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Antipodal Polygons and Their Group Properties

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We study the groups of transformations which transform antipodal polygons into antipodal ones as well as their order and the number of equivalence classes of n-gons inscribed into the regular $(2n-1)$-gon.

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1. Introduction

The two n-gons inscribed into the regular $(2n-1)$-gon are said to be *antipodal* if the total number of their diagonals and sides of the same length is n for all admissible lengthes. This notation was introduced in [1, p. 48] for studying the *half-circulant* Hadamard matrices of the form

$$H = \begin{pmatrix} A & B \\ B & -A \end{pmatrix},$$

where $A(B)$ is a matrix of order $2n$ including submatrix of order $2n-1$ which is the right (respectively the left) circulant [2, p. 459].

The question on the existence of half-circulant Hadamard matrices of order $4n$, n being an arbitrary natural number, is equivalent to the question on the existence of inscribed antipodal n-gons [1, Theorem 4]. As the question on the existence of half-circulant Hadamard matrices of order $4n$ for any n still remains open, it is important to study the properties of antipodal polygons.

In the present paper, we study the group properties of antipodal polygons, namely, the groups of transformations which transform antipodal polygons into antipodal ones as well as their order and the number of equivalence classes of n-gons inscribed into the regular $(2n-1)$-gon. We begin with the results obtained in [1] and their generalizations.

2. Self-superposition of the Regular $(2n-1)$-gon

In a complex plane, we consider a regular $(2n-1)$-gon inscribed into the unit circle with center at the origin. Any its vertex is given by a monomial z^k, $k = 0, 1, 2, \dots, 2n-2$, where $z = e^{\frac{2\pi i}{2n-1}}$ and $n \geq 3$. Hence, a convex n-gon P inscribed into the regular $(2n-1)$-gon is given by the generating polynomial $p(z) = \sum_{k=0}^{2n-2} x_k z^k$, where $x_k = 1$ if the vertex of the regular $(2n-1)$-gon with number k belongs to P, and $x_k = 0$ if otherwise. Since P is an n-gon, $\sum_{k=0}^{2n-2} x_k = n$. Thus, for the square of the modulus of the polynomial $p(z)$, the relation

$$|p|^2 = n + 2\sum_{k=1}^{n-1} d_k \cos\frac{2\pi k}{2n-1}$$

holds, where d_k is the number of equal diagonals and sides of n-gon P for which a vision angle (from the origin) is $\frac{2\pi k}{2n-1}$ [1, Lemma 1], and $\sum_{k=1}^{n-1} d_k = n(n-1)/2$. Thereby, for the antipodal n-gons P and P' with the generating polynomials p and $p' = \sum_{k=0}^{2n-2} x'_k z^k$, the equality $|p|^2 + |p'|^2 = n$ holds since, by the definition of antipodal n-gons, $d_k + d'_k = n$ for each $k = 1, 2, \dots, n-1$, and by the well-known identity, $\sum_{k=1}^{n-1} \cos\frac{2\pi k}{2n-1} = -\frac{1}{2}$ (see [1, Theorem 3]).

In [1], for convex n-gons inscribed into the regular $(2n-1)$-gon, there are studied trivial and nontrivial transformations which transform one inscribed n-gon into another and thus transform antipodal n-gons into antipodal ones. To trivial transformations we refer rotations on angle multiple $\frac{2\pi}{2n-1}$ and symmetries relative to straight lines passing through the center of the regular $(2n-1)$-gon and one of its vertices. Under such a rotation (and a specular reflection) each convex n-gon is transformed into equal convex n-gon. Since in equal polygons the corresponding diagonals are equal, antipodal n-gons transform into antipodal n-gons by definition. Moreover, their sides transform into sides, and diagonals into diagonals. But, since the sides and the diagonals have the same lengthes are equivalent by definition, one cannot exclude from consideration the nontrivial superpositions of the regular $(2n-1)$-gon when some of its sides transform into its diagonals and conversely. The same can be observed in antipodal n-gons inscribed into it (after joining diagonals!). Nontrivial self-superpositions of the regular $(2n-1)$-gon from [1] which transform some of its sides into diagonals and conversely are called inversions. We will give the definition of the inversion suitable for our case by using special transformations of the generating polynomial of antipodal n-gons.

