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On Stability of a Unit Ball in Minkowski Space with Respect to Self-Area

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The main results of the paper are the following two statements. If the length of the unit circle $\partial B = \{||x|| = 1\}$ on Minkowski plane M^2 is equal to $O(B) = 8(1 - \varepsilon)$, $0\leq \varepsilon \leq 0.04$, then there exists a parallelogram which is centrally symmetric with respect to the origin o and the sides of which lie inside an annulus $(1+18\varepsilon)^{-1} \leq ||x|| \leq 1$. If the area of the unit sphere ∂B in the Minkowski space M^n, $n \geq 3$, is equal to $O(B) = 2n\cdot\omega_{n-1}\cdot(1-\varepsilon)$, where ε is a sufficiently small nonnegative constant and ω_n is a volume of the unit ball in R^n, then in the globular layer $(1+\varepsilon^{\delta})^{-1} \leq ||x|| \leq 1$, $\delta = 2^{-n}\cdot(n!)^{-2}$ it is possible to place a parallelepiped symmetric with respect the origin o.

Key words: Minkowski space, self-perimeter, self-area, stability.

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Let B be a normalizing body of the n-dimensional Minkowski space M^n, $n \geq 2$. This body is usually called a unit ball, and its boundary ∂B is called a unit sphere in M^n. Denote by R^n a Euclidean space adjoined to M^n the distance function of which is used as an auxiliary metric [1, 2]. In its turn, the auxiliary metric is chosen in such a way that the Euclidean n-dimensional volume $V_n(B)$ of B equals the volume of the n-dimensional unit ball in R^n,

$$V_n(B) = \omega_n := \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}.$$

We identify the points in M^n with their position vectors from the origin o. Following Busemann [3], we define an $(n-1)$-dimensional area of the surface of nonempty compact convex body K. Let M^m be an m-dimensional plane in M^n. Then the m-dimensional Minkowski volume in M^m $(1 \leq m \leq n)$ is an m-dimensional Lebesgue measure of V_m^B in M^m normalized such that

$$V_m^B(B \cap M_o^m) = \omega_m,$$

where M_o^m is a translant (i.e., a result of some translation) of M^m which passes through the origin o. For any compact convex set K in M^m,

$$V_m^B(K) = \omega_m \cdot V_m(K)/V_m(B \cap M_o^m), \quad 1 \le m \le n,$$

where V_m is an arbitrary taken (affine) m-dimensional Lebesgue measure.

Isoperimetrix I in M^n is an o centrally symmetric compact convex body with the support function h_I given on the unit sphere $\Omega = \{< u,\ u >= 1\} \subset R^n$ by

$$h_I(u) = \omega_{n-1} \cdot V_{n-1}^{-1}(B \cap A_o(u)), \tag{1}$$

where V_{n-1} is a Euclidean $(n-1)$-dimensional volume and $A_o(u)$ is a hyperplane having the normal u and passing through the origin o .

Notice that the isoperimtrix I in M^n depends only on the normalizing body B and does not depend on the choice of the auxiliary metric [1, p. 279].

Let K_0 and K_1 be convex bodies in R^n. Consider a segment $K_\theta = (1-\theta) \cdot K_0 + \theta \cdot K_1$ $(0 \le \theta \le 1)$ connecting the bodies K_0 and K_1. In [4], Minkowski, introducing the notion of the mixed volumes, expressed the volume $V(K_\theta)$ as

$$V(K_\theta) = \sum_{\upsilon=0}^{n} C_n^\upsilon \cdot (1-\theta)^{n-\upsilon} \cdot \theta^\upsilon \cdot V_\upsilon(K_0, K_1), \tag{2}$$

where $V_\upsilon(K_0, K_1)$ is a mixed volume of the bodies K_0 and K_1 which corresponds to the parameter υ. Here we use the standard notations [5, p. 113]. By Minkowski, the value

$$O_B(K) = n \cdot V_1(K, I)$$

is called a *surface area* of the body K.

By a *self-area* of the surface of the unit ball B we understand the value

$$O(B) = O_B(B) = n \cdot V_1(B, I). \tag{3}$$

In the case of $n = 2$, the value $O(B)$ is called a *self-perimeter* of the unit circle. In 1932, Golab S. [6] found optimal estimations for the perimeter: $6 \le O(B) \le 8$. In 1956, Busemann H. and Petti K. [7] obtained the following result.

Theorem A. *If B is a unit ball in the n-dimensional Minkowski space M^n, then $O(B) \le 2n \cdot \omega_{n-1}$, and the equality holds only when B is a parallelepiped.*

In this paper we study a stability of the unit ball B in the case when the self-area $O(B)$ is close to the greatest possible value $2n \cdot \omega_{n-1}$. There are proved the following theorems.

Theorem 1. *Let the self-perimeter of a unit ball B on Minkowski plane M^2 be equal to $O(B) = 8 \cdot (1-\varepsilon)$, where $0 \leq \varepsilon \leq \frac{1}{25}$. Then there exists a parallelogram P which is centrally symmetric with respect to the origin o and for which the inclusions*

$$P \subset B \subset (1 + 18 \cdot \varepsilon) \cdot P \tag{4}$$

hold.

Theorem 2. *Let the self-area $O(B)$ of a unit sphere ∂B in Minkowski space M^n, $n \geq 3$, be equal to $O(B) = 2n \cdot \omega_{n-1} \cdot (1-\varepsilon)$. Then there exists a positive constant ε_0 depending only on the dimension n and the centrally symmetric w.r. to the origin o parallelepiped P for which the inclusions*

$$P \subset B \subset (1 + \varepsilon^{\delta}) \cdot P, \tag{5}$$

hold, where $0 \leq \varepsilon \leq \varepsilon_0$ and $\delta = 2^{-n} \cdot (n!)^{-2}$.

The main results of the paper can be formulated in terms of the metric $\|x\|$ of Minkowski space M^n. For example, Theorem 1 can be reformulated as follows: *if the self-area of a unit sphere is equal to $2n\omega_{n-1} \cdot (1-\varepsilon)$, where ε is a small enough nonnegative constant, then in the globular layer $(1+\varepsilon^{\delta})^{-1} \leq \|x\| \leq 1$ of the space M^n ($n \geq 3$) it is possible to place some parallelepiped P symmetric w.r. to the origin o.* And also the area of P satisfies $(1 + \varepsilon^{\delta})^{1-n} \cdot O(B) \leq O_B(P) \leq O(B)$ that follows at once from definition (3) and monotonicity of the mixed volume.

