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Space-like Submanifolds with Parallel Normalized Mean Curvature Vector Field in de Sitter Space

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Space-like submanifolds, with dimension greater than three and with negative definite normal bundle in a general de Sitter space, of any index, are studied. For the compact space-like submanifolds whose mean curvature has no zero and the corresponding normalized vector field is parallel, under natural boundedness assumptions on the lengths of the gradient of the length of the mean curvature and the covariant derivative of the second fundamental form, it is proved that they must be totally umbilical. As an application, two characterizations of totally umbilical space-like submanifolds in terms of the scalar curvature and the length of its second fundamental form are given. All the results extend the previous ones obtained by Liu for the case of space-like hypersurfaces in de Sitter space of index one. In addition, for the complete space-like submanifolds, whose normalized mean curvature vector field is parallel, two characterizations of totally umbilical space-like submanifolds and hyperbolic cylinders are obtained.

Key words: space-like submanifold, de Sitter space, normalized mean curvature vector, totally umbilical submanifold.

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1. Introduction

Let $M_p^{n+p}(c)$ be an $(n+p)$-dimensional connected semi-Riemannian manifold of constant sectional curvature c whose index is p. It is called an indefinite space form of index p and simply a space form when $p = 0$. If $c > 0$, we call it a de Sitter space of index p and denote by $S_p^{n+p}(c)$. It was pointed out by Marsden and Tipler [1] and Stumbles [2] that space-like hypersurfaces with constant mean curvature

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in arbitrary space-time got interesting in the relativity theory. Space-like hypersurfaces with constant mean curvature are convenient as initial hypersurfaces for the Cauchy problem in arbitrary space-time and for studying the propagation of gravitational radiation. Therefore, space-like hypersurfaces in a de Sitter space with constant mean curvature have recently been studied by many differential geometers from both physics and mathematical points of view. For example, one can see [3–6]. Goddard [4] conjectured that the complete constant mean curvature space-like hypersurfaces in a de Sitter space must be umbilical. Akutagawa [3] and Ramanathan [6] proved independently that a complete space-like hypersurface in a de Sitter space with constant mean curvature is totally umbilical if the mean curvature H satisfies $H^2 \leq c$ when $n = 2$ and $n^2H^2 < 4(n-1)c$ when $n \geq 3$. The well-known examples with $H^2 = 4(n-1)/n^2$ are the umbilical sphere $S^n((n-2)^2/n^2))$ and the hyperbolic cylinder $H^1(c_1) \times S^{n-1}(c_2)$, $c_1 = (2-n)$ and $c_2 = (n-2)/(n-1)$. Later, Cheng [7] generalized the result of [3] and [6] to general submanifolds with higher codimension in a de Sitter space $S_p^{n+p}(c)$.

On the other hand, there are some interesting results related to the study of space-like hypersurfaces in a de Sitter space with constant scalar curvature, see, for instance [8–10]. Recently, Camargo, Chaves and De Sousa Jr. [11] have studied the complete space-like submanifolds with higher codimension in a de Sitter space $S_p^{n+p}(c)$. If the normalized mean curvature vector field is parallel, the scalar curvature $n(n-1)R$ is constant and $R \leq c$, they obtain some interesting results.

We should notice that the investigation on space-like hypersurfaces with the scalar curvature $n(n-1)R$ and the mean curvature H being linearly related is also interesting, see, for instance, [8, 9, 12, 13]. Cheng [12] and Li [8] obtained some characteristic theorems of such hypersurfaces in terms of the sectional curvature. Recently, the author [13] proved a characteristic theorem of such hypersurfaces in terms of the mean curvature H. The well-known complete space-like hypersurfaces with constant mean curvature are given by

$$M^n = \{p \in S_1^{n+1} \mid p_{k+1}^2 + \cdots + p_{n+1}^2 = \cosh^2 r\},$$

with $r \in R^1$ and $1 \leq k \leq n$, where R^1 is the set of all real numbers. We can prove that M^n is isometric to the Riemannian product $H^k(\sinh r) \times S^{n-k}(\cosh r)$ of a k-dimensional hyperbolic space and a $(n-k)$-dimensional sphere of radii $\sinh r$ and $\cosh r$, respectively. M^n has k principal curvatures equal to $\coth r$ and $(n-k)$ principal curvatures equal to $\tanh r$, so the mean curvature is given by $nH = k \coth r + (n-k) \tanh r$. If $k = 1$, the Riemannian product $H^1(\sinh r) \times S^{n-1}(\cosh r)$ is called a hyperbolic cylinder.

Let $|\nabla h|^2 = \sum_{i,j,k,\alpha}(h_{ijk}^\alpha)^2$ and $|\nabla H|^2 = \sum_{i,\alpha}(H_{,i}^\alpha)^2$. From Proposition 3.1 and Proposition 3.2 in Section 3, we should notice that the condition $|\nabla h|^2 \geq$

$n^2|\nabla H|^2$ is the natural generalization of one of the following three conditions: (i) $H = constant$, (ii) the scalar curvature $n(n-1)R$ is constant and $R \leq c$, (iii) the scalar curvature $n(n-1)R$ is proportional to the mean curvature H, that is, $n(n-1)R = kH$.

For compact space-like hypersurfaces in a de Sitter space $S_1^{n+1}(1)$ with $|\nabla h|^2 \geq n^2|\nabla H|^2$, Liu [13] has recently proved the following results:

Theorem 1.1. *Let M^n be an n-dimensional $(n \geq 3)$ compact space-like hypersurface in an $(n+1)$-dimensional de Sitter space $S_1^{n+1}(1)$. If $|\nabla h|^2 \geq n^2|\nabla H|^2$ and*

$$|h|^2 \leq 2\sqrt{n-1},$$

then M^n is a totally umbilical hypersurface, where $|h|^2$ is the squared norm of the second fundamental form and H is the mean curvature of M^n.