Let m and $2n-1$ be mutually prime integers denoted by $(m, 2n-1) = 1$. In a generating polynomial $p(z) = \sum_{k=0}^{2n-2} x_k z^k$ of the n-gon P replacing the argument z by z^m, we obtain a convex n-gon P_m with generating polynomial $p_m = \sum_{k=0}^{2n-2} x_k z^{|mk|}$, where $|mk|$ is the least nonnegative residue of number mk modulo $2n-1$. In [1, p. 49], it is shown that if s is a solution of the comparison equation $ms \equiv 1, \pmod{2n-1}$ which is unique by property IV [3, p. 49], then the polynomial p_m can be written in the form $p_m = \sum_{k=0}^{2n-2} x_{|ks|} z^k$. Thus the n-gon P_m is obtained from P by multiplying by modulo $2n-1$ the indexes of the coefficients x_k of its generating polynomial, which correspond to vertices P, by the same number s, $(s, 2n-1) = 1$. The set of all these numbers is the group of residue classes modulo $2n-1$ which are mutually prime. Their total number is determined by the Euler function $\varphi(2n-1) = \prod(1-\frac{1}{p})$, where the product is extended to all proper prime divisors of the number $2n-1$ (see [3, p. 59]). The transformation of n-gons inscribed into a regular $(2n-1)$-gon under which the numbers of their vertices are multiplied modulo $2n-1$ by the same natural number, which is mutually prime with module, is called *inversion*. The set of all inversions forms the group possessing the properties of the residue classes which are mutually prime with module $2n-1$. In particular, the product of two inversions is an inversion. And all inversions form the Abelian group Γ_{2n-1} of order $\varphi(2n-1)$. By Theorem 1 [1], any inversion transformation transforms an antipodal n-gon into antipodal n-gon.

We may summarize the above as follows. Rotations and specular reflections, transforming one inscribed n-gon into another, transforms simultaneously the regular $(2n-1)$-gon they are inscribed in into itself. Together they form a self-superposition group D_{2n-1} of order $4n-2$ which is known to be called as the *dihedral* group. In fact, it is a permutation group over the set of vertices of the regular $(2n-1)$-gon and thus is a subgroup of the symmetrical group of degree $2n-1$. The group of inversions Γ_{2n-1}, transforming one inscribed n-gon into another, also transforms the regular $(2n-1)$-gon they are inscribed in into itself. Let Γ_{2n-1} be a subgroup of the symmetric group of degree $2n-1$ generated by its subgroups D_{2n-1} and Γ_{2n-1}. This subgroup (further called permutation group) consists only of those members of the symmetric group of degree $2n-1$ that are equal to the product of finite number of group members D_{2n-1} and Γ_{2n-1}. The group D_{2n-1} is the group of self-superposition (with diagonals!) of the regular $(2n-1)$-gon.

