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Radon–Nikodým Theorems for Multimeasures in Non-Separable Spaces

B. Cascales

Departamento de Matemáticas, Universidad de Murcia
30100 Espinardo (Murcia), Spain
E-mail: beca@um.es

V. Kadets

Department of Mechanics and Mathematics, V.N. Karazin Kharkov National University
4 Svobody sq., Kharkiv 61022, Ukraine
E-mail: vova1kadets@yahoo.com

J. Rodríguez

Departamento de Matemática Aplicada, Facultad de Informática, Universidad de Murcia
30100 Espinardo (Murcia), Spain
E-mail: joserr@um.es

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We prove two Radon–Nikodým theorems for multimeasures using set-valued Pettis integrable derivatives. The first one works for dominated strong multimeasures taking convex compact values in a locally convex space. The second one works for strong multimeasures taking bounded convex closed values in a Banach space with the RNP (and for Bochner integral of the Radon–Nikodým derivative as well). The main advantage of our results is the absence of any separability assumptions.

Key words: multimeasure, Radon–Nikodým theorem.

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Dedicated to the memory of Mikhail Kadets

1. Introduction

The first Radon–Nikodým theorems for multimeasures go back to the 1970's where pioneering results were established amongst others by Debreu and Schmeidler [9], Artstein [1], and Costé and Pallu de la Barrière [8]; whereas the first two

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papers deal with multimeasures with values in finite dimensional spaces the last one deals with multimeasures with values in Souslin infinite dimensional spaces. As presented in the introduction of [1] these original results in finite dimensional spaces were motivated for their applications to mathematical economics, control theory and other mathematical fields: Chapter 20 in [19] is a good reading for the origins and applicability of these results.

We are here interested about the mathematical ideas behind Radon–Nidodým theorem for multimeasures and our starting point is the remarkable result that follows:

Theorem 1.1 (Costé and Pallu de la Barrière [8], Thm. 3.1)**.** *Let* (E, Y) *be a dual pair such that* $(E, \sigma(E, Y))$ *is a Souslin space,* M *be a weak multimeasure defined on a complete finite measure space* (Ω, Σ, μ)*. Let* M *take* $\sigma(E, Y)$*-locally compact closed convex values, and let there be a* $\sigma(E, F)$*-locally compact set* Q *such that* $M(A) \subset \mu(A)Q$ *for all* $A \in \Sigma$*. Then there is a weakly integrable multifunction* F *whose indefinite weak integral is* M*.*

Note that the theorem above is applicable for instance when X is a separable Banach space and $Q \subset X$ is a weakly compact set, because in this case the weak topology $\sigma(X, X^*)$ is Souslin. The aim of this article is to show that in reality in the case of Q being compact any topological restriction about Q or about the pair (E, Y) is unnecessary. Our proof is completely different from that of [8, Theorem 3.1] and yields a widely applicable technique that allows us to obtain Radon–Nikodým type results, amongst other, for multimeasures in Banach and dual Banach spaces without separability assumptions.

Note, that there is a number of papers (see, e.g., [7, 16, 15] and the references therein) devoted to the Radon–Nikodým theorem for multimeasures searching for a Radon–Nikodým derivative in separable Banach spaces, where the integral of the corresponding multifunction is defined as (the closure of) the set of all Bochner integrable selectors. Our method of constructing Radon–Nikodým derivatives enables us to get rid of the separability restriction in that type of results as well.

2. Definitions and Terminology

Throughout this paper (Ω, Σ, μ) is a complete finite measure space. The indicator function of $A \in \Sigma$ is denoted by $\mathbf{1}_A$. By X we denote a (real) locally convex space. X^* stands for the topological dual space of X. By 2^X we denote the family of all non-empty subsets of X. We consider the following subfamilies of 2^X:

$$\begin{aligned} \mathrm{bcc}(X) &:= \{A \in 2^X : A \text{ is bounded, convex and closed}\}; \\ \mathrm{ck}(X) &:= \{A \in 2^X : A \text{ is convex and compact}\}; \\ \mathrm{cwk}(X) &:= \{A \in 2^X : A \text{ is convex and weakly compact}\}. \end{aligned}$$