Corollary 1.1. *Let M^n be an n-dimensional $(n \geq 3)$ compact space-like hypersurface with constant scalar curvature $n(n-1)R$ in an $(n+1)$-dimensional de Sitter space $S_1^{n+1}(1)$. If $R \leq 1$ and*

$$|h|^2 \leq 2\sqrt{n-1},$$

then M^n is a totally umbilical hypersurface.

Corollary 1.2. *Let M^n be an n-dimensional $(n \geq 3)$ compact space-like hypersurface in an $(n+1)$-dimensional de Sitter space $S_1^{n+1}(1)$. Suppose that the scalar curvature $n(n-1)R$ is proportional to the mean curvature H of M^n, that is, there exists a constant k such that $n(n-1)R = kH$. If*

$$|h|^2 \leq 2\sqrt{n-1},$$

then M^n is a totally umbilical hypersurface.

It is natural and interesting to study the n-dimensional compact space-like submanifolds in a de Sitter space $S_p^{n+p}(c)$ with $|\nabla h|^2 \geq n^2|\nabla H|^2$. We should point out that the normalized mean curvature vector field is defined by $\frac{\xi}{H}$, where ξ and H denote the mean curvature vector field and the mean curvature of M^n, respectively. It is well known that submanifolds with nonzero parallel mean curvature vector field also have parallel normalized mean curvature vector field. The condition to have parallel normalized mean curvature vector field is much weaker than the condition to have parallel mean curvature vector field. If the mean curvature vector field is parallel, that is, $\nabla H = 0$, we have H constant.

In this paper, by using Cheng-Yau's self-adjoint operator, we generalize Liu's results to general submanifolds in a de Sitter space $S_p^{n+p}(c)$ with parallel normalized mean curvature vector field. We shall prove the following:

Theorem 1.2. *Let M^n be an n-dimensional $(n \geq 3)$ compact space-like submanifold in an $(n+p)$-dimensional de Sitter space $S_p^{n+p}(c)$. Suppose that the normalized mean curvature vector field is parallel. If $|\nabla h|^2 \geq n^2|\nabla H|^2$ and*

$$|h|^2 \leq nc/[(1+\frac{1}{p})\frac{\sqrt{n-1}}{n} + (1-\frac{1}{p})\frac{n-1}{n} + \frac{(n-2)^2}{2n\sqrt{n-1}}],$$

then M^n is a totally umbilical submanifold, where $|h|^2$ is the squared norm of the second fundamental form and H is the mean curvature of M^n.

Since we know that submanifolds with nonzero parallel mean curvature vector field also have parallel normalized mean curvature vector field and $\nabla H = 0$, we can easily see that

Corollary 1.3. *Let M^n be an n-dimensional $(n \geq 3)$ compact space-like submanifold with nonzero parallel mean curvature vector field in an $(n+p)$-dimensional de Sitter space $S_p^{n+p}(c)$. If*

$$|h|^2 \leq nc/[(1+\frac{1}{p})\frac{\sqrt{n-1}}{n} + (1-\frac{1}{p})\frac{n-1}{n} + \frac{(n-2)^2}{2n\sqrt{n-1}}],$$

then M^n is a totally umbilical submanifold.

We also have the following:

Corollary 1.4. *Let M^n be an n-dimensional $(n \geq 3)$ compact space-like submanifold with constant scalar curvature $n(n-1)R$ in an $(n+p)$-dimensional de Sitter space $S_p^{n+p}(c)$. Suppose that the normalized mean curvature vector field is parallel. If $R \leq c$ and*

$$|h|^2 \leq nc/[(1+\frac{1}{p})\frac{\sqrt{n-1}}{n} + (1-\frac{1}{p})\frac{n-1}{n} + \frac{(n-2)^2}{2n\sqrt{n-1}}],$$

then M^n is a totally umbilical submanifold.

Corollary 1.5. *Let M^n be an n-dimensional $(n \geq 3)$ compact space-like submanifold in an $(n+p)$-dimensional de Sitter space $S_p^{n+p}(c)$. Suppose that the normalized mean curvature vector field is parallel and the scalar curvature*

$n(n-1)R$ is proportional to the mean curvature H of M^n, that is, there exists a constant k such that $n(n-1)R = kH$. If

$$|h|^2 \leq nc/[(1+\frac{1}{p})\frac{\sqrt{n-1}}{n} + (1-\frac{1}{p})\frac{n-1}{n} + \frac{(n-2)^2}{2n\sqrt{n-1}}],$$

then M^n is a totally umbilical submanifold.

R e m a r k 1.1. If $p = 1$ and $c = 1$, we have

$$|h|^2 \leq nc/[(1+\frac{1}{p})\frac{\sqrt{n-1}}{n} + (1-\frac{1}{p})\frac{n-1}{n} + \frac{(n-2)^2}{2n\sqrt{n-1}}] = 2\sqrt{n-1},$$

then Theorem 1.2, Corollary 1.3 and Corollary 1.4 reduce to Theorem 1.1, Corollary 1.1 and Corollary 1.2, respectively. Therefore, we generalize the previous results obtained by Liu [9] to general submanifolds with higher codimension.