Studying the possibility of the equality $O(B) = 2n \cdot \omega_{n-1}$, Busemann H. and Petti K. used the fact that the body B, being a cylindrical one, possesses n linearly independent one-dimensional generators. Discussing the results obtained in this paper, Diskant V.I. drew my attention that I used only one such a generator in the proof of Theorem 2. In fact, it is proved by induction over the dimension m of M^m ($n \geq m \geq 2$) by constructing a cylinder in Minkowski space, which approximates a unit ball with a given accuracy. In our opinion, this construction is of independent interest.

If K is a convex body in M^n, then there are two supporting hyperplanes H_K^+ and H_K^- parallel to any given $(n-1)$-dimensional hyperplane H. By Minkowski, the value

$$\Delta_B(K, H) = \min\left\{\|x_1 - x_2\| : x_1 \in H_K^+,\ x_2 \in H_K^-\right\}$$

is called *the width of the convex body K* in M^n w.r. to H [2, p. 106], [8]. Since the isoperimetrix I is symmetric w.r. to the origin o, its width satisfies the equality $\Delta_B(I, H) = 2 \cdot \min\{\|x\| : x \in H_I\}$, where H_I is one of two supporting hyperplanes. Consider the body B as the one located in some adjoint space R^n and specify a unit vector u normal to $H_I = H_I(u)$. Let $h_I(u)$ and $h_B(u)$ be the supporting numbers of I and B. Then $\Delta_B(I, H) = 2 \cdot h_I(u) \cdot h_B^{-1}(u)$. There follows the theorem on the stability of the unit ball B w.r. to the width of isoperimetrix.

Theorem 3. *If $\Delta_B(I,H) = 4(1-\varepsilon)\cdot\omega_{n-1}/\omega_n,\ \ 0 \le \varepsilon \le 10^{-4n^3}$, then there exists a cylinder $C_n(D)$ with one-dimensional generators such that:*

1. *$C_n(D)$ is centrally symmetric w.r.t. to the origin o;*
2. *$C_n(D)$ cross-section D is parallel to H;*
3. $C_n(D) \subset B \subset C_n(D)\cdot\left(1+\varepsilon^{\frac{1}{2n^2}}\right).$ (6)

This result is close to that obtained by Diskant V.I. on the estimation from above for the width of the isoperimetrix $\Delta_B(I,H) \le 4\omega_{n-1}\cdot\omega_n^{-1}$, where the equality holds only when B is a cylinder [8].

P r o o f of the Theorem 1. Let Q_2 be a parallelogram of the smallest area and let it be centered at o and circumscribed around B. The midpoints of the Q_2 sides necessarily lie on ∂B [1, p. 121]. On M^2, chose an auxiliary Euclidean metric such that on the adjoint plane R^2 with the Cartesian system xoy the parallelogram Q_2 becomes a square $abcd$ with the vertices $a(-1;1)$, $b(1;1)$, $c(1;-1)$, $d(-1;-1)$. The points $e(0;1)$, $f(1;0)$, $g(0;-1)$, $p(-1;0)$ lie on ∂Q_2 and $efgp \subset B$. Denote by n and m the points of intersection of straight lines $y=x$ and $y=-x$ with ∂B in a half-plane $y>0$. Let $0<\xi<\frac{1}{2}$ and $0<\eta<\frac{1}{2}$ be the parameters that determine n and m by $n(1-\xi,1-\xi)$ and $m(-1+\eta;1-\eta)$. From the symmetry $B=-B$, the points $-n(-1+\xi;-1+\xi)$ and $-m(1-\eta;-1+\eta)$ lie on ∂B. Draw the straight lines (pm), (ab) and denote their intersection by $a_2=(pm)\cap(ab)$; draw the straight lines (em), (da) and denote their intersection by $a_1=(em)\cap(da)$. Set $b_2=(en)\cap(bc)$, $b_1=(fn)\cap(ab)$, $c_{1,2}=-a_{1,2}$, $d_{1,2}=-b_{1,2}$. Since B is convex, its line of support at m crosses the segments $[a_2e]$ and $[pa_1]$, and hence the segment $[a_1a_2]$ does not have common points with the interior $\overset{\circ}{B}$. Therefore, $B\subset a_1a_2b_1b_2c_1c_2d_1d_2$, and it follows then that

$$8\cdot(1-\varepsilon) \le O(B) \le O_B(a_1a_2b_1b_2c_1c_2d_1d_2) \le O_B(Q_2) = 8. \tag{7}$$

Denote by $\|x\|$ the length of a vector x on M^2 with a normalizing body B and by $|x|$, its Euclidean length on R^2. Taking into account (7), we have

$$\begin{cases} \|pa_1\|+\|a_1a_2\|+\|a_2e\| \le \|ap\|+\|ae\| = 2, \\ \|eb_1\|+\|b_1b_2\|+\|b_2f\| \le 2, \\ 4-4\varepsilon \le (\|pa_1\|+\|a_1a_2\|+\|a_2e\|)+(\|eb_1\|+\|b_1b_2\|+\|b_2f\|) \le 4. \end{cases}$$

Hence,

$$\begin{cases} 0 \le 2-(\|pa_1\|+\|a_1a_2\|+\|a_2e\|) \le 4\varepsilon, \\ 0 \le 2-(\|eb_1\|+\|b_1b_2\|+\|b_2f\|) \le 4\varepsilon. \end{cases}$$

By calculating

$$|aa_2| = |a_1a| = \frac{\eta}{1-\eta},$$

we can see that

$$\|a_2e\| = \|pa_1\| = 1 - \frac{\eta}{1-\eta}$$

and

$$\|a_1a_2\| = \frac{|a_1a_2|}{|on|} = \frac{|aa_2|}{n_x} = \frac{\eta}{(1-\eta)(1-\xi)}.$$

Consequently,

$$2 - 4\varepsilon \leq \|pa_1\| + \|a_1a_2\| + \|a_2e\| = 2 - \frac{\eta}{1-\eta}\left(2 - \frac{1}{1-\xi}\right).$$

After the similar calculations for n, compose the system

$$\begin{cases} \eta(1-2\xi) \leq 4\varepsilon(1-\eta)(1-\xi), \\ \xi(1-2\eta) \leq 4\varepsilon(1-\eta)(1-\xi), \end{cases}$$

where $0 \leq \xi,\ \eta \leq \frac{1}{2}$.