R e m a r k. Attention should be payed to the fact that the group G_{2n-1} does not include all self-superpositions of the regular $(2n-1)$ that transform antipodal n-gons into antipodal ones. For example, previously it was proved that if number $2n-1$ is prime, then there exists a Hadamard matrix of half-circulant type of order $4n$ [2, Theorems 1 and 2]. To these matrix there corresponds a pair of antipodal n-gons, the vertices of one of which are the vertices of the regular $(2n-1)$-gon, numbers of which are quadratic residues modulo $2n-1$ (and null-vertex), and the vertices of other are quadratic non-residues modulo $2n-1$ (and again null-vertex). It is evident that any permutation, transforming quadratic residues into quadratic ones and quadratic non-residues into quadratic non-residues or transforming quadratic residues into quadratic non-residues and conversely, transforms the pair of antipodal n-gons into itself (each into itself or one into another). At the same time, for some n there may exist other pairs of antipodal n-gons not transformed into antipodal n-gons under this permutation. As it is in the case when $n = 6$, for antipodal 6-gons with the numbers of vertices $0, 1, 2, 3, 5, 6$ and $0, 1, 3, 5, 7, 8$ (natural numbers $1, 3, 5$ are quadratic residues modulo 11, and $2, 6$ and $7, 8$ are quadratic non-residues modulo 11,), the numbers of equal diagonals and sides for the first one are: $d_1 = 4, d_2 = 3, d_3 = 3, d_4 = 2, d_5 = 3$ and for the second one: $d_1 = 2, d_2 = 3, d_3 = 3, d_4 = 4, d_5 = 3$. Therefore, these transformations are not included into our self-superposition group of the regular $(2n-1)$-gon. We are focused only on the self-superpositions of the regular $(2n - 1)$-gon that transform any antipodal polygons into antipodal ones.

3. Equivalence Classes of Inscribed n-gons

Let us renumber the vertices of the regular $(2n - 1)$-gon by integer from 0 to $2n - 2$ inclusive in counterclockwise order. The total number of the convex n-gons inscribed into this polygon is equal, evidently, C^n_{2n-1}. Among them there are either the ones that superimpose one another by a rotation (or, possibly, by a specular reflection), belonging to the dihedral group D_{2n-1} of order $4n-2$, or the ones that superimpose each other by an inversion γ_m, $(m, 2n - 1) = 1$, belonging to the inversion group Γ_{2n-1} of order $\varphi(2n - 1)$. They all belong to the group of self-superpositions G_{2n-1} of the regular $(2n - 1)$-gon generated by them.

Theorem 1. *The order of the permutation group G_{2n-1} is $(2n-1)\varphi(2n-1)$.*

P r o o f. Let $\alpha = (012 \dots 2n - 2)$ be a permutation generating the rotation group of the regular $(2n - 1)$-gon denoted by A_{2n-1}. The group A_{2n-1} is cyclic with $\alpha^{2n-1} = e$, where e is its unit member. And let $\beta = (0)(1\,2n - 2)(2\,2n - 3) \dots (n - 1\,n)$ be a transformation of the symmetry of the regular $(2n - 1)$-gon relatively to the straight line, passing through its center and the vertex with number null, decomposed into the product of cycles. Since $(\alpha^k\beta)^2 = e$, then $\alpha^k\beta = \beta\alpha^{-k}$, where $\alpha^{-k} = \alpha^{2n-1-k}$. Therefore, any permutation of dihedral group D_{2n-1} can be represented in the form α^k or $\alpha^k\beta$, $k = 1, 2, \dots, 2n - 1$. Any permutation of the group D_{2n-1} can also be represented in the form $\alpha^k\gamma_1$ or $\alpha^k\gamma_{2n-2}$, $k = 1, 2, \dots, 2n - 1$, since the inversion γ_1, obtained by multiplying indexes of the generating polynomial $p(z)$ by 1, coincides with the identity element of the rotation group A_{2n-1}, and the inversion γ_{2n-2}, obtained by multiplying the same indexes by $2n - 2$, coincides with β.

Next we verify that $\alpha\gamma_m = \gamma_m\alpha^m$, $m = 2, 3, \dots, 2n - 3$. Let us prove that $\alpha^k\gamma_m = \gamma_m\alpha^{|km|}$ assuming by induction that it is valid for $\alpha^2, \alpha^3, \dots, \alpha^{k-1}$. Indeed,

$$\alpha^k\gamma_m = \alpha(\alpha^{k-1}\gamma_m) = \alpha(\gamma_m\alpha^{|(k-1)m|}) = (\alpha\gamma_m)\alpha^{|(k-1)m|} = \gamma_m\alpha^{|km|}.$$

