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Characterization of Hyperbolic Cylinders in a Lorentzian Space Form

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We give a characterization of the n-dimensional ($n \geq 3$) hyperbolic cylinders in a Lorentzian space form. We show that the hyperbolic cylinders are the only complete space-like hypersurfaces in an $(n+1)$-dimensional Lorentzian space form $M_1^{n+1}(c)$ with non-zero constant mean curvature H whose two distinct principal curvatures λ and μ satisfy $\inf(\lambda-\mu)^2 > 0$ for $c \leq 0$ or $\inf(\lambda-\mu)^2 > 0$, $H^2 \geq c$, for $c > 0$, where λ is of multiplicity $n-1$ and μ of multiplicity 1 and $\lambda < \mu$.

Key words: space-like hypersurface, Lorentzian space form, mean curvature, principal curvature, hyperbolic cylinder.

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1. Introduction

By an $(n+1)$-dimensional Lorentzian space form $M_1^{n+1}(c)$ we mean a Minkowski space R_1^{n+1}, a de Sitter space $S_1^{n+1}(c)$ or an anti-de Sitter space $H_1^{n+1}(c)$, according to $c > 0$, $c = 0$ or $c < 0$, respectively. That is, a Lorentzian space form $M_1^{n+1}(c)$ is a complete simply connected $(n+1)$-dimensional Lorentzian manifold with constant curvature c. A hypersurface in a Lorentzian manifold is said to be space-like if the induced metric on the hypersurface is positive definite.

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In connection with the negative settlement of the Bernstein problem due to Calabi [1] and Cheng-Yau [2], Choquet-Bruhat *et al.* [3] proved the following theorem:

Theorem 1.1 ([3]). *Let M be a complete space-like hypersurface in an $(n+1)$-dimensional Lorentzian space form $M_1^{n+1}(c)$, $c \geq 0$. If M is maximal, then it is totally geodesic.*

T. Ishihara [4] also proved the following well-known result:

Theorem 1.2 ([4]). *Let M be an n-dimensional $(n \geq 2)$ complete maximal space-like hypersurface in anti-de Sitter space $H_1^{n+1}(-1)$, then*

$$S \leq n, \tag{1.1}$$

and $S = n$ if and only if $M = H^m(-\frac{n}{m}) \times H^{n-m}(-\frac{n}{n-m})$,$(1 \leq m \leq n-1)$, where S denotes the square of the norm of the second fundamental form of M.

Recently, L. Cao and G. Wei [5] gave a new characterization of hyperbolic cylinders in anti-de Sitter space $H_1^{n+1}(-1)$ as follows:

Theorem 1.3 ([5]). *Let M be an n-dimensional $(n \geq 3)$ complete maximal space-like hypersurface with two distinct principal curvatures λ and μ in anti-de Sitter space $H_1^{n+1}(-1)$. If $\inf(\lambda-\mu)^2 > 0$, then $M = H^m(-\frac{n}{m}) \times H^{n-m}(-\frac{n}{n-m})$, $(1 \leq m \leq n-1)$.*

As a generalization of Theorem 1.1, the complete space-like hypersurfaces with constant mean curvature in a Lorentz manifold were studied by many mathematicians, see, for instance, ([6–12]). We should note that two types of the well-known standard models of complete space-like hypersurfaces with non-zero constant mean curvature in an $(n+1)$-dimensional Lorentzian space form $M_1^{n+1}(c)$ are the totally umbilical space-like hypersurfaces and the following product manifolds:

$$H^k(c_1) \times S^{n-k}(c_2) \ \text{ in } \ S_1^{n+1}(c), (\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}, c_1 < 0, c_2 > 0),$$

$$H^k(c_1) \times R^{n-k} \ \text{ in } \ R_1^{n+1}, \ \ (c_1 < 0, c = c_2 = 0),$$

$$H^k(c_1) \times H^{n-k}(c_2) \ \text{ in } \ H_1^{n+1}(c), (\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}, c_1 < 0, c_2 < 0),$$