Combining the inequalities, we get

$$(1+8\varepsilon)(\xi+\eta) \leq (1+2\varepsilon)4\xi\eta + 8\varepsilon.$$

Since $4\xi\eta \leq (\xi+\eta)^2$, the value $z = \xi + \eta$ satisfies the square inequality

$$(1+2\varepsilon)z^2 - (1+8\varepsilon)z + 8\varepsilon \geq 0.$$

It is obvious that either

$$0 \leq \xi + \eta \leq \frac{1+8\varepsilon-\sqrt{1-16\varepsilon}}{2(1+2\varepsilon)} \quad \text{or} \quad \frac{1+8\varepsilon+\sqrt{1-16\varepsilon}}{2(1+2\varepsilon)} \leq \xi + \eta \leq 1.$$

As a consequence, either

$$\max\{\xi;\eta\} \leq \frac{1+8\varepsilon-\sqrt{1-16\varepsilon}}{2(1+2\varepsilon)} \quad \text{or} \quad \max\left\{\frac{1}{2}-\xi;\frac{1}{2}-\eta\right\} \leq \frac{1-4\varepsilon-\sqrt{1-16\varepsilon}}{2(1+2\varepsilon)}.$$

If $0 \leq \varepsilon \leq \frac{1}{25}$, then $\sqrt{1-16\varepsilon} \geq 1 - 10\varepsilon$. There are two cases:

$$1)\ \max\{\xi;\eta\} \leq \frac{9\varepsilon}{1+2\varepsilon} \leq 9\varepsilon; \qquad 2)\ \max\left\{\frac{1}{2}-\xi;\frac{1}{2}-\eta\right\} \leq \frac{3\varepsilon}{1+2\varepsilon} \leq 3\varepsilon.$$

Consider each case separately. Suppose (1) holds. Chose a square $r_1r_2r_3r_4$ with the vertices at points $r_1(-1+9\varepsilon;1-9\varepsilon)$, $r_2(1-9\varepsilon;1-9\varepsilon)$, $r_3(1-9\varepsilon;-1+9\varepsilon)$,

$r_4(-1+9\varepsilon;-1+9\varepsilon)$ to be a parallelogram P in (4). By the construction, $P \subset B \subset Q_2$. Since $Q_2 = \frac{1}{1-9\varepsilon}P$, we have $Q_2 \subset (1+18\varepsilon)P$.

Suppose (2) holds. Chose a square $efgp$ to be P in (4). As noticed above, $[a_1a_2] \cap \overset{o}{B} = \emptyset$. The points $a_1(-1;1-\frac{\eta}{1-\eta})$ and $a_2(-1+\frac{\eta}{1-\eta};1)$ lie on a straight line $y = x+2-\frac{\eta}{1-\eta}$. For $\frac{1}{2}-\eta \le 3\varepsilon$ we have

$$2 - \frac{\eta}{1-\eta} \le 1 + \frac{12\varepsilon}{1+6\varepsilon} \le 1+12\varepsilon,$$

and hence the figure B is under a straight line $y = x+1+12\varepsilon$. For the segments $[b_1b_2]$, $[c_1c_2]$, $[d_1d_2]$ we draw the straight lines $y = -x+1+12\varepsilon$, $y = x-1-12\varepsilon$, $y = -x-1-12\varepsilon$. Denote by S_2 a square with vertices at $e_1(0;1+12\varepsilon)$, $f_1(1+12\varepsilon;0)$, $g_1(0;-1-12\varepsilon$, $p_1(-1-12\varepsilon;0)$. Then $B \subset S_2 = (1+12\varepsilon)\cdot P$. The proof is complete. ■

To prove Theorem 3 we need some auxiliary statements. Without loss of generality, further we will consider a proper convex compact body B symmetric w.r. to the origin o and located in the corresponding adjoint Euclidean space R^n $(n \ge 2)$.

Proposition 1. *Let K_0 and K_1 be convex compact bodies in R^m, $m \ge 2$, with the m-dimensional Euclidean volumes satisfying $V(K_0) \le V(K_1)$. Let V_0 be a constant such that $V(K_\theta) \le V_0$, $0 \le \theta \le 1$. Then*

$$V_1(K_0,K_1) - V(K_0) \le e(V_0 - V(K_0)). \tag{8}$$

P r o o f. The Brunn inequality implies

$$V^{\frac{1}{m}}(K_\theta) \ge (1-\theta)V^{\frac{1}{m}}(K_0) + \theta V^{\frac{1}{m}}(K_1) \ge V^{\frac{1}{m}}(K_0),$$

and hence $V(K_\theta) \ge V(K_0)$.

Using the identity

$$1 = \sum_{\upsilon=0}^{m} C_m^\upsilon (1-\theta)^{m-\upsilon}\theta^\upsilon,$$

rewrite (2) in the form of

$$V(K_\theta) - V(K_0) = \sum_{\upsilon=0}^{m} C_m^\upsilon (1-\theta)^{m-\upsilon}\theta^\upsilon \left[V_\upsilon(K_0,K_1) - V(K_0)\right]. \tag{9}$$

Write down the inequality for the mixed volumes

$$V_\upsilon^m(K_0,K_1) \ge V^{m-\upsilon}(K_0)V^\upsilon(K_1),$$

which is a consequence of a more general A.D. Aleksandrov's inequality [9, p. 78]. Then $V_v^m(K_0, K_1) \geq V^m(K_0)$ and $V_v(K_0, K_1) - V(K_0) \geq 0$. Since all terms in the right-hand side of (9) are nonnegative, then

$$m(1-\theta)^{m-1}\theta\,[V_1(K_0, K_1) - V(K_0)] \leq V(K_\theta) - V(K_0) \leq V_0 - V(K_0).$$

The inequality holds for all $0 \leq \theta \leq 1$. For $\theta = \frac{1}{m}$ we get

$$\left(1 - \frac{1}{m}\right)^{m-1} [V_1(K_0, K_1) - V(K_0)] \leq V_0 - V(K_0).$$

Since the Euler sequence $a_n = \left(1 + \frac{1}{n}\right)^n < e$ is monotonously increasing, then

$$\left(1 - \frac{1}{m}\right)^{m-1} = \left(1 + \frac{1}{m-1}\right)^{1-m} > \frac{1}{e}\,.$$

Therefore,

$$\frac{1}{e}\,[V_1(K_0, K_1) - V(K_0)] \leq V_0 - V(K_0),$$

which completes the proof of Proposition 1. ■

Further we will use a method suggested by V.I. Diskant [10, 11] for studying a stability in the theory of convex bodies. Denote by $q = q(K_0, K_1)$ a capacity coefficient of K_1 w.r. K_0, i.e., the greatest of γ's for which the body $\gamma \cdot K_1$ is embedded into K_0 by a translation. Recall one of Diskant's inequalities for the mixed volumes [10, p. 101]:

$$V_1^{\frac{m}{m-1}}(K_0, K_1) - V(K_0) V^{\frac{1}{m-1}}(K_1) \geq \left[V_1^{\frac{1}{m-1}}(K_0, K_1) - q V^{\frac{1}{m-1}}(K_1) \right]^m . \quad (10)$$