Then by the above, we obtain that any permutation $\gamma_m\alpha^s$, $s = 1, 2, \ldots, 2n-2$ of the group G_{2n-1} can be represented in the form $\alpha^k\gamma_m$ since for a given m, mutually prime with $2n-1$, the comparison equation $mk \equiv s \pmod{2n-1}$ always has a solution for any integer s and it is unique (by property IV [3, p. 49]). Hence the product of permutations $\alpha^{k_1}\gamma_s$ and $\alpha^{k_2}\gamma_s$ is a permutation of the same kind $\alpha^{k_3}\gamma_{|ms|}$, where the residue is equal to the product residues m and s mutually prime with module $2n-1$, and the power k_3 is uniquely determined by k_1 and k_2. Therefore, a product of any finite number of permutations of the kind $\alpha^k\gamma_m$ is a permutation of the same kind. Thus, $\alpha^k\gamma_m$ is a general element of the group Γ_{2n-1} and, consequently, the order of G_{2n-1} is equal to the product of the integers $2n-1$ and $\varphi(2n-1)$, which was to be proved.

As it follows from the proof of Theorem 1, the permutation group G_{2n-1}, operating on the set of vertices of the regular $(2n-1)$-gon z^s, $s = 0, 1, 2, \ldots, 2n-2$, is a common (non-direct) product of its subgroups A_{2n-1} and Γ_{2n-1} [4, p. 485]. Thus, if not taking into account the passage to the residues modulo $2n-1$, any rotation α^k from the first subgroup leads to the increasing of the number of each vertex on k, and any inversion γ_m from the second subgroup leads to multiplying of the same numbers by m. On a set of convex n-gons inscribed into the regular $(2n-1)$-gon, we introduce the relation of equivalence. Namely, the two inscribed n-gons P_1 and P_2 are said to be *equivalent relative to the group* G_{2n-1} (or any its subgroup) if in this group (subgroup) there can be found a permutation g such that $P_2 = P_1g$. Evidently, $P_1 = P_2g^{-1}$. All inscribed n-gons that are equivalent to each other form an equivalence class. Denote the number of equivalence classes relative to a group G by K_G.

Lemma 1. *The number of equivalence classes $K_{A_{2n-1}}$ of the set of convex n-gons inscribed into the regular $(2n-1)$-gon relative to subgroup A_{2n-1} of the group G_{2n-1} is equal to $C^n_{2n-1}/(2n-1)$. As a representative of every equivalence class there can always be chosen an n-gon with generating polynomial $\sum_{k=0}^{2n-2} x_kz^k$ for which $\sum_{k=0}^{2n-2} kx_k \equiv 0 \pmod{2n-1}$.*

P r o o f. The first statement of the lemma follows from the fact that the order of the rotation group A_{2n-1} is equal to $2n-1$. Let us prove the second statement.

Let an n-gon P with generating polynomial $\sum_{k=0}^{2n-2} x_kz^k$, where $\sum_{k=0}^{2n-2} x_k = n$, be a representative of some equivalence class. And let $\sum_{k=0}^{2n-2} kx_k = q \not\equiv 0 \pmod{2n-1}$. Turn P on the angle $\frac{2\pi r}{2n-1}$ counterclockwise. For the n -gon P in a new position P', $\sum_{k=0}^{2n-2} x'_{|k+r|}z^{|k+r|}$ is a generating polynomial. However, by the construction, $x'_{|k+r|} = x_k$ for all k . Therefore,

$$\sum_{k=0}^{2n-2}(k+r)x'_{|k+r|}=\sum_{k=0}^{2n-2}(k+r)x_k=q+r\sum_{k=0}^{2n-2}x_k=q+rn.$$

Since the integers n and $2n-1$ are mutually prime, then by known property there is one and only one residue class r satisfying the comparison equation $nr+q\equiv 0 \pmod{2n-1}$, from which the second statement follows. Lemma 1 is proved completely.