where $k = 1, \ldots, n-1$. These three product hypersurfaces are respectively called the hyperbolic cylinders in $S_1^{n+1}(c)$, R_1^{n+1} or $H_1^{n+1}(c)$. From U-H. Ki *et al.* [9], we know that the hyperbolic cylinder $H^1(c_1) \times S^{n-1}(c_2)$ in $S_1^{n+1}(c)$ has two distinct principal curvatures $\sqrt{c-c_1}$ with multiplicity 1 and $\sqrt{c-c_2}$ with multiplicity $n-1$; the hyperbolic cylinder $H^1(c_1) \times R^{n-1}$ in R_1^{n+1} has two distinct principal curvatures $\sqrt{-c_1}$ with multiplicity 1 and 0 with multiplicity $n-1$; the hyperbolic cylinder $H^1(c_1) \times H^{n-1}(c_2)$ in $H_1^{n+1}(c)$ has two distinct principal curvatures $\pm\sqrt{c-c_1}$ with multiplicity 1 and $\mp\sqrt{c-c_2}$ with multiplicity $n-1$. The square of the norm of the second fundamental form satisfies (1.3).

U-H. Ki *et al.* [9] proved that

Theorem 1.4 ([9]). *Let M be a complete space-like hypersurface with constant mean curvature in an $(n+1)$-dimensional Lorentzian space form $M_1^{n+1}(c)$. If one of the following properties holds*

(1) $c \leq 0$,

(2) $c > 0$, $n \geq 3$ *and* $n^2H^2 \geq 4(n-1)c$,

(3) $c > 0$, $n = 2$ *and* $H^2 > c$,

then

$$S \leq -nc + \frac{n^3H^2}{2(n-1)} + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 - 4(n-1)cH^2}, \tag{1.2}$$

where S denotes the square of the norm of the second fundamental form of M.

As an application to Theorem 1.4, the authors of [9] gave a characterization of hyperbolic cylinders of a Lorentzian space form $M_1^{n+1}(c)$ as follows:

Theorem 1.5 ([9]). *The hyperbolic cylinders are the only complete space-like hypersurfaces of $M_1^{n+1}(c)$ with non-zero mean curvature H and such that the square of the norm of the second fundamental form satisfies*

$$S = -nc + \frac{n^3H^2}{2(n-1)} + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 - 4(n-1)cH^2}. \tag{1.3}$$

About the same time, R. Aiyama [13] obtained a characterization of hyperbolic cylinders of a Lorentzian 3-space form $M_1^3(c)$ as follows:

Theorem 1.6 ([13]). *The hyperbolic cylinders are the only complete space-like surfaces in $M_1^3(c)$ with non-zero constant mean curvature whose principal curvatures λ and μ satisfy* $\inf(\lambda-\mu)^2 > 0$.

It is natural for us to pose the following problem:

Problem 1.1. *Are the hyperbolic cylinders the only complete space-like hypersurfaces in an $(n+1)$-dimensional Lorentzian space form $M_1^{n+1}(c)$ with non-zero constant mean curvature and two distinct principal curvatures λ and μ satisfying* $\inf(\lambda-\mu)^2 > 0$*?*

We should note that for $c \leq 0$, L. Cao and G. Wei [5] posed the same problem as above, but they did not solve it. So the problem is still open.

In this paper, we will give a characterization of the hyperbolic cylinders of $(n+1)$-dimensional Lorentzian space form $M_1^{n+1}(c)$, which implies that the above

Problem 1.1 can be solved affirmatively for $c \le 0$. For $c > 0$, we should note that the condition $H^2 \ge c$ is necessary. We state our result as follows:

Main Theorem. *Let M be an n-dimensional ($n \ge 3$) complete space-like hypersurface in an $(n+1)$-dimensional Lorentzian space form $M_1^{n+1}(c)$ with non-zero constant mean curvature and two distinct principal curvatures λ and μ of multiplicities $n-1$ and 1 and $\lambda < \mu$. Then:*

(1) *For $c = 0$, if $\inf(\lambda - \mu)^2 > 0$, then M is the hyperbolic cylinder $H^1(c_1) \times R^{n-1}$, where $c_1 < 0$;*

(2) *For $c < 0$, if $\inf(\lambda - \mu)^2 > 0$, then M is the hyperbolic cylinder $H^1(c_1) \times H^{n-1}(c_2)$, where $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$, $c_1 < 0$, $c_2 < 0$;*

(3) *For $c > 0$, if $\inf(\lambda - \mu)^2 > 0$ and $H^2 \ge c$, then M is the hyperbolic cylinder $H^1(c_1) \times S^{n-1}(c_2)$, where $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$, $c_1 < 0$, $c_2 > 0$.*

R e m a r k 1.1. For the case $n = 2$, this Main Theorem was proved by R. Aiyama [13](see Theorem 1.6).