R e m a r k 1.2. We should notice that L.J. Alias and A. Romero [14] proved an integral formula for the compact space-like n-submanifolds in de Sitter spaces $S_q^{n+p}(c)$, $1 \leq q \leq p$, by calculating the divergence of certain tangent vector fields and using the divergence theorem. They obtained a Bernstein type result for the complete maximal submanifolds in $S_q^{n+p}(c)$, $1 \leq q \leq p$. From [15], if $p = q$, we know that the complete maximal space-like submanifolds in $S_p^{n+p}(c)$ or R_p^{n+p} are totally geodesic. Therefore, the class of all these submanifolds is very small. But if $q < p$, we see that the class of complete maximal space-like submanifolds is very large (see [16]). Thus, it is very interesting to study the n-dimensional space-like submanifolds in $S_q^{n+p}(c)$, $1 \leq q < p$. The Simons' formulas of the n-dimensional space-like submanifolds in $S_q^{n+p}(c)$, $1 \leq q < p$, from those in $S_p^{n+p}(c)$. Thus, the results will be different.

2. Preliminary

Let $S_p^{n+p}(c)$ be an $(n+p)$-dimensional de Sitter space with index p. Let M^n be an n-dimensional connected space-like submanifold immersed in $S_p^{n+p}(c)$. We choose a local field of the semi-Riemannian orthonormal frames $e_1, \ldots, e_{n+p}$ in $S_p^{n+p}(c)$ such that at each point of M^n, $e_1, \ldots, e_n$ span the tangent space of M^n and form an orthonormal frame there. We use the following convention on the range of indices:

$$1 \leq A, B, C, \ldots \leq n+p; \quad 1 \leq i, j, k, \ldots \leq n, \quad n+1 \leq \alpha, \beta, \gamma, \ldots \leq n+p.$$

Let $\omega_1, \ldots, \omega_{n+p}$ be its dual frame field so that the semi-Riemannian metric of $S_p^{n+p}(c)$ is given by $d\bar{s}^2 = \sum_i \omega_i^2 - \sum_\alpha \omega_\alpha^2 = \sum_A \varepsilon_A \omega_A^2$, where $\varepsilon_i = 1$ and $\varepsilon_\alpha = -1$.

Then the structure equations of $S_p^{n+p}(c)$ are given by

$$d\omega_A = \sum_B \varepsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} = \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \varepsilon_C \varepsilon_D K_{ABCD} \omega_C \wedge \omega_D, \tag{2.2}$$

$$K_{ABCD} = c\varepsilon_A \varepsilon_B (\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{2.3}$$

If we restrict these form to M^n, then

$$\omega_\alpha = 0, \quad n+1 \le \alpha \le n+p. \tag{2.4}$$

From Cartan's lemma we have

$$\omega_{\alpha_i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha. \tag{2.5}$$

The connection forms of M^n are characterized by the structure equations

$$d\omega_i = \sum_{j=1}^n \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.6}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \tag{2.7}$$

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \tag{2.8}$$

where R_{ijkl} are the components of the curvature tensor of M^n.

Denote by h the second fundamental form of M^n. Then

$$h = \sum_{i,j,\alpha} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha. \tag{2.9}$$

Denote by ξ, H and $|h|^2$ the mean curvature vector field, the mean curvature and the squared norm of the second fundamental form of M^n, respectively. Then they are defined by

$$\xi = \frac{1}{n} \sum_\alpha (\sum_i h_{ii}^\alpha) e_\alpha, \;\; H = |\xi| = \frac{1}{n} \sqrt{\sum_\alpha (\sum_i h_{ii}^\alpha)^2}, \;\; |h|^2 = \sum_{i,j,\alpha} (h_{ij}^\alpha)^2. \tag{2.10}$$

Moreover, the normal curvature tensor $R_{\alpha\beta kl}$, the Ricci curvature tensor R_{ik} and the scalar curvature $n(n-1)R$ are expressed as

$$R_{\alpha\beta kl} = \sum_m (h_{km}^\alpha h_{ml}^\beta - h_{lm}^\alpha h_{mk}^\beta), \tag{2.11}$$

$$R_{ik} = (n-1)c\delta_{ik} - \sum_{\alpha}(\sum_{l} h^{\alpha}_{ll})h^{\alpha}_{ik} + \sum_{\alpha,j} h^{\alpha}_{ij}h^{\alpha}_{jk}, \tag{2.12}$$

$$n(n-1)R = n(n-1)c + |h|^2 - n^2H^2, \tag{2.13}$$

where R is the normalized scalar curvature.

Define the first and the second covariant derivatives of h^{α}_{ij}, say h^{α}_{ijk} and h^{α}_{ijkl}, by

$$\sum_{k} h^{\alpha}_{ijk}\omega_k = dh^{\alpha}_{ij} + \sum_{k} h^{\alpha}_{ik}\omega_{kj} + \sum_{k} h^{\alpha}_{jk}\omega_{ki} - \sum_{\beta} h^{\beta}_{ij}\omega_{\beta\alpha}, \tag{2.14}$$

$$\sum_{l} h^{\alpha}_{ijkl}\omega_l = dh^{\alpha}_{ijk} + \sum_{m} h^{\alpha}_{mjk}\omega_{mi} + \sum_{m} h^{\alpha}_{imk}\omega_{mj} + \sum_{m} h^{\alpha}_{ijm}\omega_{mk} - \sum_{\beta} h^{\beta}_{ijk}\omega_{\beta\alpha}. \tag{2.15}$$

We obtain the Codazzi equation by straightforward computations

$$h^{\alpha}_{ijk} = h^{\alpha}_{ikj}. \tag{2.16}$$

It follows that the Ricci identities hold

$$h^{\alpha}_{ijkl} - h^{\alpha}_{ijlk} = \sum_{m} h^{\alpha}_{im}R_{mjkl} + \sum_{m} h^{\alpha}_{jm}R_{mikl} + \sum_{\beta} h^{\beta}_{ij}R_{\alpha\beta kl}. \tag{2.17}$$