Corollary. *In each equivalence class relative to the rotation subgroup of* G_{2n-1} *there is only one inscribed* n*-gon for the coefficients* x_k *of the generating polynomial of which the condition* $\sum_{k=0}^{2n-2}kx_k\equiv 0 \pmod{2n-1}$ *is fulfilled.*

The convex n-gons inscribed into the regular $(2n-1)$-gon, for the coefficients of the generating polynomial $\sum_{k=0}^{2n-2}x_kz^k$ of which the condition $\sum_{k=0}^{2n-2}kx_k\equiv 0$ $\pmod{2n-1}$ is fulfilled, are called *basic* n-gons.

Lemma 2. *Each inversion* γ_m $(m,2n-1)=1$ *transforms any basic* n*-gon into itself or into another basic* n*-gon.*

P r o o f. Let P be a basic n-gon with generating polynomial $\sum_{k=0}^{2n-2}x_kz^k$, where $\sum_{k=0}^{2n-2}kx_k\equiv 0 \pmod{2n-1}$. By the definition, the inversion γ_m transforms it into an n-gon with vertex numbers $|km|$, for which $x_k=1$, $0\le k\le 2n-2$. But $\sum_{x_k=1}|km|\equiv m\sum_{x_k=1}k\equiv 0 \pmod{2n-1}$, from which the desired statement follows. Lemma 2 is proved.

Notice that the inversion γ_m transforms the basic n-gon P into itself if numbers of all its vertices enter into several cycles of the kind $(s\,|sm|\,\dots|sm^{i-1}|)$, where $m^i\equiv 1 \pmod{2n-1}$, for which the total sum of lengthes is n, moreover, the power i is a divisor of order $\varphi(2n-1)$ of the inversion group. Any n-gon transformed into itself by permutation $\alpha^k\gamma_m$ of the group G_{2n-1} also possesses this property. It is possible to show that if the inversion γ_m transforms P into itself, then the length of each cycle, in product of which γ_m decomposes, is smaller than n.

The group properties established above allow to obtain the results being of great importance for the development of the computer search algorithm of antipodal polygons.

Theorem 2. *The number of equivalence classes of the set of* C_{2n-1}^n *convex* n*-gons inscribed into the regular* $(2n-1)$*-gon relative to the group* G_{2n-1} *equals*

$$K_{G_{2n-1}}=\left(\frac{C_{2n-1}^n}{2n-1}+\sum_{m=2}^{(m,2n-1)=1}F(\gamma_m)\right)/\varphi(2n-1),$$

where $F(\gamma_m)$ *is the number of basic* n*-gons transformed into themselves by the inversion* γ_m.

P r o o f. By Lemma 1, every equivalence class of the set of n-gons inscribed into the regular $(2n-1)$-gon relative to G_{2n-1} contains necessarily basic n-gons. From Lemma 2, it follows that the set of all basic n-gons, contained in each equivalence class relative to G_{2n-1}, is the equivalence class of all basic n-gons inscribed into the regular $(2n-1)$-gon relative to the inversion group Γ_{2n-1}. Therefore, the numbers of the equivalence classes relative to the groups G_{2n-1} and γ_{2n-1} equal one another. By the well-known Burnside lemma [5, p. 68], we have

$$K_{G_{2n-1}} = \frac{1}{\varphi(2n-1)}\left(\sum_{m=1}^{(m,2n-1)=1} F(\gamma_m)\right),$$

where $\varphi(2n-1)$ is the order of the group Γ_{2n-1} and $F(\gamma_m)$ is the number of basic n-gons transformed into themselves by the inversion γ_m. Taking into account that by Lemma 1 $F(\gamma_1) = \frac{C^n_{2n-1}}{2n-1}$, we obtain the desired statement from the previous relation.