2. Preliminaries

Let M be an n-dimensional space-like hypersurface in an $(n+1)$-dimensional Lorentzian space form $M_1^{n+1}(c)$. We choose a local field of the semi-Riemannian orthonormal frames $\{e_1, \ldots, e_{n+1}\}$ in $M_1^{n+1}(c)$ such that at each point of M, $\{e_1, \ldots, e_n\}$ span the tangent space of M and form an othonormal frame there. We use the following convention on the range of indices:

$$1 \le A, B, C, \ldots, \le n+1; \quad 1 \le i, j, k, \ldots, \le n.$$

Let $\{\omega_1, \ldots, \omega_{n+1}\}$ be the dual frame field so that the semi-Riemannian metric of $M_1^{n+1}(c)$ is given by $d\bar{s}^2 = \sum\limits_i \omega_i^2 - \omega_{n+1}^2 = \sum\limits_A \epsilon_A \omega_A^2$, where $\epsilon_i = 1$ and $\epsilon_{n+1} = -1$.

The structure equations of $M_1^{n+1}(c)$ are given by

$$d\omega_A + \sum_B \epsilon_B \omega_{AB} \wedge \omega_B = 0, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} + \sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} = \Omega_{AB}, \tag{2.2}$$

where

$$\Omega_{AB} = -\frac{1}{2} \sum_{C,D} K_{ABCD} \omega_C \wedge \omega_D, \tag{2.3}$$

$$K_{ABCD} = \epsilon_A \epsilon_B c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{2.4}$$

If we restrict these forms to M, we have

$$\omega_{n+1} = 0. \tag{2.5}$$

Cartan's Lemma implies that

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{2.6}$$

The structure equations of M are

$$d\omega_i + \sum_j \omega_{ij} \wedge \omega_j = 0, \qquad \omega_{ij} + \omega_{ji} = 0, \tag{2.7}$$

$$d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} = -\frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.8}$$

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.9}$$

where R_{ijkl} are the components of the curvature tensor of M, and

$$h = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j \tag{2.10}$$

is the second fundamental form of M.
From the above equation, we have

$$n(n-1)(R-c) = S - n^2H^2, \tag{2.11}$$

where $n(n-1)R$ is the scalar curvature of M, H is the mean curvature, and $S = \sum_{i,j} h_{ij}^2$ is the square of the norm of the second fundamental form of M.

The Codazzi equations are

$$h_{ijk} = h_{ikj}, \tag{2.12}$$

where the covariant derivative of h_{ij} is defined by

$$\sum_k h_{ijk}\omega_k = dh_{ij} - \sum_m h_{mj}\omega_{mi} - \sum_m h_{im}\omega_{mj}. \tag{2.13}$$

The second covariant derivative of h_{ij} is defined by

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} - \sum_m h_{mjk}\omega_{mi} - \sum_m h_{imk}\omega_{mj} - \sum_m h_{ijm}\omega_{mk}. \tag{2.14}$$

Then we have the following Ricci identities

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}. \tag{2.15}$$

In a neighbourhood of a point x of M, we may choose orthonormal frame field $\{e_1, \ldots, e_n\}$ such that $h_{ij} = \lambda_i \delta_{ij}$ at x. We introduce the operator ϕ given by

$$\langle \phi X, Y\rangle = \langle hX, Y\rangle - H\langle X, Y\rangle. \tag{2.16}$$

Putting $\phi = \sum\limits_{i,j} \phi_{ij}\omega_i \otimes \omega_j$, where $\phi_{ij} = h_{ij} - H\delta_{ij}$, we can easily see that ϕ is traceless, that the basis $\{e_1, \ldots, e_n\}$ also diagonalizes ϕ at x with eigenvalues $\mu_i = \lambda_i - H$, and that

$$|\phi|^2 = \sum_i \mu_i^2 = \frac{1}{2n}\sum_{i,j}(\lambda_i - \lambda_j)^2 = S - nH^2.$$