The Laplacian of h^{α}_{ij} is defined by $\Delta h^{\alpha}_{ij} = \sum_{k} h^{\alpha}_{ijkk}$. From (2.17), for any $\alpha, n+1 \le \alpha \le n+p$, we obtain

$$\Delta h^{\alpha}_{ij} = \sum_{k} h^{\alpha}_{kkij} + \sum_{k,m} h^{\alpha}_{km}R_{mijk} + \sum_{k,m} h^{\alpha}_{im}R_{mkjk} + \sum_{k,\beta} h^{\beta}_{ik}R_{\alpha\beta jk}. \tag{2.18}$$

In the case when the mean curvature vector ξ has no zero, we know that ξ/H is a normal vector field defined globally on M^n. We define $|\mu|^2$ and $|\tau|^2$ by

$$|\mu|^2 = \sum_{i,j}(h^{n+1}_{ij} - H\delta_{ij})^2, \quad |\tau|^2 = \sum_{\alpha>n+1}\sum_{i,j}(h^{\alpha}_{ij})^2, \tag{2.19}$$

respectively. Then $|\mu|^2$ and $|\tau|^2$ are functions defined on M^n globally, which do not depend on the choice of the orthonormal frame $\{e_1, \ldots, e_n\}$. We have

$$|h|^2 = nH^2 + |\mu|^2 + |\tau|^2. \tag{2.20}$$

Since the normalized mean curvature vector field is parallel, we choose $e_{n+1} = \xi/H$. Then

$$\mathrm{tr}H^{n+1} = \sum_{i} h^{n+1}_{ii} = nH, \quad \mathrm{tr}H^{\alpha} = \sum_{i} h^{\alpha}_{ii} = 0 \ \ (\alpha \ge n+2). \tag{2.21}$$

From (2.8), (2.11), (2.18) and (2.21), by direct calculation we get (see [11])

$$\begin{aligned}\frac{1}{2}\Delta|h|^2 = &\sum_{i,j,k,\alpha}(h^{\alpha}_{ijk})^2 + \sum_{i,j} h^{n+1}_{ij}(nH)_{ij} + nc(|h|^2 - nH^2) \\ &- nH\sum_{\alpha}\mathrm{tr}(H^2_{\alpha}H_{n+1}) + \sum_{\alpha,\beta}[\mathrm{tr}(H_{\alpha}H_{\beta})]^2 \\ &+ \sum_{\alpha,\beta} N(H_{\alpha}H_{\beta} - H_{\beta}H_{\alpha}),\end{aligned} \tag{2.22}$$

where H_α denotes the matrix (h^{α}_{ij}) for all α, $N(A) = \mathrm{tr}(AA^t)$ for any matrix $A = (a_{ij})$.

We need the following lemma

Lemma 2.1 ([17]). *Let A, B be symmetric $n \times n$ matrices satisfying $AB = BA$ and $\mathrm{tr}A = \mathrm{tr}B = 0$. Then*

$$|\mathrm{tr}A^2B| \le \frac{n-2}{\sqrt{n(n-1)}}(\mathrm{tr}A^2)(\mathrm{tr}B^2)^{1/2}, \tag{2.23}$$

and the equality holds if and only if $(n-1)$ of the eigenvalues x_i of B and the corresponding eigenvalues y_i of A satisfy $|x_i| = (\mathrm{tr}B^2)^{1/2}/\sqrt{n(n-1)}$, $x_ix_j \ge 0$, $y_i = (\mathrm{tr}A^2)^{1/2}/\sqrt{n(n-1)}$.

3. Proof of Theorem

For a C^2-function f defined on M^n, we define its gradient and Hessian (f_{ij}) by $df = \sum_i f_i\omega_i$, $\sum_j f_{ij}\omega_j = df_i + \sum_j f_j\omega_{ji}$. Let $T = \sum_{i,j} T_{ij}\omega_i \otimes \omega_j$ be a symmetric tensor on M^n defined by $T_{ij} = nH\delta_{ij} - h^{n+1}_{ij}$. Following Cheng-Yau [18], we introduce an operator $\Box$ associated to T acting on f by

$$\Box f = \sum_{i,j} T_{ij}f_{ij} = \sum_{i,j}(nH\delta_{ij} - h^{n+1}_{ij})f_{ij}. \tag{3.1}$$

Since M^n is compact, the operator $\Box$ is self-adjoint (see [18]) if and only if

$$\int_M (\Box f)g\,dv = \int_M f(\Box g)\,dv,$$

where f and g are any smooth functions on M^n.

By a simple calculation and from (2.13), we obtain

$$\begin{aligned}\Box(nH) =&\sum_{i,j}(nH\delta_{ij}-h_{ij}^{n+1})(nH)_{ij} \\ =&\frac{1}{2}\Delta(n^2H^2)-n^2|\nabla H|^2-\sum_{i,j}h_{ij}^{n+1}(nH)_{ij} \\ =&-\frac{1}{2}n(n-1)\Delta R+\frac{1}{2}\Delta|h|^2-n^2|\nabla H|^2-\sum_{i,j}h_{ij}^{n+1}(nH)_{ij}.\end{aligned} \tag{3.2}$$

Set $\phi_{ij}^{\alpha}=h_{ij}^{\alpha}-\frac{1}{n}\mathrm{tr}H^{\alpha}\delta_{ij}$ and consider the symmetric tensor $\phi=\sum_{i,j,\alpha}\phi_{ij}^{\alpha}\omega_i\omega_j e_{\alpha}$. We can easily know that ϕ is traceless and

$$N(\Phi_\alpha)=N(H_\alpha)-\frac{1}{n}(\mathrm{tr}H_\alpha)^2,\quad |\phi|^2=\sum_{\alpha}N(\Phi_\alpha)=|h|^2-nH^2, \tag{3.3}$$

where Φ_α denotes the matrix (ϕ_{ij}^{α}).