4. Antipodal Basic n-gons

From the proof of Lemma 1 it follows that the basic n-gon is obtained from a non-basic n-gon with the help of some rotation α^k, where $\alpha = (012\ldots 2n-2)$ and $1 \le k \le 2n-2$. Since at any rotation a convex n-gon transforms into the one equal to it, antipodal n-gons P and P' can be transformed into basic n-gons by a proper rotation not changing their quantities d_k and d'_k of equal diagonals and sides (recall that by the definition of antipodal n-gons, $d_k + d'_k = n$ for all $k = 1, 2, \ldots, n-1$). Thus, at finding a pair of antipodal n-gons, it is possible to restrict ourselves only by basic n-gons for which the coefficients x_k and x'_k of their generating polynomials satisfy the conditions $\sum_{k=0}^{2n-2} kx_k \equiv 0 \pmod{2n-1}$, $\sum_{k=0}^{2n-2} kx'_k \equiv 0 \pmod{2n-1}$, respectively.

Theorem 3. *Let $p(z) = \sum_{k=0}^{2n-2} x_k z^k$ and $p'(z) = \sum_{k=0}^{2n-2} x'_k z^k$ be the generating polynomials of antipodal basic n-gons P and P'. Then $\sum_{k=0}^{2n-2} k^2(x_k + x'_k) \equiv 0 \pmod{2n-1}$ if $(3, 2n-1) = 1$ and $\sum_{k=0}^{2n-2} k^2(x_k + x'_k) \equiv \frac{2n-1}{3} \pmod{2n-1}$, otherwise.*

P r o o f. Since P is a basic n-gon, $\sum_{k=0}^{2n-2} kx_k \equiv 0 \pmod{2n-1}$. Next, since x_k equals 1 or 0, that is, $x_k^2 = x_k$, then

$$\left(\sum_{k=0}^{2n-2} kx_k\right)^2 = \sum_{k=0}^{2n-2} k^2 x_k + 2\sum_{k>m} kmx_k x_m$$

$$= \sum_{k=0}^{2n-2} k^2 x_k + \sum_{k>m} [k^2 x_k + m^2 x_m - (kx_k - mx_m)^2]$$

$$= n \sum_{k=0}^{2n-2} k^2 x_k - \sum_{x_k = x_m = 1}^{k>m} (kx_k - mx_m)^2 \equiv n \sum_{k=0}^{2n-2} k^2 x_k - \sum_{s=1}^{n-1} s^2 d_s \pmod{2n-1}.$$

Here on the last step, our passing to the comparison equation is determined by the fact that for the diagonal connecting the vertices of the n-gon P with numbers k and m and visible from the center of the regular $(2n-1)$-gon under the angle, say, $\frac{2\pi s}{2n-1}$, the difference $kx_k - mx_m$ may be equal either to s or $2n-1-s$, but $(2n-1-s)^2 \equiv s^2 \pmod{2n-1}$.

Similarly, for the basic n-gon P' we obtain

$$\left(\sum_{k=0}^{2n-2} kx'_k\right)^2 \equiv n \sum_{k=0}^{2n-2} k^2 x'_k - \sum_{s=1}^{n-1} s^2 d_s \pmod{2n-1}.$$

Summing up the concluding comparisons for the inscribed n-gons P and P' termwise and taking into account that they are basic and antipodal by the conditions of the theorem, we obtain

$$\left(\sum_{k=0}^{2n-2} kx_k\right)^2 + \left(\sum_{k=0}^{2n-2} kx'_k\right)^2 \equiv n \sum_{k=0}^{2n-2} k^2 (x_k + x'_k) - \sum_{s=1}^{n-1} s^2 (d_s + d'_s)$$

$$\equiv n \sum_{k=0}^{2n-2} k^2 (x_k + x'_k) - n \sum_{s=1}^{n-1} s^2 \equiv 0 \pmod{2n-1}.$$

Since the integers n and $2n-1$ are mutually prime, and $\sum_{s=1}^{n-1} s^2 = \frac{n(n-1)(2n-1}{6}$ [6, p. 89], for the basic antipodal n-gons we finally obtain the relation

$$\sum_{k=0}^{2n-2} k^2 (x_k + x'_k) \equiv \frac{n(n-1)(2n-1)}{6} \pmod{2n-1}.$$