Proposition 2. *Let the bodies K_0 and K_1 meet the requirements of Proposition 1. Set $\alpha = 3(V_0/V(K_0) - 1) \leq \frac{1}{4}$. Then the capacity coefficient q satisfies*

$$q(K_0, K_1) \geq 1 - 2\alpha^{\frac{1}{m}}. \quad (11)$$

P r o o f. To estimate $q(K_0, K_1)$ from below, we use inequality (10) (see formula (2.1) in [10, p. 110])

$$q \geq \left[\frac{V_1(K_0, K_1)}{V(K_1)} \right]^{\frac{1}{m-1}} - \left[V_1^{\frac{m}{m-1}}(K_0, K_1) - V(K_0) V^{\frac{1}{m-1}}(K_1) \right]^{\frac{1}{m}} \cdot V^{\frac{-1}{m-1}}(K_1).$$

Transform this inequality

$$q \geq \left[\frac{V_1(K_0,K_1)}{V(K_1)}\right]^{\frac{1}{m-1}} - \left[\frac{V_1(K_0,K_1)}{V(K_1)}\right]^{\frac{1}{m}} \left\{\left(\frac{V_1(K_0,K_1)}{V(K_1)}\right)^{\frac{1}{m-1}} - \frac{V(K_0)}{V_1(K_0,K_1)}\right\}^{\frac{1}{m}}. \quad (12)$$

The inequality $V_1(K_0,K_1) \geq V(K_0)$ implies

$$\frac{V_1(K_0,K_1)}{V(K_1)} \geq \frac{V(K_0)}{V(K_1)} \geq \frac{V(K_0)}{V_0} = \frac{1}{1+\frac{\alpha}{3}} \geq 1 - \frac{\alpha}{3}. \quad (13)$$

By (8), we have $V_1(K_0,K_1) - V(K_0) \leq 3 \cdot (V_0 - V(K_0))$, and hence

$$\frac{V_1(K_0,K_1)}{V(K_1)} \leq \frac{V_1(K_0,K_1)}{V(K_0)} \leq 1 + 3(\frac{V_0}{V(K_0)} - 1) = 1 + \alpha. \quad (14)$$

Besides,

$$\frac{V(K_0)}{V_1(K_0,K_1)} \geq \frac{1}{1+\alpha} \geq 1 - \alpha. \quad (15)$$

Substituting (13), (14), (15) into (12), we obtain

$$q \geq \left(1 - \frac{\alpha}{3}\right)^{\frac{1}{m-1}} - (1+\alpha)^{\frac{1}{m}} \left\{(1+\alpha)^{\frac{1}{m-1}} - (1-\alpha)\right\}^{\frac{1}{m}}.$$

For $p \geq 1$ we have

$$\begin{array}{lll} (1) & (1+x)^{\frac{1}{p}} \leq 1 + \frac{x}{p}, & 0 \leq x \leq 1; \\ (2) & (1-x)^{\frac{1}{p}} \geq 1 - \frac{12}{11}x, & 0 \leq x \leq \frac{1}{12}. \end{array}$$

Therefore,

$$q \geq 1 - \frac{4}{11}\alpha - \left(1 + \frac{\alpha}{m}\right)\left\{\frac{m}{m-1}\alpha\right\}^{\frac{1}{m}} \geq 1 - \frac{4}{11}\alpha - \frac{9}{8}\left(\frac{m}{m-1}\right)^{\frac{1}{m}} \alpha^{\frac{1}{m}}.$$

The conditions $m \geq 2$ and $0 \leq \alpha \leq \frac{1}{4}$ provide

$$\alpha \leq \frac{1}{2}\alpha^{\frac{1}{m}} \text{and} \left(\frac{m}{m-1}\right)^{\frac{1}{m}} \leq \sqrt{2}.$$

Finally,

$$q \geq 1 - \frac{2}{11}\alpha^{\frac{1}{m}} - \frac{9}{8}\sqrt{2}\alpha^{\frac{1}{m}} \geq 1 - 2\alpha^{\frac{1}{m}}.$$

■

Denote by $A_t(u)$ a hyperplane in R^n which is parallel to $A_o(u)$ and is at the distance t in the direction of the vector u. If $t < 0$, then $A_t(u)$ is at the same distance from $A_o(u)$ in the direction of the vector $-u$.

We denote by $h_B = h(u)$ $(u \in \Omega)$ a supporting function of the normalizing body B. Denote by $H(u)$ the hyperplanes of support that correspond to $h(u)$.

Let $B_t(u) = B \cap A_t(u)$. If $-h(u) \le t \le h(u)$, then $B_t(u) \ne \emptyset$. The central symmetry of the unit ball $B = -B$ provides the equalities $B_{-t}(u) = -B_t(u)$.

Consider the function

$$\phi_u(t) = V_{n-1}^{\frac{1}{n-1}}(B_t(u)), \quad t \in [-h(u); h(u)] \, .$$

The function is even, $\phi_u(-t) = \phi_u(t)$, and by the Brunn inequality it is convex upwards. Then $\max_t \phi_u(t) = \phi_u(0)$, and this provides the estimation

$$V_n(B) \le 2h(u) \cdot V_{n-1}(B_0(u)).$$

Denote by $\Delta V(u)$ the difference

$$\Delta V(u) = 2h(u)V_{n-1}(B_0(u)) - V_n(B).$$

Proposition 3. *Let u_0 be a unit normal vector of some hyperplane of support $H_0 = H_I(u_0)$ for the isoperimetrix I. If a Minkowski width of I in the direction u_0 is equal to $\Delta_B(I, H_0) = 4(1-\varepsilon)\omega_{n-1}\omega_n^{-1}$, $0 \le \varepsilon < 1$, then*

$$\Delta V(u_0) = \varepsilon 2h(u_0)V_{n-1}(B_0(u_0)). \tag{16}$$

P r o o f. Indeed, from the expression in the terms of supporting numbers for the Minkowski width of the body I in the adjoint space R^n and the explicit expression for the isoperimetrix I supporting function h_I given by (1), we get

$$\Delta_B(I, H_0) = 2\frac{h_I(u_0)}{h_B(u_0)} = 2\frac{\omega_{n-1}}{h(u_0)V_{n-1}(B_0(u_0))}.$$

Taking into account the normalization $V_n(B) = \omega_n$, we have

$$\Delta_B(I, H_0) = 4\frac{\omega_{n-1}}{\omega_n}\frac{V_n(B)}{2h(u_0)V_{n-1}(B_0(u_0))}.$$

Together with the condition imposed on Δ_B by the hypothesis, the latter equality provides (16). ■

Set $V_0 = V_{n-1}(B_0(u_0))$, $h_0 = h_B(u_0)$, $\phi_0(t) = \phi_{u_0}(t)$ and $\Delta V(u_0) = 2h_0V_0\varepsilon$. Denote by B^* a Schwartz-symmetrized body B w.r. to a straight line $L(u_0)$ which is parallel to u_0 and passes through the origin o. By the construction,

$V_n(B^*) = V_n(B)$. By the Brunn theorem, the body of rotation B^* is convex [5, p. 89]. On R^2 with the Cartesian coordinates xoy, define the function

$$x(y) = \phi_0(y)\omega_{n-1}^{-\frac{1}{n-1}}, \quad -h_0 \le y \le h_0.$$

Set for brevity $x(0) = r$. The function $x = x(y)$ defines the radii of the $(n-1)$-dimensional balls that generate B^*. On the graph of this function, mark the point $M_0(x_0; y_0)$ which is an intersection point of the graph and a straight line $y = \frac{h_0}{r}x$. We have $0 < x_0 \le r$, $0 < y_0 \le h_0$. It is convenient to use a parameter $\tau = r - x_0$. Then $M_0(r-\tau, h_0 - \frac{h_0}{r}\tau)$.