Since the normalized mean curvature vector field is parallel, choosing $e_{n+1}=\xi/H$, from (2.21), we infer that

$$\begin{aligned}&\phi_{ij}^{n+1}=h_{ij}^{n+1}-H\delta_{ij},\quad \phi_{ij}^{\alpha}=h_{ij}^{\alpha},(\alpha\geq n+2),\\ &N(\Phi_{n+1})=N(H_{n+1})-nH^2,\quad N(\Phi_\alpha)=N(H_\alpha),(\alpha\geq n+2),\\ &\mathrm{tr}(H_{n+1})^3=\mathrm{tr}(\Phi_{n+1})^3+3HN(\Phi_{n+1})+nH^3.\end{aligned} \tag{3.4}$$

From (2.22), (3.3) and (3.4), we have

$$\begin{aligned}\frac{1}{2}\Delta|h|^2\geq&\sum_{i,j,k,\alpha}(h_{ijk}^{\alpha})^2+\sum_{i,j}h_{ij}^{n+1}(nH)_{ij}+n(c-H^2)|\phi|^2\\ &-nH\sum_{\alpha}\mathrm{tr}(\Phi_\alpha^2\Phi_{n+1})+\sum_{\alpha,\beta}[\mathrm{tr}(\Phi_\alpha\Phi_\beta)]^2.\end{aligned} \tag{3.5}$$

Since we choose $e_{n+1}=\xi/H$, we have $\omega_{\alpha n+1}=0$ for all α. Consequently, $R_{\alpha n+1jk}=0$, from (2.11), we have $\sum_i h_{ij}^{\alpha}h_{ik}^{n+1}=\sum_i h_{ik}^{\alpha}h_{ij}^{n+1}$, that is, $H_\alpha H_{n+1}=H_{n+1}H_\alpha$. Thus $\Phi_\alpha\Phi_{n+1}=\Phi_{n+1}\Phi_\alpha$. Since the matrices Φ_α and Φ_{n+1} are traceless, by Lemma 2.1, we have

$$\sum_{\alpha}\mathrm{tr}(\Phi_\alpha^2\Phi_{n+1})\leq\frac{n-2}{\sqrt{n(n-1)}}|\mu||\phi|^2\leq\frac{n-2}{\sqrt{n(n-1)}}|\phi|^3, \tag{3.6}$$

where the following

$$|\mu|^2\leq|h|^2-nH^2=|\phi|^2 \tag{3.7}$$

is used. By the Cauchy–Schwarz inequality, we have

$$\sum_{\alpha,\beta}[\mathrm{tr}(\Phi_\alpha\Phi_\beta)]^2 \geq \sum_{\alpha}[N(\Phi_\alpha)]^2 \geq \frac{1}{p}|\phi|^4. \tag{3.8}$$

From (3.5), (3.6) and (3.8), we have

$$\begin{aligned}\frac{1}{2}\Delta|h|^2 \geq & \sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2 + \sum_{i,j} h^{n+1}_{ij}(nH)_{ij} \\ & + |\phi|^2\{nc - nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi| + \frac{1}{p}|\phi|^2\}.\end{aligned} \tag{3.9}$$

From (3.2) and (3.9), we have

$$\begin{aligned}\Box(nH) \geq & -\frac{1}{2}n(n-1)\Delta R + (|\nabla h|^2 - n^2|\nabla H|^2) \\ & + |\phi|^2\{nc - nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi| + \frac{1}{p}|\phi|^2\}.\end{aligned} \tag{3.10}$$

P r o o f of Theorem 1.2. Since M^n is compact and the operator $\Box$ is self-adjoint, by $|\nabla h|^2 \geq n^2|\nabla H|^2$ and Stokes formula, we have

$$\begin{aligned}0 \geq & \int_{M^n} |\phi|^2\{nc - nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi| + \frac{1}{p}|\phi|^2\}dv \\ = & \int_{M^n} |\phi|^2 P_H(|\phi|)dv,\end{aligned} \tag{3.11}$$

where $P_H(|\phi|) = n(c - H^2) - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi| + \frac{1}{p}|\phi|^2$.

Considering the quadratic form $Q(u,t) = \frac{1}{p}u^2 - \frac{n-2}{\sqrt{n-1}}ut - t^2$ and by the orthogonal transformation

$$\begin{aligned}\tilde{u} &= \frac{1}{\sqrt{2n}}\{(1+\sqrt{n-1})u + (1-\sqrt{n-1})t\},\\ \tilde{t} &= \frac{1}{\sqrt{2n}}\{(\sqrt{n-1}-1)u + (\sqrt{n-1}+1)t\},\end{aligned}$$