Therefore, we know that $|\phi|^2 \equiv 0$ if and only if M is totally umbilical. We shall prove the following Lemma:

Lemma 2.1. *Let M be an n-dimensional space-like hypersurface in an $(n+1)$-dimensional Lorentzian space form $M_1^{n+1}(c)$ with constant mean curvature and two distinct principal curvatures λ and μ of multiplicities $n-1$ and 1 and $\lambda < \mu$. Then*

$$\frac{1}{2}\Delta|\phi|^2 = |\nabla\phi|^2 + |\phi|^2\{|\phi|^2 - \frac{n(n-2)H}{\sqrt{n(n-1)}}|\phi| + n(c - H^2)\}. \tag{2.17}$$

P r o o f. We firstly need the following result due to [14] and [15] : Let $\mu_1, \mu_2, \ldots, \mu_n$ be real numbers such that $\sum\limits_i \mu_i = 0$ and $\sum\limits_i \mu_i^2 = \beta^2$, where $\beta =$ const ≥ 0, then

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}\beta^3, \tag{2.18}$$

and equality holds in the right-hand(left-hand) side if and only if $(n-1)$ of the $\mu_i's$ are non-positive and equal ($(n-1)$ of the $\mu_i's$ are non-negative and equal).

Now we put $\mu_i = \lambda_i - H$, then $\sum\limits_i \mu_i = 0$ and $\sum\limits_i \mu_i^2 = |\phi|^2$. Since M has two distinct principal curvatures λ and μ of multiplicities $n-1$ and 1, without loss of generality, we may assume $\lambda_1 = \cdots = \lambda_{n-1} = \lambda$, $\mu = \lambda_n$, where λ_i for $i = 1, 2, \ldots, n$ are the principal curvatures of M. Therefore, we know that

$$(n-1)\lambda + \mu = nH, \quad S = (n-1)\lambda^2 + \mu^2. \tag{2.19}$$

Since we assume that $\lambda < \mu$, from (2.19), we have $n(\lambda - H) = \lambda - \mu < 0$. Therefore, we know that $\mu_1 = \cdots = \mu_{n-1} = \lambda - H < 0$. We infer that the equality holds in the right-hand side of (2.18), that is,

$$\sum_i \mu_i^3 = \frac{n-2}{\sqrt{n(n-1)}}|\phi|^3. \tag{2.20}$$

From [8] or [10], we have the well-known Simons' formula of Lorentzian version as follows

$$\frac{1}{2}\Delta|\phi|^2 = |\nabla\phi|^2 + (|\phi|^2)^2 - nH\,\mathtt{tr}\phi^3 + n(c - H^2)|\phi|^2. \tag{2.21}$$

From (2.20) and (2.21), we see that Lemma 2.1 is true.

The following generalized maximum principle will be important in the sequel.

Proposition 2.1 ([16, 17])**.** *Let M be a complete Riemannian manifold with Ricci curvature bounded from below and f a C^2-function which is bounded from below on M. Then there is a point sequence x_k in M such that*

$$\lim_{k\to\infty} f(x_k) = \inf(f), \quad \lim_{k\to\infty} |\nabla f(x_k)| = 0, \quad \lim_{k\to\infty} \inf \Delta f(x_k) \geq 0.$$

Now we state a proposition which can be proved by making use of the similar method due to Otsuki [18].

Proposition 2.2. *Let M be a hypersurface in an $(n+1)$-dimensional Lorentzian space form $M_1^{n+1}(c)$ such that the multiplicities of the principal curvatures are constant. Then the distribution of the space of the principal vectors corresponding to each principal curvature is completely integrable. In particular, if the multiplicity of a principal curvature is greater than 1, then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of the principal vectors.*

3. Proof of Main Theorem

We denote the integral submanifold through $x \in M^n$ corresponding to λ by $M_1^{n-1}(x)$. Putting

$$d\lambda = \sum_{k=1}^{n} \lambda_{,k}\,\omega_k, \qquad d\mu = \sum_{k=1}^{n} \mu_{,k}\,\omega_k. \tag{3.1}$$

From Proposition 2.2, we have

$$\lambda_{,1} = \lambda_{,2} = \cdots = \lambda_{,n-1} = 0 \quad \text{on} \quad M_1^{n-1}(x). \tag{3.2}$$