It is seen that if 3 is not a divisor of module $2n-1$, then $\sum_{k=0}^{2n-2} k^2 (x^k + x'_k) \equiv 0 \pmod{2n-1}$ which is stated by the theorem. And if 3 is a divisor of integer $2n-1$, then it is easy to check that

$$\frac{n(n-1)(2n-1)}{6} = \frac{(n+1)(n-2)(2n-1)}{6} + \frac{2n-1}{3},$$

where $n+1 = 3n - (2n-1)$ is divided by three. And thus in this case, $\sum_{k=0}^{2n-2} k^2 (x^k + x'_k) \equiv \frac{2n-1}{3} \pmod{2n-1}$. The theorem is proved completely.

Let W_{2n-1} be a quadratic residue subgroup of the group of residue classes modulo $2n-1$, w be its order, and $w_1, w_2, w_3, \ldots$ be its members in ascending order beginning with $w_1 = 1$. Let Q and Q' be basic antipodal n-gons with generating polynomials $q = \sum_{k=0}^{2n-2} x_k z^k$ and $q' = \sum_{k=0}^{2n-2} x'_k z^k$. Denote by q_2 a nonnegative residue of number $\sum_{k=0}^{2n-2} k^2 x_k$ modulo $2n-1$, and by q'_2 a nonnegative residue of number $\sum_{k=0}^{2n-2} k^2 x'_k$ the same modulo. By Theorem 3, $q_2 + q'_2 \equiv \frac{2n-1}{3}$ (mod $2n-1$) or $q_2 + q'_2 \equiv 0 \pmod{2n-1}$ depending on whether 3 is a divisor of integer $2n-1$ or not. Under the condition $q_2 \leq q'_2$, these comparison equations have evidently exactly n solutions. The pair $(\bar{q}_2, \bar{q}'_2)$ is said to be *equivalent* to the pair (q_2, q'_2) if there exists a quadratic residue w_s, $2 \leq s \leq w$ such that the nonnegative residue of number $w_s\bar{q}_2$ is equal to q_2, and nonnegative residue of number $w_s\bar{q}'_2$ is equal to q'_2 or nonnegative residue of number $w_s\bar{q}_2$ is equal to q'_2 and nonnegative residue of number $w_s\bar{q}'_2$ is equal to q_2. Thus, the solution (q_2, q'_2) is equivalent to the solution $(\bar{q}_2, \bar{q}'_2)$ with respect to w_r, where $s \cdot r \equiv 1$ (mod $2n-1$), i.e., the equivalent solutions of our comparison form some class relatively to the quadratic residue group W_{2n-1}.

Theorem 4. *Let $\sum_{k=0}^{2n-2} x_k z^k$ and $\sum_{k=0}^{2n-2} x'_k z^k$ be the generating polynomials of any antipodal basic n-gons Q and Q'. And let K be the number of pairwise nonequivalent pairs (q_2, q'_2), which can be the suitable nonnegative residues modulo $2n-1$ for Q and Q', namely, $q_2 \equiv \sum_{k=0}^{2n-2} k^2 x_k$, $q'_2 \equiv \sum_{k=0}^{2n-2} k^2 x'_k$. Then*

$$K = \left(n + \sum_{s=2}^{w} F(w_s)\right)/w,$$

where $F(w_s)$ is the number of pairs (q_2, q'_2) which are transformed into themselves by quadratic residue w_s.

To prove the theorem, it is sufficiently to notice that $F(w_1) = n$ and to apply the well-known Burnside lemma.

Notice that by property III [3, pp. 99–100], the order of the quadratic residue group W_{2n-1} is equal to $\frac{\varphi(2n-1)}{2^k}$, where k is a quantity of various prime divisors of number $2n-1$.

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