we have

$$\begin{aligned}
Q(u,t) = &\frac{1}{2n}\{[\frac{1}{p}(n+2\sqrt{n-1})+\frac{(n-2)^2}{\sqrt{n-1}}-(n-2\sqrt{n-1})]\tilde{u}^2\\
&-2(1-\frac{1}{p})(n-2)\tilde{u}\tilde{t}+[\frac{1}{p}(n-2\sqrt{n-1})-\frac{(n-2)^2}{\sqrt{n-1}}-(n+2\sqrt{n-1})]\tilde{t}^2\}\\
=&-\frac{1}{2n}[\frac{1}{p}(2\sqrt{n-1}-n)+\frac{(n-2)^2}{\sqrt{n-1}}+(n+2\sqrt{n-1})](\tilde{u}^2+\tilde{t}^2)\\
&+\frac{1}{2n}[(\frac{1}{p}+1)4\sqrt{n-1}+\frac{2(n-2)^2}{\sqrt{n-1}}]\tilde{u}^2-\frac{1}{2n}(1-\frac{1}{p})(n-2)2\tilde{u}\tilde{t}\\
\geq&-\frac{1}{2n}[\frac{1}{p}(2\sqrt{n-1}-n)+\frac{(n-2)^2}{\sqrt{n-1}}+(n+2\sqrt{n-1})\\
&+(1-\frac{1}{p})(n-2)](\tilde{u}^2+\tilde{t}^2)+\frac{1}{2n}[(\frac{1}{p}+1)4\sqrt{n-1}+\frac{2(n-2)^2}{\sqrt{n-1}}]\tilde{u}^2\\
=&-[(1+\frac{1}{p})\frac{\sqrt{n-1}}{n}+(1-\frac{1}{p})\frac{n-1}{n}+\frac{(n-2)^2}{2n\sqrt{n-1}}](\tilde{u}^2+\tilde{t}^2)\\
&+\frac{1}{n}[(1+\frac{1}{p})2\sqrt{n-1}+\frac{(n-2)^2}{\sqrt{n-1}}]\tilde{u}^2\\
\geq&-[(1+\frac{1}{p})\frac{\sqrt{n-1}}{n}+(1-\frac{1}{p})\frac{n-1}{n}+\frac{(n-2)^2}{2n\sqrt{n-1}}](\tilde{u}^2+\tilde{t}^2),
\end{aligned}$$

where $\tilde{u}^2+\tilde{t}^2=u^2+t^2$.

Take $u=|\phi|, t=\sqrt{n}H$, then

$$P_H(|\phi|)=nc+Q(u,t)\geq nc-[(1+\frac{1}{p})\frac{\sqrt{n-1}}{n}+(1-\frac{1}{p})\frac{n-1}{n}+\frac{(n-2)^2}{2n\sqrt{n-1}}]|h|^2.$$

From (3.11) and the assumption of Theorem 1.2, we have

$$0\geq\int_{M^n}|\phi|^2\{nc-[(1+\frac{1}{p})\frac{\sqrt{n-1}}{n}+(1-\frac{1}{p})\frac{n-1}{n}+\frac{(n-2)^2}{2n\sqrt{n-1}}]|h|^2\}dv\geq 0. \quad (3.12)$$

Therefore, we see that

$$|\phi|^2\{nc-[(1+\frac{1}{p})\frac{\sqrt{n-1}}{n}+(1-\frac{1}{p})\frac{n-1}{n}+\frac{(n-2)^2}{2n\sqrt{n-1}}]|h|^2\}=0.$$

This implies that either $|\phi|^2=0$ or M^n is totally umbilical, or

$$nc-[(1+\frac{1}{p})\frac{\sqrt{n-1}}{n}+(1-\frac{1}{p})\frac{n-1}{n}+\frac{(n-2)^2}{2n\sqrt{n-1}}]|h|^2=0.$$

In the latter case, we infer that the equalities hold in (3.12), (3.11), (3.7) and (2.23) of Lemma 2.1. If the equality holds in (3.7), we have $|\mu|^2 = |h|^2 - nH^2$, this implies that $|\tau| = 0$. Since e_{n+1} is parallel on the normal bundle $T^{\perp}(M^n)$ of M^n, by using the method of B.Y. Chen [19] or Yau [20], we know that M^n lies in a totally geodesic submanifold $S_1^{n+1}(c)$ of $S_p^{n+p}(c)$. If the equality holds in Lemma 2.1, then $(n-1)$ of the numbers $\lambda_i - H$ are equal to $\frac{N(\Phi_{n+1})}{\sqrt{n(n-1)}} = \frac{|\mu|}{\sqrt{n(n-1)}}$, or equal to the negative of this last expression, where $\lambda_i \delta_{ij} = h_{ij}^{n+1}$. It follows that M^n has at most two distinct constant principle curvatures. We conclude that M^n is totally umbilical from the compactness of M^n. This completes the proof of Theorem 1.2.

From [21], we have the following:

Proposition 3.1. *Let M^n be an n-dimensional space-like submanifold in an $(n+p)$-dimensional de Sitter space $S_p^{n+p}(c)$. If the scalar curvature $n(n-1)R$ is constant and $R \leq c$, then we have*

$$|\nabla h|^2 \geq n^2 |\nabla H|^2.$$

We may also prove the following:

Proposition 3.2. *Let M^n be an n-dimensional space-like submanifold in an $(n+p)$-dimensional de Sitter space $S_p^{n+p}(c)$. If the scalar curvature $n(n-1)R$ is proportional to the mean curvature H of M^n, that is, there exists a constant k such that $n(n-1)R = kH$, then we have*

$$|\nabla h|^2 \geq n^2 |\nabla H|^2.$$

P r o o f. For a fixed α, we choose an orthonormal frame field $\{e_i\}$ at each point on M^n so that $h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}$. Then we have $|h|^2 = \sum_{i,j,\alpha} (h_{ij}^\alpha)^2 \neq 0$. In fact, if $|h|^2 = \sum_{i,\alpha} (\lambda_i^\alpha)^2 = 0$ at a point of M^n, then $\lambda_i^\alpha = 0$ for all i and α at this point. This implies $H = 0$ and $R = 0$ at this point. From (2.13), we have $n(n-1)c = 0$. This is impossible. From (2.13) and $n(n-1)R = kH$, we have

$$k \nabla_i H = -2n^2 H \nabla_i H + 2 \sum_{j,k,\alpha} h_{kj}^\alpha h_{kji}^\alpha,$$

$$(\frac{k}{2} + n^2 H)^2 |\nabla H|^2 = \sum_i (\sum_{j,k,\alpha} h_{kj}^\alpha h_{kji}^\alpha)^2 \leq \sum_{i,j,\alpha} (h_{ij}^\alpha)^2 \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 = |h|^2 |\nabla h|^2.$$

