + sx^3,$ $e(y) = -qd_0^{-1}y^2 + tx^4,$ $f(x) = 0, \quad f(y) = d_0$	$h(x) = x, \quad h(y) = 2y,$ $e(x) = -d_0^{-1}xy + sx^3,$ $e(y) = -d_0^{-1}y^2 + tx^4,$ $f(x) = 0, \quad f(y) = d_0$
$\left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_0 ; \begin{pmatrix} 0 & \star \\ \star & 0 \end{pmatrix}_1\right]$	$k(x) = qx,$ $k(y) = q^{-1}y,$ $e(x) = 0, \quad e(y) = \tau x,$ $f(x) = \tau^{-1}y,\ f(y) = 0$	$h(x) = x,$ $h(y) = -y,$ $e(x) = 0, \quad e(y) = \tau x,$ $f(x) = \tau^{-1}y,\ f(y) = 0$

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