Proposition 4. *If the conditions of Proposition 3 hold, then*

$$\tau \le r\sqrt{\frac{\varepsilon}{2}}. \tag{17}$$

P r o o f. If $\tau = 0$, then inequality (17) is trivial. Notice that by the Minkowski–Brunn theorem, the equality $\tau = 0$ holds only when the body B is a cylinder with the generators parallel to u_0.

Suppose $\tau > 0$. Draw a supporting straight line to $x = x(y)$ at M_0. The intersection points of this line and straight lines $y = h_0$ and $x = r$ denote by P and Q, respectively. The points $P_1 = (0; h_0)$, $Q_1 = (r; 0)$ and the whole segment $[P_1Q_1]$ are to the left of the convex curve $x = x(y)$, $0 \le y \le h_0$. Therefore, $0 < \tau \le \frac{r}{2}$. Rewrite the coordinates of P and Q in the terms of a and b, namely, $P = (r-a; h_0)$ and $Q = (r; h_0 - b)$.

Define the function $r_1 = r_1(y)$, $y \in [-h_0; h_0]$ by

$$r_1(y) = \begin{cases} r, & \text{if} \quad b - h_0 \le y \le h_0 - b; \\ r - a - \frac{a}{b}(y - h_0), & \text{if} \quad h_0 - b \le y \le h_0; \\ r - a + \frac{a}{b}(y + h_0), & \text{if} \quad -h_0 \le y \le b - h_0. \end{cases}$$

In R^n, construct a rotation body $\widehat{B}$ with the axis $L(u_0)$ and the radii of the $(n-1)$-dimensional spheres given by the function $r_1 = r_1(y)$. By the construction, $x(y) \le r_1(y)$, which provides $B^* \subset \widehat{B}$. Estimate from below a difference $\Delta V(u_0)$ in the terms of $V_n(\widehat{B})$

$$\begin{aligned} \Delta V(u_0) &= 2h_0V_0 - V_n(B^*) \ge 2h_0V_0 - V_n(\widehat{B}) \\ &= 2\omega_{n-1}r^{n-1}b - 2\omega_{n-1}\int_0^b (r - \tfrac{a}{b}z)^{n-1}dz \\ &= 2\omega_{n-1}\frac{b}{na}\left[(r-a)^n - r^n + nar^{n-1}\right]. \end{aligned}$$

It is easy to verify that the function

$$\phi(s) = (1-s)^n - 1 + ns - \frac{n}{2}s^2, \quad 0 \le s \le 1, \quad n \ge 2,$$

is monotonously increasing. Multiplying the inequality

$$(1-s)^n - 1 + ns \ge \frac{n}{2}s^2$$

by r^n and denoting $rs = a$, we obtain

$$(r-a)^n - r^n + nr^{n-1}a \ge \frac{n}{2}r^{n-2}a^2.$$

Thus,

$$\Delta V(u_0) \ge \omega_{n-1} r^{n-2} ab. \tag{18}$$

The chosen point $M_0(r-\tau; h_0 - \frac{h_0}{r}\tau)$ lies on the supporting straight line, therefore a and b are connected by the equation

$$\frac{\tau}{a} + \frac{h_0}{r}\frac{\tau}{b} = 1.$$

The product ab in the right-hand side of (18) can be expressed in the terms of b

$$ab = \tau(b + \frac{h_0}{r}a) = \frac{rb^2\tau}{rb - h_0\tau} = f(b).$$

Estimate ab from below by $\min f(b) = f(b_0)$, where $b_0 = 2\frac{h_0}{r}\tau$, $a_0 = 2\tau$. Then

$$ab \ge a_0 b_0 = 4\frac{h_0}{r}\tau^2.$$

The hypotheses of Proposition 3, (16) and (18) imply

$$\varepsilon 2h_0 V_0 = \Delta V(u_0) \ge 4\omega_{n-1} r^{n-3} h_0 \tau^2 = 4\frac{V_0}{r^2} h_0 \tau^2, \quad \text{or} \quad \varepsilon \ge 2\left(\frac{\tau}{r}\right)^2.$$

■

Corollary 1. *A cross-section $B_t = B \cap A_t(u_0)$, which corresponds to M_0, is defined by $T = h_0 - \frac{h_0}{r}\cdot\tau$. Besides, the distance between $A_T(u_0)$ and $A_0(u_0)$ is*

$$T \ge t_0 = h_0\left(1 - \sqrt{\frac{\varepsilon}{2}}\right). \tag{19}$$

The Euclidean $(n-1)$-dimensional volume of the section B_T satisfies

$$\begin{aligned} V_{n-1}(B_T) &= \omega_{n-1}(r-\tau)^{n-1} \\ &\ge \omega_{n-1} r^{n-1}\left(1 - \sqrt{\frac{\varepsilon}{2}}\right)^{n-1} = V_0\left(1 - \sqrt{\frac{\varepsilon}{2}}\right)^{n-1}. \end{aligned} \tag{20}$$

The following theorem (the analog of Theorem 3 for the plane) illustrates the importance of inequalities (19) and (20).

Theorem 4. *If the width of the isoperimetrix I on M^2, which corresponds to a straight line H_0, satisfies $\Delta_B(I, H_0) = \frac{8}{\pi}(1-\varepsilon)$, $0 \leq \varepsilon \leq \frac{1}{6}$, then there exists a symmetric w.r. to the origin o parallelogram P with a side parallel to H_0 such that $P \subset B \subset (1+2\sqrt{\varepsilon}) \cdot P$.*

P r o o f. As above, denote by u_0 a normal vector of the isoperimetrix I supporting the straight line $H_I = H_I(u_0)$, $H_I || H_0$ in the adjoint plane R^2. If $n = 2$, then the section B_T is a segment $[ab]$. Set $c = -a$, $d = -b$. Then $B_{-T} = -B_T = [cd]$. Denote $B_0 = [ef]$. Then $|oe| = |of| = r$. We assume that the points a, b, f, c, d, e are on ∂B in the cyclic order and clockwise.