Thus, we have

$$|\nabla h|^2 - n^2|\nabla H|^2 \geq [(\frac{k}{2} + n^2H)^2 - n^2|h|^2]|\nabla H|^2 \frac{1}{|h|^2}$$
$$= [\frac{(k)^2}{4} + n^3(n-1)c]|\nabla H|^2 \frac{1}{|h|^2} \geq 0.$$

The proof of Proposition 3.2 is completed.

From Theorem 1.2, Proposition 3.1 and Proposition 3.2, we can easily see that Corollary 1.4 and Corollary 1.5 are true.

4. Some Related Results for Complete Cases

In this section, we study the complete space-like submanifolds in a de Sitter space $S_p^{n+p}(c)$ with parallel normalized mean curvature vector field. We obtain the following:

Theorem 4.1. *Let M^n be an n-dimensional $(n \geq 3)$ complete space-like submanifold with constant scalar curvature $n(n-1)R$ in an $(n+p)$-dimensional de Sitter space $S_p^{n+p}(c)$. Suppose that the normalized mean curvature vector field is parallel and the mean curvature H obtains its supremum on M^n. If $R < c$ and*

$$|h|^2 \leq nc/[(1+\frac{1}{p})\frac{\sqrt{n-1}}{n} + (1-\frac{1}{p})\frac{n-1}{n} + \frac{(n-2)^2}{2n\sqrt{n-1}}],$$

then M^n is totally umbilical, or M^n is isometric to a hyperbolic cylinder $H^1(\sinh r)\times S^{n-1}(\cosh r)$.

Theorem 4.2. *Let M^n be an n-dimensional $(n \geq 3)$ complete space-like submanifold in an $(n+p)$-dimensional de Sitter space $S_p^{n+p}(c)$. Suppose that the normalized mean curvature vector field is parallel and the mean curvature H obtains its supremum on M^n. If there exists a constant k such that $n(n-1)R = kH$ and*

$$|h|^2 \leq nc/[(1+\frac{1}{p})\frac{\sqrt{n-1}}{n} + (1-\frac{1}{p})\frac{n-1}{n} + \frac{(n-2)^2}{2n\sqrt{n-1}}],$$

then M^n is totally umbilical, or M^n is isometric to a hyperbolic cylinder $H^1(\sinh r)$ $\times S^{n-1}(\cosh r)$.

We prove the following Lemma:

Lemma 4.1. *Let M^n be an n-dimensional space-like submanifold in a de Sitter space $S_p^{n+p}(c)$. Then the following properties hold:*

(i) if $R < c$, then the operator $\Box$ defined by (3.1) is elliptic;

(ii) if $n(n-1)R = kH$ and $H > 0$, then the operator $L = \Box + (k/2n)\Delta$ is elliptic.

P r o o f. (i) Choosing a local orthonormal frame field $\{e_1, \ldots, e_n\}$ such that $h_{ij}^{n+1} = \lambda_i \delta_{ij}$, we get $\Box f = \sum_i (nH - \lambda_i) f_{ii}$. Since $R < c$, from (2.13), we have $|h|^2 < n^2H^2$. If there is one i such that $nH - \lambda_i \leq 0$, then $n^2H^2 \leq \lambda_i^2 \leq |h|^2$. This is a contradiction. Thus, we have $nH - \lambda_i > 0$ for any i and the operator $\Box$ is elliptic.

(ii) For a fixed α, we choose a local orthonormal frame field $\{e_1, \ldots, e_n\}$ at each point on M^n so that $h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}$. From $H > 0$, $nH = \sum_i h_{ii}^{n+1}$ and $\sum_i h_{ii}^\alpha = 0$ for $n+2 \leq \alpha \leq n+p$ on M^n, we have for any i:

$$
\begin{aligned}
(nH - \lambda_i^{n+1} + k/2n) &= \sum_j \lambda_j^{n+1} - \lambda_i^{n+1} \\
&\quad + (1/2)[\sum_{j,\alpha} (\lambda_j^\alpha)^2 - n^2H^2 + n(n-1)c]/(nH) \\
&\geq \sum_j \lambda_j^{n+1} - \lambda_i^{n+1} \\
&\quad + (1/2)[\sum_j (\lambda_j^{n+1})^2 - (\sum_j \lambda_j^{n+1})^2 + n(n-1)c]/(nH) \\
&=[(\sum_j \lambda_j^{n+1})^2 - \lambda_i^{n+1}(\sum_j \lambda_j^{n+1}) \\
&\quad - (1/2)\sum_{l \neq j} \lambda_l^{n+1}\lambda_j^{n+1} + (1/2)n(n-1)c](nH)^{-1} \\
&=[\sum_j (\lambda_j^{n+1})^2 + (1/2)\sum_{l \neq j} \lambda_l^{n+1}\lambda_j^{n+1} \\
&\quad - \lambda_i^{n+1}(\sum_j \lambda_j^{n+1}) + (1/2)n(n-1)c](nH)^{-1} \\
&=[\sum_{j \neq i} (\lambda_j^{n+1})^2 + (1/2)\sum_{\substack{l \neq j \\ l,j \neq i}} \lambda_l^{n+1}\lambda_j^{n+1} + (1/2)n(n-1)c](nH)^{-1} \\
&=(1/2)[\sum_{j \neq i} (\lambda_j^{n+1})^2 + (\sum_{j \neq i} \lambda_j^{n+1})^2 + n(n-1)c](nH)^{-1} > 0.
\end{aligned}
\tag{4.1}
$$

Thus, L is an elliptic operator. The proof of Lemma 4.1 is completed.