For any set $C \subset X$ and any $x^* \in X^*$, we write

$$\delta^*(x^*, C) := \sup\{x^*(x) : x \in C\} \quad \text{and} \quad C|^{x^*} := \{x \in C : x^*(x) = \delta^*(x^*, C)\}.$$

Definition 2.1. *A multifunction $F : \Omega \to \mathrm{bcc}(X)$ is called* Pettis integrable *if*

(i) $\delta^*(x^*, F)$ *is integrable for every* $x^* \in X^*$;

(ii) for each $A \in \Sigma$, there is $\int_A F\, d\mu \in \mathrm{bcc}(X)$ such that

$$\delta^*\left(x^*, \int_A F\, d\mu\right) = \int_A \delta^*(x^*, F)\, d\mu \quad \textit{for every } x^* \in X^*.$$

Here the function $\delta^*(x^*, F) : \Omega \to \mathbb{R}$ is defined by $\delta^*(x^*, F)(t) := \delta^*(x^*, F(t))$.

The Pettis integral for multifunctions was first considered by Castaing and Valadier [6, Chapter V, § 4] and has been widely studied in recent years, see, e.g., [3, 4, 12, 23].

Given a sequence (A_n) of subsets of X, we write $\sum_n A_n$ to denote the set of all elements of X which can be written as the sum of an unconditionally convergent series $\sum_n x_n$, where $x_n \in A_n$ for every $n \in \mathbb{N}$.

Definition 2.2. *A multifunction $M : \Sigma \to 2^X$ is called a* strong multimeasure *if* :

(i) $M(\emptyset) = \{0\}$;

(ii) for each disjoint sequence (E_n) in Σ, we have $M(\bigcup_n E_n) = \sum_n M(E_n)$.

We say that the strong multimeasure $M : \Sigma \to 2^X$ is *μ-continuous* (shortly $M \ll \mu$) if $M(A) = \{0\}$ whenever $A \in \Sigma$ satisfies $\mu(A) = 0$. A selector m of M is a vector-valued function $m : \Sigma \to X$ such that $m(A) \in M(A)$ for every $A \in \Sigma$.

For the concept of multimeasure and historical references we refer to [15, Chapter 7] and the references therein. For the terminology of vector measure and integration theory, in particular for definition and properties of Bochner integral and Banach spaces with the Radon–Nikodým Property (RNP) we refer to [11].

3. Radon–Nikodým Theorem for Dominated Multimeasures

This section is devoted to proving the following Radon–Nikodým theorem for strong multimeasures. To be able to provide a proof for this result we will have to establish first a few preliminary results.

Theorem 3.1. *Let $M : \Sigma \to \mathrm{ck}(X)$ be a strong multimeasure for which there is a set $Q \in \mathrm{ck}(X)$ such that $M(A) \subset \mu(A)Q$ for all $A \in \Sigma$. Then there is a Pettis integrable multifunction $F : \Omega \to \mathrm{ck}(X)$ such that:*

(*i*) *For every countably additive selector m of M there is a Pettis integrable selector f of F such that $m(A) = \int_A f\, d\mu$ for all $A \in \Sigma$.*

(*ii*) *For every $A \in \Sigma$ the following equalities hold:*

$$M(A) = \int_A F\, d\mu = \left\{ \int_A f\, d\mu : \ f \text{ is a Pettis integrable selector of } F \right\}.$$

Any strong multimeasure as in Theorem 3.1 has bounded variation, in the sense of the following definition. Given a continuous seminorm p on X and $A \subset X$, we write

$$\|A\|_p := \sup\{p(x) : \ x \in A\}.$$

Definition 3.2. *Let $M : \Sigma \to 2^X$ be a strong multimeasure. For each continuous seminorm p on X and each $E \in \Sigma$, we define*

$$|M|_p(E) := \sup \sum_i \|M(E_i)\|_p \,,$$

where the supremum is taken over all finite partitions (E_i) of E in Σ.

We say that M has bounded variation *if $|M|_p(\Omega) < \infty$ for every continuous seminorm p on X.*

We start by recalling the following result that is part of the folklore.

Proposition 3.3. *Let $M : \Sigma \to 2^X$ be a strong multimeasure of bounded variation. Then:*

(*i*) *for every continuous seminorm p on X, $|M|_p$ is a countably additive finite measure;*

(*ii*) *every finitely additive selector $m : \Sigma \to X$ of M is countably additive.*

P r o o f. Statement (i) can be proved in the same way that the case of signed (single-valued) measures; its validity for Banach space-valued strong multimeasures was already pointed out in [16, Proposition 1.1].