Show that $P = abcd$ can be taken as a required parallelogram. The inclusion $P \subset B$ is obvious. Prove now the inclusion $B \subset (1+2\sqrt{\varepsilon})P$. Denote $a_1 = (de) \cap (ab)$; $d_1 = (ae) \cap (cd)$; $b_1 = (cf) \cap (ab)$; $c_1 = (bf) \cap (dc)$. The segments $[ea_1]$, $[ed_1]$, $[fb_1]$, $[fc_1]$ do not have any common points with $\overset{\circ}{B}$. Therefore, the figure B is in a strip bounded by the two parallel straight lines (d_1a_1) and (c_1b_1). Denote $a_2 = (d_1a_1) \cap H(u_0)$, $b_2 = (c_1b_1) \cap H(u_0)$, $c_2 = (c_1b_1) \cap H(-u_0)$, $d_2 = (d_1a_1) \cap H(-u_0)$. Mark also the points $f_1 = (cb) \cap (ef)$ and $e_1 = (da) \cap (ef)$. Due to (20), we have

$$|ab| = 2\,|of_1| = 2(r-\tau) \geq 2r\left(1 - \sqrt{\frac{\varepsilon}{2}}\right) \text{ and } |f_1f| = \tau \leq r\sqrt{\frac{\varepsilon}{2}}.$$

A similarity of the triangles Δcf_1f and Δcbb_1 implies $|bb_1| = 2|f_1f| \leq r\sqrt{2\varepsilon}$. Besides, $|cc_1| = |dd_1| = |aa_1| = |bb_1| \leq r\sqrt{2\varepsilon}$. It is easy to see that

$$\frac{|a_2b_2|}{|ab|} \leq \frac{|ab| + 2r\sqrt{2\varepsilon}}{|ab|} \leq 1 + \frac{\sqrt{2\varepsilon}}{1 - \sqrt{\frac{\varepsilon}{2}}} \leq 1 + 2\sqrt{\varepsilon}.$$

The estimation (19) provides

$$\frac{|c_2b_2|}{|cb|} \leq \frac{h_0}{h_0\left(1 - \sqrt{\frac{\varepsilon}{2}}\right)} \leq 1 + \sqrt{2\varepsilon}.$$

By the construction, the parallelogram $a_2b_2c_2d_2 \supset B$, hence $B \subset (1+2\sqrt{\varepsilon})P$. The theorem is proved. ∎

To prove Theorem 3 for the case of $n \geq 3$ we need an estimation from below for the capacity coefficient of B_T w.r. to B_{-T}.

Corollary 2. *If $0 \le \varepsilon \le 10^{-2}$, then the capacity coefficient satisfies*

$$q(-B_T; B_T) \ge 1 - 5\varepsilon^{\frac{1}{2(n-1)}}, \quad n \ge 3. \tag{21}$$

P r o o f. By means of translation, place the convex bodies B_T and B_{-T} in the $(n-1)$-dimensional hyperplane $A_o(u_0)$. Denote by $B'_T = B_T - Tu_0$, $B'_{-T} = B_{-T} + Tu_0$ the corresponding traslants. The equalities $V_{n-1}(B'_T) = V_{n-1}(B'_{-T})$ and $q(-B_T, B_T) = q(B'_{-T}, B'_T)$ are obvious. The line segment $K_\theta = (1-\theta)B_T + \theta B_{-T}$, $0 \le \theta \le 1$ is in the section $B_{(1-2\theta)T} = B \cap A_{(1-2\theta)T}(u_0)$. Thus, for $K'_\theta = (1-\theta)B'_T + \theta B'_{-T}$, we have

$$V_{n-1}(K'_\theta) = V_{n-1}(K_\theta) \le V_0 = V_{n-1}(B_0(u_0)).$$

Take $K_0 = B'_{-T}$ and $K_1 = B'_T$ from the hypothesis of Proposition 2. Then from (20) we get

$$\alpha = 3\left(\frac{V_0}{V(K_0)} - 1\right) \le 3\left(\left(1 - \sqrt{\frac{\varepsilon}{2}}\right)^{1-n} - 1\right),$$

and for the capacity coefficient

$$q(-B_T; B_T) \ge 1 - 2 \times 3^{\frac{1}{n-1}} \left(1 - \left(1 - \sqrt{\frac{\varepsilon}{2}}\right)^{n-1}\right)^{\frac{1}{n-1}} \left(1 - \sqrt{\frac{\varepsilon}{2}}\right)^{-1}.$$

For $m \ge 2$, $0 \le x \le \frac{1}{2}$, $0 \le \varepsilon \le 10^{-2}$ the inequalities

$$\left(\frac{3m}{\sqrt{2}}\right)^{\frac{1}{m}} \le \frac{9}{4}, \qquad (1-(1-x)^m)^{\frac{1}{m}} \le m^{\frac{1}{m}} x^{\frac{1}{m}}, \qquad 1 - \sqrt{\frac{\varepsilon}{2}} \ge \frac{10}{11}$$

hold. The estimation (21) follows at once from the above. ■

P r o o f of the Theorem 3. Let B_T be defined as in Corollary 1 and the relations (19)–(21) be fulfilled. Let B'_T and B'_{-T} respectively denote the tranlants of B_T and B_{-T} after a translation on $A_o(u)$. Notice that $B'_{-T} = -B'_T$. Let $\gamma = q(B'_T, -B'_T)$. Then there is a vector a in the hyperplane $A_o(u_0)$ such that $a + \gamma(-B'_T) \subset B'_T$. On $A_o(u)$ consider a mapping $\varphi(x) = a - \gamma x$, $x \in A_o(u)$. Evidently, $\phi(B'_T) = a - \gamma B'_T = a + \gamma(-B'_T) \subset B'_T$. Since $V_{n-1}(B'_T) = V_{n-1}(B'_{-T})$, then $\gamma \le 1$. If $\gamma = 1$, then the bodies B'_T and B'_{-T} coincide after a translation on the vector a. In general, $\gamma < 1$. Denote by x_0 a solution of the equation $x_0 = a - \gamma x_0$, i.e., $x_0 = (1+\gamma)^{-1}a$.

By the choice, $\varphi(x_0) = x_0$. Let $\widetilde{B}_T = B'_T - x_0$, $\widetilde{B}_{-T} = B'_{-T} + x_0$. Then $\widetilde{B}_{-T} = -\widetilde{B}_T$. It is easy to check that after this replacement we have $\gamma\widetilde{B}_{-T} \subset \widetilde{B}_T$.