From (2.19), we have

$$d\mu = -(n-1)d\lambda. \tag{3.3}$$

Hence, we also have

$$\mu_{,1} = \mu_{,2} = \cdots = \mu_{,n-1} = 0 \quad \text{on} \quad M_1^{n-1}(x). \tag{3.4}$$

From (2.13), we have

$$\sum_k h_{ijk}\omega_k = d\lambda_i\delta_{ij} + (\lambda_j - \lambda_i)\omega_{ij}. \tag{3.5}$$

We infer that

$$h_{ijk} = 0, \quad \text{for any} \quad k, \quad \text{if} \quad i \neq j, \quad 1 \leq i \leq n-1 \quad \text{and} \quad 1 \leq j \leq n-1. \tag{3.6}$$

From (3.1), (3.2) and (3.5), we have for $1 \leq j \leq n-1$,

$$\begin{aligned} d\lambda &= d\lambda_j = \sum_{k=1}^{n} h_{jjk}\omega_k \\ &= \sum_{k=1}^{n-1} h_{jjk}\omega_k + h_{jjn}\omega_n = \lambda_{,n}\,\omega_n. \end{aligned} \tag{3.7}$$

Therefore, we have for $1 \leq j \leq n-1$,

$$h_{jjk} = 0, \quad 1 \leq k \leq n-1, \quad \text{and} \quad h_{jjn} = \lambda_{,n}\,. \tag{3.8}$$

From (3.1), (3.4) and (3.5), we have

$$\begin{aligned} d\mu &= d\lambda_n = \sum_{k=1}^{n} h_{nnk}\omega_k \\ &= \sum_{k=1}^{n-1} h_{nnk}\omega_k + h_{nnn}\omega_n = \sum_{i=1}^{n} \mu_{,i}\,\omega_i = \mu_{,n}\,\omega_n. \end{aligned} \tag{3.9}$$

Hence, we obtain

$$h_{nnk} = 0, \quad 1 \leq k \leq n-1, \quad \text{and} \quad h_{nnn} = \mu_{,n}\,. \tag{3.10}$$

Now we prove the following Lemma:

Lemma 3.1. *Let M be an n-dimensional space-like hypersurface in an $(n+1)$-dimensional Lorentzian space form $M_1^{n+1}(c)$ with a constant mean curvature and two distinct principal curvatures λ and μ of multiplicities $n-1$ and 1. Then*

$$|\nabla|\phi|^2|^2 = \frac{4n|\phi|^2}{n+2}|\nabla\phi|^2, \tag{3.11}$$

where ϕ is defined by (2.16).

P r o o f. From (2.19), we have

$$\begin{aligned} |\phi|^2 &= S - nH^2 = n(n-1)\lambda^2 - 2n(n-1)\lambda H + n(n-1)H^2 \\ &= n(n-1)(\lambda - H)^2. \end{aligned} \tag{3.12}$$

Hence, from (3.2) we obtain

$$|\nabla|\phi|^2|^2 = \sum_k (|\phi|^2_{,k})^2 = \sum_k [2n(n-1)(\lambda - H)\lambda_{,k}]^2 \tag{3.13}$$
$$= 4n^2(n-1)^2(\lambda - H)^2(\lambda_{,n})^2.$$

Since $\phi_{ij} = h_{ij} - H\delta_{ij}$, from (3.3), (3.6), (3.8) and (3.10), we have

$$|\nabla\phi|^2 = |\nabla h|^2 = \sum_{i,j,k} h^2_{ijk} = \sum_{i,j,k=1}^{n-1} h^2_{ijk} + 3\sum_{i,j=1}^{n-1} h^2_{ijn} + 3\sum_{i=1}^{n-1} h^2_{inn} + h^2_{nnn} \tag{3.14}$$
$$= 3\sum_{i=1}^{n-1} h^2_{iin} + h^2_{nnn} = 3(n-1)(\lambda_{,n})^2 + (\mu_{,n})^2$$
$$= 3(n-1)(\lambda_{,n})^2 + (n-1)^2(\lambda_{,n})^2 = (n-1)(n+2)(\lambda_{,n})^2.$$

From (3.12), (3.13) and (3.14), we have

$$|\nabla|\phi|^2|^2 = 4n^2(n-1)^2(\lambda - H)^2\frac{|\nabla\phi|^2}{(n-1)(n+2)}$$
$$= \frac{4n^2(n-1)(\lambda - H)^2}{n+2}|\nabla\phi|^2 = \frac{4n|\phi|^2}{n+2}|\nabla\phi|^2.$$

So, the proof of Lemma 3.1 is completed.