To prove statement (ii), take a disjoint sequence (E_n) in Σ and fix a continuous seminorm p on X. Then

$$p\left(m\left(\bigcup_n E_n\right)-\sum_{n=1}^{k} m(E_n)\right)=p\left(m\left(\bigcup_{n>k} E_n\right)\right)$$
$$\leq \left\| M\left(\bigcup_{n>k} E_n\right)\right\|_p \leq |M|_p\left(\bigcup_{n>k} E_n\right)\to 0 \quad \text{as } k\to\infty,$$

because $|M|_p$ is a countably additive finite measure (by (i)). Since p is arbitrary, the series $\sum_n m(E_n)$ converges to $m(\bigcup_n E_n)$ in X. ■

The following result on countably additive selectors of multimeasures can be extracted from [24, Theorem 3]; cf. [16, Proposition 2.1] for an analogous result dealing with exposed points and Banach spaces: the proof is included here because it might be difficult for some readers to get a copy of the original paper [24].

Lemma 3.4 (Pallu de la Barrière). *Let $M:\Sigma\to 2^X$ be a convex valued strong multimeasure of bounded variation.*

(i) If $x\in \mathrm{ext}(M(\Omega))$, then there is a countably additive selector m of M such that $m(\Omega)=x$.

(ii) If M takes values in $\mathrm{ck}(X)$, then for every $x\in M(\Omega)$ there is a countably additive selector m of M such that $m(\Omega)=x$.

P r o o f. (i) Observe first that if $C_1, C_2\subset X$ are convex and $x\in\mathrm{ext}(C_1+C_2)$, then there exist unique $x_i\in C_i$ such that $x=x_1+x_2$; moreover, $x_i\in\mathrm{ext}(C_i)$.

For each $A\in\Sigma$, we have $x\in\mathrm{ext}(M(\Omega))=\mathrm{ext}\big(M(A)+M(\Omega\setminus A)\big)$ and so there exist unique $m(A)\in M(A)$ and $m(\Omega\setminus A)\in M(\Omega\setminus A)$ such that $x=m(A)+m(\Omega\setminus A)$.

We claim that $m:\Sigma\to X$ is *finitely* additive. Indeed, take disjoint $A_1,A_2\in\Sigma$ and set $A:=A_1\cup A_2$. Since $m(A)\in M(A)=M(A_1)+M(A_2)$, we can write $m(A)=x_1+x_2$ for some $x_i\in M(A_i)$. Since

$$x=m(A)+m(\Omega\setminus A)=x_1+\big(x_2+m(\Omega\setminus A)\big)$$

and $x_2+m(\Omega\setminus A)\in M(A_2)+M(\Omega\setminus A)=M(\Omega\setminus A_1)$, we obtain $x_1=m(A_1)$. In a similar manner, $x_2=m(A_2)$. Hence $m(A)=m(A_1)+m(A_2)$. Now we use (ii) in Proposition 3.3 to conclude that m is *countably* additive.

(ii) Let $\mathcal{S}$ be the set of all finitely additive selectors of M and consider

$$R := \Big\{ \big(m(A)\big)_{A\in\Sigma} : m \in \mathcal{S} \Big\} \subset \prod_{A\in\Sigma} M(A) \subset X^{\Sigma}$$

equipped with the product topology $\mathfrak{T}$. Since R is $\mathfrak{T}$-closed and $\prod_{A\in\Sigma} M(A)$ is $\mathfrak{T}$-compact, R is $\mathfrak{T}$-compact as well. Since the mapping

$$\varphi : R \to X, \quad \varphi(m) := m(\Omega),$$

is $\mathfrak{T}$-continuous, the set $\varphi(R) \subset M(\Omega)$ is compact. By (i) and the convexity of $\varphi(R)$ we have

$$\mathrm{co}\big(\mathrm{ext}(M(\Omega))\big) \subset \varphi(R)$$

and an appeal to the Krein–Milman theorem [20, §25.1.(4)] ensures us that $\varphi(R) = M(\Omega)$.