Hence, the inclusions $\gamma(-\widetilde{B}_T) \subset \widetilde{B}_T \subset \frac{1}{\gamma}\left(-\widetilde{B}_T\right)$ hold. Moreover, $q(\widetilde{B}_T, -\widetilde{B}_T) = q(-\widetilde{B}_T, \widetilde{B}_T) = \gamma$. On the hyperplane $A_o(u_0)$, construct a body $D = \widetilde{B}_T \cap \left(-\widetilde{B}_T\right)$ which is centrally symmetric w.r. to the origin o. Set $\nu = x_0 + Tu_0$. By the construction, $D_T \equiv D + v \subset B_T$ and $D_{-T} \equiv D - v \subset B_{-T}$. Notice also that

$$D \supset \gamma(\widetilde{B}_{_T} \cup \widetilde{B}_{_{-T}}). \tag{22}$$

Denote by $C_n(D) \subset R^n$ a cylinder whose cross-sections coincide with D and whose 1-dimensional generators are parallel to v and bounded by the hyperplanes $A_T(u_0)$ and $A_{-T}(u_0)$. This cylinder is symmetric w.r. to the origin o: $C_n(D) = -C_n(D)$. Since the symmetric body B is convex, the inclusion $C_n(D) \subset B$ holds.

Estimate from below the capacity coefficient $q(C_n(D), B)$. By formula (22),

$$V_n(B) \geq V_n(C_n(D)) = 2TV_{n-1}(D) \geq 2TV_{n-1}(\gamma B_T) = 2T\gamma^{n-1}V_{n-1}(B_T).$$

Using estimations (19)–(21), we conclude

$$V_n(B) \geq 2h_0V_0\left(1 - 5\varepsilon^{\frac{1}{2(n-1)}}\right)^{n-1}\left(1 - \sqrt{\frac{\varepsilon}{2}}\right)^n.$$

For further calculations to be substantial, we assume $0 \leq \varepsilon \leq (10(n-1))^{-2(n-1)}$. Initially, $2h_0V_0 \geq V_n(B)$ (see, for example, equality (16)). Hence,

$$V_n(B) \geq V_n(C_n(D)) \geq V_n(B)\left(1 - 5(n-1)\varepsilon^{\frac{1}{2(n-1)}}\right)\left(1 - n\sqrt{\frac{\varepsilon}{2}}\right),$$

or

$$1 \leq \frac{V_n(B)}{V_n(C_n(D))} \leq \left(1 - 7(n-1)\varepsilon^{\frac{1}{2(n-1)}}\right)^{-1} \leq 1 + 14(n-1)\varepsilon^{\frac{1}{2(n-1)}}.$$

Consider a segment $K_\theta = (1-\theta)C_n(D) + \theta \cdot B$, $0 \leq \theta \leq 1$ inside B. In Proposition 2 assume that $K_0 = C_n(D)$, $K_1 = B$, $V_0 = V_n(B)$, where $V_n(K_\theta) \leq V_n(B)$. Then $\alpha \leq 50(n-1)\varepsilon^{\frac{1}{2(n-1)}}$, and the capacity coefficient $q(C_n(D), B)$ can be estimated by (11),

$$q_1 = q(C_n(D), B) \geq 1 - 10\varepsilon^{\frac{1}{2n(n-1)}}, \quad n \geq 3.$$

The bodies $C_n(D)$ and B being centrally symmetric w.r. to the origin o, we will get $q_1B \subset C_n(D)$ and $B \subset \frac{1}{q_1}C_n(D)$. Since $\frac{1}{1-x} \leq 1 + 2x$, $0 \leq x \leq \frac{1}{2}$, we have

$$C_n(D) \subset B \subset \left(1 + 20\varepsilon^{\frac{1}{2n(n-1)}}\right)C_n(D). \tag{23}$$

Finally, if $0 \leq \varepsilon \leq 10^{-4n^3}$, then the inclusions (6) hold. The theorem is proved. ■

P r o o f of the Theorem 2. The proof is based on the idea of Busemann–Petti (see Theorem 7.4.1 [2]) and on the properties of a superficial function of convex body introduced by Aleksandrov A.D. [9, p. 39].

For a convex body B, the *superficial function* $F(B,\omega)$ on a unit sphere Ω is defined by the following construction. Let a Lebesgue measurable set ω be given on Ω. Denote by $\sigma(\omega)$ a set of all points on the surface of the convex body B having a normal u directed to ω. The superficial function $F(\sigma(\omega))$ is the area of $\sigma(\omega)$.

Write down the first mixed volume from definition (3) in the terms of the Stieltjes–Radon integral for the continuous isoperimetrix I supporting function $h_I(u)$ over a unit sphere,

$$O(B) = \int\limits_{\Omega} h_I(u)F(B,d\omega).$$

Since the origin o is inside of B, then $h_B(u) > 0$, and hence the ratio $h_I(u)/h_B(u)$ is a continuous function on Ω. By the integral mean value theorem, there is a vector u_0 on Ω such that

$$O(B) = \int\limits_{\Omega} \frac{h_I(u)}{h_B(u)} h_B(u)F(B,d\omega) \\ = \frac{h_I(u_0)}{h_B(u_0)} \int\limits_{\Omega} h_B(u)F(B,d\omega) = \frac{h_I(u_0)}{h_B(u_0)} nV_n(B).$$

The plane of support $H_0 = H_I(u_0)$ for I is given by the supporting number $h_I(u_0)$. By Theorem 2, the area $O(B) = 2n\omega_{n-1}(1-\varepsilon)$, and hence the width

$$\Delta_B(I,H_0) = 2h_I(u_0)/h_B(u_0) = 4(1-\varepsilon)\omega_{n-1}/\omega_n.$$

By Theorem 3, in M^n there is a cylinder with the cross-section perpendicular to u_0 for which (6) holds.