From (3.10) and the proof of Theorem 1.2, we have

$$\begin{aligned}\Box(nH) \geq & -\frac{1}{2}n(n-1)\Delta R + (|\nabla h|^2 - n^2|\nabla H|^2) \\ & + |\phi|^2\{nc - [(1+\frac{1}{p})\frac{\sqrt{n-1}}{n} + (1-\frac{1}{p})\frac{n-1}{n} + \frac{(n-2)^2}{2n\sqrt{n-1}}]|h|^2\}.\end{aligned} \tag{4.2}$$

P r o o f of Theorem 4.1. Since the scalar curvature $n(n-1)R$ is constant and $R < c$, from Proposition 3.1, (4.2) and the assumption of Theorem 4.1, we have

$$\Box(nH) \geq |\phi|^2\{nc - [(1+\frac{1}{p})\frac{\sqrt{n-1}}{n} + (1-\frac{1}{p})\frac{n-1}{n} + \frac{(n-2)^2}{2n\sqrt{n-1}}]|h|^2\} \geq 0. \tag{4.3}$$

Since H obtains its supremum on M^n and $\Box$ is elliptic, we see that H is constant. Thus, from (4.3), we get

$$|\phi|^2\{nc - [(1+\frac{1}{p})\frac{\sqrt{n-1}}{n} + (1-\frac{1}{p})\frac{n-1}{n} + \frac{(n-2)^2}{2n\sqrt{n-1}}]|h|^2\} = 0.$$

It follows that $|\phi|^2 = 0$, and M^n is totally umbilical, or

$$nc - [(1+\frac{1}{p})\frac{\sqrt{n-1}}{n} + (1-\frac{1}{p})\frac{n-1}{n} + \frac{(n-2)^2}{2n\sqrt{n-1}}]|h|^2 = 0.$$

In the latter case, we know that the equalities hold in (4.3), (4.2), (3.7) and (2.23) of Lemma 2.1. By the same method as in the proof of Theorem 1.2, we see that M^n lies in a totally geodesic submanifold $S_1^{n+1}(c)$ of $S_p^{n+p}(c)$ and has two distinct constant principle curvatures. Therefore, we know that M^n is isometric to a hyperbolic cylinder $H^1(\sinh r) \times S^{n-1}(\cosh r)$ from the congruence theorem in [22]. This completes the proof of Theorem 4.1.

P r o o f of Theorem 4.2. Applying the operator $L = \Box + (k/2n)\Delta$ to nH and by Proposition 3.2, (4.2) and the assumption of Theorem 4.2, we have

$$\begin{aligned}L(nH) = & \Box(nH) + \frac{k}{2n}\Delta(nH) = \Box(nH) + \frac{1}{2}n(n-1)\Delta R \\ \geq & |\phi|^2\{nc - [(1+\frac{1}{p})\frac{\sqrt{n-1}}{n} + (1-\frac{1}{p})\frac{n-1}{n} + \frac{(n-2)^2}{2n\sqrt{n-1}}]|h|^2\} \geq 0.\end{aligned} \tag{4.4}$$

Since the normalized mean curvature vector field is parallel and $H \neq 0$, from (2.10) it follows that $H > 0$. From Lemma 4.1, we know that L is elliptic as H

obtains its supremum on M^n, we can see that H is constant. Thus, from (4.4), we get

$$|\phi|^2\{nc - [(1+\frac{1}{p})\frac{\sqrt{n-1}}{n} + (1-\frac{1}{p})\frac{n-1}{n} + \frac{(n-2)^2}{2n\sqrt{n-1}}]|h|^2\} = 0.$$

It follows that $|\phi|^2 = 0$, and M^n is totally umbilical, or

$$nc - [(1+\frac{1}{p})\frac{\sqrt{n-1}}{n} + (1-\frac{1}{p})\frac{n-1}{n} + \frac{(n-2)^2}{2n\sqrt{n-1}}]|h|^2 = 0.$$

By the same method as in the proof of Theorem 4.1, we see that Theorem 4.2 is true.

If $c = 1$ and $p = 1$, we can easily see that there holds the following:

Corollary 4.1. *Let M^n be an n-dimensional ($n \geq 3$) complete space-like hypersurface with constant scalar curvature $n(n-1)R$ in an $(n+1)$-dimensional de Sitter space $S_1^{n+1}(1)$. Suppose that the mean curvature H obtains its supremum on M^n. If $R \leq 1$ and*

$$|h|^2 \leq 2\sqrt{n-1},$$

then M^n is totally umbilical, or M^n is isometric to a hyperbolic cylinder $H^1(\sinh r)\times S^{n-1}(\cosh r)$.

Corollary 4.2. *Let M^n be an n-dimensional ($n \geq 3$) compact space-like hypersurface in an $(n+1)$-dimensional de Sitter space $S_1^{n+1}(1)$. Suppose that the mean curvature H obtains its supremum on M^n. If there exists a constant k such that $n(n-1)R = kH$ and*

$$|h|^2 \leq 2\sqrt{n-1},$$

then M^n is totally umbilical, or M^n is isometric to a hyperbolic cylinder $H^1(\sinh r)\times S^{n-1}(\cosh r)$.

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