Hence for every $x \in M(\Omega)$ there is a finitely additive selector m of M such that $m(\Omega) = x$. Again, statement (ii) in Proposition 3.3 can be used to conclude that m is countably additive. ■

Given a set $B \subset X$, we denote by $\mathrm{att}(B)$ the set of those $x^* \in X^*$ that attain their supremum on B (i.e. $B|^{x^*} \neq \emptyset$). The result isolated in Lemma 3.5 below was proved in [8, Proposition 5.1] under the assumption that X is finite-dimensional, but in fact that assumption was not used in the proof.

Lemma 3.5 (Coste and Pallu de la Barrière). *Let $M : \Sigma \to 2^X$ be a strong multimeasure. If $x^* \in \mathrm{att}(M(\Omega))$ then:*

(i) $x^* \in \mathrm{att}(M(A))$ *for all* $A \in \Sigma$;

(ii) the mapping $M|^{x^*} : \Sigma \to 2^X$, $M|^{x^*}(A) := (M(A))|^{x^*}$, *is a strong multimeasure.*

P r o o f. (i) Fix $A \in \Sigma$. Pick $x \in M|^{x^*}(\Omega) \subset M(\Omega) = M(A) + M(\Omega \setminus A)$ and write $x = y + z$ for some $y \in M(A)$ and $z \in M(\Omega \setminus A)$. Suppose if possible that $y \notin (M(A))|^{x^*}$. Then $x^*(v) > x^*(y)$ for some $v \in M(A)$ and so the vector $v + z \in M(A) + M(\Omega \setminus A) = M(\Omega)$ satisfies $x^*(v+z) > x^*(y+z) = x^*(x)$, which contradicts the choice of x. Thus $y \in (M(A))|^{x^*}$ and so $x^* \in \mathrm{att}(M(A))$.

(ii) Clearly $M|^{x^*}(\emptyset) = \{0\}$. Now let (A_n) be a disjoint sequence in Σ and set $A := \bigcup_n A_n$. To prove $M|^{x^*}(A) \subset \sum_n M|^{x^*}(A_n)$, pick $x \in M|^{x^*}(A) \subset M(A) = \sum_n M(A_n)$ and write $x = \sum_n x_n$, where $x_n \in M(A_n)$. By the argument used in the proof of (i), we have $x_n \in M|^{x^*}(A_n)$ for every $n \in \mathbb{N}$, hence $x \in \sum_n M|^{x^*}(A_n)$.

To prove $\sum_n M|^{x^*}(A_n) \subset M|^{x^*}(A)$, take any $x \in \sum_n M|^{x^*}(A_n)$ and write $x = \sum_n x_n$, where $x_n \in M|^{x^*}(A_n)$. Each $y \in M(A)$ can be written as $y = \sum_n y_n$ for some $y_n \in M(A_n)$, so that

$$x^*(y) = \sum_n x^*(y_n) \leq \sum_n x^*(x_n) = x^*(x).$$

It follows that $x \in M|^{x^*}(A)$ and the proof is over. ■

Let $\rho : \Sigma \to \Sigma$ be a lifting on (Ω, Σ, μ) (see, e.g., [17, p. 46, Theorem 3] or [14, 341K]). Note that ρ satisfies the following properties:

1. If $A, B \in \Sigma$ and $\mu(A \Delta B) = 0$ then $\rho(A) = \rho(B)$.
2. $\mu(\rho(A)\Delta A) = 0$ for every $A \in \Sigma$.
3. $\rho(A \cap B) = \rho(A) \cap \rho(B)$ for every $A, B \in \Sigma$.
4. $\rho(\emptyset) = \emptyset$, $\rho(\Omega) = \Omega$.
5. $\rho(\Omega \backslash A) = \Omega \backslash \rho(A)$ for every $A \in \Sigma$.
6. $\rho(A \cup B) = \rho(A) \cup \rho(B)$ for every $A, B \in \Sigma$.

Then $\rho(\Sigma)$ is a subalgebra of Σ such that $\mu(A) > 0$ whenever $A \in \rho(\Sigma) \setminus \{\emptyset\}$.

We consider the collection $\mho$ of all finite partitions of Ω into elements of $\rho(\Sigma) \setminus \{\emptyset\}$, equipped with the natural ordering (defined by saying that $\Gamma_1 \succ \Gamma_2$ if and only if Γ_1 is finer than Γ_2). Then $(\mho, \succ)$ is a directed set.