Now we study the cross-section D of the cylinder $C_n = C_n(D)$. Show that the body D, by analogy with (6), can be approximated by some "$(n-1)$ - dimensional" cylinder $C_{n-1} = C_{n-1}(D_{n-2})$ with the cross-section D_{n-2}. Denote by $Q = 1 + 20 \cdot \varepsilon^{\frac{1}{2n(n-1)}}$ the factor from (23). Without loss of generality, assume that the generators of the cylinder $C_n(D)$ are perpendicular to the cross-section D, i.e., $v \parallel u_0$. The latter is based on the affine invariancy of the definition of self-area of the surface $O(B)$ and on the free choosing of the auxiliary metric in M^n. Notice that the inclusions (23) provide

$$O(B) \leq O_B(QC_n) = \int\limits_{\Omega} \frac{\omega_{n-1}}{V_{n-1}(B \cap A_0(u))} F(QC_n, d\omega)$$
$$\leq Q^{n-1} \int\limits_{\Omega} \frac{\omega_{n-1}}{V_{n-1}(C_n \cap A_0(u))} F(C_n, d\omega) = Q^{n-1} O_{C_n}(C_n(D)).$$

From the conditions imposed on $O(B)$ in Theorem 2, we have

$$O_{C_n}(C_n) \geq Q^{1-n} O(B) = Q^{1-n}(1-\varepsilon) 2n\omega_{n-1} \geq Q^{-n} 2n\omega_{n-1}.$$

Using the inequalities $\frac{1}{1+x} \geq 1 - x$ and $(1-x)^n \geq 1 - nx$ for

$$0 \leq x = 20\varepsilon^{\frac{1}{2n(n-1)}} \leq \frac{1}{2n},$$

we obtain

$$O_{C_n}(C_n) \geq 2n\omega_{n-1}\left(1 - 20n\varepsilon^{\frac{1}{2n(n-1)}}\right). \tag{24}$$

The surface of the cylinder $C_n(D)$ consists of two bases D_T, D_{-T} that are equal to D and of a lateral surface C'_n. Hence,

$$O_{C_n}(C_n) = 2O_{C_n}(D) + O_{C_n}(C'_n) = 2\omega_{n-1} + O_{C_n}(C'_n). \tag{25}$$

Denote by Ω' an intersection of the unit sphere Ω and the $(n-1)$ dimensional hyperplane R^{n-1} which corresponds to $A_0(u_0)$. Recall the equalities

$$\begin{cases} V_{n-1}(C_n \cap A_0(w)) = 2h_0 V_{n-2}(D \cap A_0(w)), \ w \in \Omega'; \\ F_{n-1}(C'_n, d\omega) = 2h_0 F_{n-2}(D, d'\omega), \end{cases},$$

where $d'\omega$ is a restriction of $d\omega$ on Ω'. Thus,

$$O_{C_n}(C'_n) = \int\limits_{\Omega} \frac{\omega_{n-1}}{V_{n-1}(C_n \cap A_0(w))} F_{n-1}(C'_n, d\omega)$$
$$= \int\limits_{\Omega'} \frac{\omega_{n-1}}{V_{n-2}(D \cap A_0(w))} F_{n-2}(D, d'\omega) = \frac{\omega_{n-1}}{\omega_{n-2}} O_D(D).$$

Imposing the condition $\varepsilon \leq (20n)^{-2n^3}$ and taking into account (24) and (25), we obtain

$$O_D(D) \geq 2(n-1)\omega_{n-2}\left(1 - \frac{20n^2}{n-1}\varepsilon^{\frac{1}{2n(n-1)}}\right) \geq 2(n-1)\omega_{n-2}\left(1 - \varepsilon^{\frac{1}{2n^2}}\right).$$

Set $\varepsilon_1 = \varepsilon^{\frac{1}{2n^2}}$. Then for the compact proper central symmetric $(n-1)$-dimensional body D the estimate

$$O(D) \geq 2(n-1)\omega_{n-2}(1-\varepsilon_1)$$

holds.

Remark. *Calculations in the proof of Theorem 3 up to formula* (23) *remain valid for the dimension* $(n-1) \geq 3$.

Taking initially a body D instead of B, which is in the space R^{n-1} adjoint to $A_0(u_0)$, we can construct a centrally symmetric cylinder $C_{n-1} = C_{n-1}(D_{n-2})$ with the cross-section $D_{n-2} \subset R^{n-2}$ satisfying the inclusions similar to (6). Namely,

$$C_{n-1}(D_{n-2}) \subset D \subset \left(1 + \varepsilon_1^{\frac{1}{2(n-1)^2}}\right) C_{n-1}(D_{n-2}).$$

In R^n consider a cylinder $C_n(C_{n-1}(D_{n-2}))$ whose cross-sections coincide with the "$(n-$ 1)-dimensional" cylinder $C_{n-1}(D_{n-2})$; the one-dimensional generators are parallel to u_0 and bounded by the hyperplanes $A_T(u_0)$ and $A_{-T}(u_0)$. The cylinder possesses a specific property

$$C_n(C_{n-1}(D_{n-2})) \subset B \subset \left(1 + \varepsilon^{\frac{1}{2n^2}}\right)\left(1 + \varepsilon^{\frac{1}{2^2 n^2 (n-1)^2}}\right) C_n(C_{n-1}(D_{n-2})). \quad (26)$$

Using recurrently $(n-2)$ times the specified above constructions that correspond to the pass from formula (23) to formula (26), we get a cylinder $\widetilde{C} = C_n(C_{n-1}(\ldots(C_3(D_2))\ldots))$ which approximates the initial normalizing body B as follows:

$$\widetilde{C} \subset B \subset \left(1 + \varepsilon^{\frac{1}{2n^2}}\right)\left(1 + \varepsilon^{\frac{1}{2^2 n^2 (n-1)^2}}\right) \ \ldots \ \left(1 + \varepsilon^{\frac{1}{2^{n-2} n^2 (n-1)^2 \ \ldots \ 3^2}}\right) \widetilde{C}.$$

The cylinder C_2 on the plane M^2 is a parallelogram. Approximate a figure D_2 by the parallelogram; the approximation order is defined on the $(n-1)$-step by $\varepsilon_{n-1} = \varepsilon^{2^{4-n}(n!)^{-2}}$. Recall that on the plane M^2 there is formula (4) from Theorem 1, where ε_{n-1} appears to be in the first degree. Thus, it is possible to approximate the body B by the parallelepiped P for which the inclusions

$$P \subset B \subset \left(1 + \varepsilon^{\frac{1}{2n^2}}\right)\left(1 + \varepsilon^{\frac{1}{2^2 n^2 (n-1)^2}}\right) \times \ \ldots$$
$$\times \left(1 + \varepsilon^{\frac{1}{2^{n-4}(n!)^2}}\right)\left(1 + 18\varepsilon^{\frac{1}{2^{n-4} t (n!)^2}}\right) P \quad (27)$$

hold. There is such a sufficiently small positive $\varepsilon_0(n)$ depending only on the dimension n that the inequalities

$$\left(1+18\varepsilon^{2^{4-n}(n!)^{-2}}\right)^{n-1} \leq 1+18n\varepsilon^{2^{4-n}(n!)^{-2}} \leq 1+\varepsilon^{2^{-n}(n!)^{-2}} \qquad (28)$$

hold for $0\leq \varepsilon \leq \varepsilon_0$. Put $\delta = 2^{-n}(n!)^{-2}$. Then from (27) and (28) we derive formula (5). The theorem is proved. ■

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