P r o o f of Main Theorem. Since we assume that $\inf(\lambda-\mu)^2 > 0$, we have $(\lambda-\mu)^2 > 0$. Putting $(\lambda-\mu)^2 = \kappa > 0$, we have $[n(\lambda - H)]^2 = (\lambda-\mu)^2 = \kappa > 0$. Therefore, we know that

$$|\phi|^2 = n(n-1)(\lambda - H)^2 = \frac{n-1}{n}\kappa > 0,$$

that is, M is not umbilical. From Lemma 2.1 and Lemma 3.1, we have

$$\frac{1}{2}\Delta|\phi|^2 = \frac{n+2}{4n|\phi|^2}|\nabla|\phi|^2|^2 + |\phi|^2\{|\phi|^2 - \frac{n(n-2)H}{\sqrt{n(n-1)}}|\phi| + n(c - H^2)\}. \tag{3.15}$$

Since the Ricci curvature $R_{ii} \geq (n-1)c - \frac{n^2H^2}{4}$ and $|\phi|^2 = n(n-1)(\lambda-H)^2 = \frac{n-1}{n}\kappa > 0$ are bounded from below, from Proposition 2.1, we have that there is a point sequence x_k in M such that

$$\lim_{k\to\infty} |\phi|^2(x_k) = \inf(|\phi|^2), \quad \lim_{k\to\infty} |\nabla|\phi|^2(x_k)| = 0, \quad \lim_{k\to\infty} \inf \Delta|\phi|^2(x_k) \geq 0.$$

By (3.15), we have

$$\inf|\phi|^2\{\inf|\phi|^2 - \frac{n(n-2)H}{\sqrt{n(n-1)}}\inf|\phi| + n(c - H^2)\} \geq 0.$$

Since $\inf|\phi|^2 > 0$, we have

$$\inf|\phi|^2 - \frac{n(n-2)H}{\sqrt{n(n-1)}}\inf|\phi| + n(c-H^2) \geq 0. \tag{3.16}$$

Since for $c > 0$, $H^2 \geq c$ implies $n^2H^2 \geq 4(n-1)c$, we know that the discriminant of (3.16) is non-negative for all c. From (3.16), we have

$$\inf|\phi| \leq \frac{1}{2}\sqrt{\frac{n}{n-1}}[(n-2)H - \sqrt{n^2H^2 - 4(n-1)c}], \tag{3.17}$$

or

$$\inf|\phi| \geq \frac{1}{2}\sqrt{\frac{n}{n-1}}[(n-2)H + \sqrt{n^2H^2 - 4(n-1)c}]. \tag{3.18}$$

Assume that (3.17) holds, if $c \leq 0$, we have $\inf|\phi| \leq \frac{1}{2}\sqrt{\frac{n}{n-1}}[(n-2)H - nH] < 0$, which contradicts $\inf|\phi|^2 > 0$; if $c > 0$, since we assume that $H^2 \geq c$, we have $\inf|\phi| \leq \frac{1}{2}\sqrt{\frac{n}{n-1}}[(n-2)H - \sqrt{n^2H^2 - 4(n-1)c}] \leq 0$, this is also in contradiction to $\inf|\phi|^2 > 0$. Therefore, we know that (3.18) holds, we have

$$|\phi|^2 \geq \frac{1}{4}\frac{n}{n-1}[(n-2)H + \sqrt{n^2H^2 - 4(n-1)c}]^2,$$

and this is equivalent to

$$S \geq -nc + \frac{n^3H^2}{2(n-1)} + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 - 4(n-1)cH^2}.$$

From Theorem 1.4, we have

$$S = -nc + \frac{n^3H^2}{2(n-1)} + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 - 4(n-1)cH^2}.$$

By Theorem 1.5, we see that Main Theorem is true.

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