The notion of Pettis integrable vector-valued function $f : \Omega \to X$ as can be found in the literature (see, e.g., [11, II.3] for the Banach space case) corresponds to Definition 2.1 for $F(t) := \{f(t)\}$ when the integral $\int_A F\, d\mu$ is a singleton.

Recall that a function $f : \Omega \to X$ is strongly measurable if it is the μ-a.e. limit of a sequence of simple functions. A celebrated Pettis' result establishes that, when X is a Banach space, a function $f : \Omega \to X$ is strongly measurable if, and only if, f is weakly measurable (i.e., $x^* \circ f$ is measurable for every $x^* \in X$) and $f(\Omega \setminus A)$ is separable for some $A \in \Sigma$ with $\mu(A) = 0$, see [11, Theorem 2, p. 42].

Definition 3.6. *For every Pettis integrable function $f : \Omega \to X$ and every $\Gamma \in \mho$, we define $f_\Gamma : \Omega \to X$ by*

$$f_\Gamma := \sum_{A \in \Gamma} \Big(\frac{1}{\mu(A)} \int_A f\, d\mu \Big) \mathbf{1}_A.$$

The following lemma can be deduced from a result by Kupka, see [21, Lemma 4.3]. For the readers convenience we prefer to give a direct proof.

Lemma 3.7. *Suppose X is a Banach space and $f : \Omega \to X$ is strongly measurable and Pettis integrable. Then* $\lim_\Gamma f_\Gamma = f$ *μ-a.e. Moreover, for every $\varepsilon > 0$ there is $U \in \Sigma$ with $\mu(\Omega \setminus U) < \varepsilon$ such that* $\lim_\Gamma f_\Gamma = f$ *uniformly on U.*

P r o o f. Without loss of generality we may assume that $f(\Omega)$ is separable. Fix $\varepsilon > 0$ and a sequence (ε_n) of positive real numbers converging to 0.

Fix $n \in \mathbb{N}$. We can find a disjoint covering $\{D_{n,k}\}_{k\in\mathbb{N}}$ of $f(\Omega)$ by Borel sets with $\mathrm{diam}(D_{n,k}) < \varepsilon_n$. Since $\{f^{-1}(D_{n,k})\}_{k\in\mathbb{N}}$ is a partition of Ω into measurable sets, we can choose $j_n \in \mathbb{N}$ large enough such that

$$\mu\Big(f^{-1}\Big(\bigcup_{k>j_n} D_{n,k}\Big)\Big) < \frac{\varepsilon}{2^n}.$$

Define $B_{n,k} = f^{-1}(D_{n,k})$ and $A_{n,k} := \rho(B_{n,k})$ for all $k = 1, 2, \ldots, j_n$. Observe that $V_n := \bigcup_{k=1}^{j_n}(A_{n,k} \bigcap B_{n,k}) \in \Sigma$ satisfies

$$\mu(V_n) = \sum_{k=1}^{j_n} \mu(A_{n,k} \cap B_{n,k}) = \sum_{k=1}^{j_n} \mu(B_{n,k}) = \mu\Big(f^{-1}\Big(\bigcup_{k=1}^{j_n} D_{n,k}\Big)\Big) > \mu(\Omega) - \frac{\varepsilon}{2^n}.$$

Set

$$A_{n,j_n+1} := \Omega \setminus \bigcup_{k=1}^{j_n} A_{n,k} = \rho\Big(f^{-1}\Big(\bigcup_{k>j_n} D_{n,k}\Big)\Big)$$

and let $\Gamma_n \in \mho$ be the partition of Ω consisting of all non-empty $A_{n,k}$'s.

We claim that $\|f_\Gamma(t) - f(t)\| < \varepsilon_n$ for every $\Gamma \in \mho$ with $\Gamma \succ \Gamma_n$ and every $t \in V_n$. Indeed, let $k \in \{1, \ldots, j_n\}$ be such that $t \in A_{n,k} \cap B_{n,k}$. Then $A_{n,k} \in \Gamma_n$ and there is $A \in \Gamma$ such that $t \in A \subset A_{n,k}$, so that $\mu(A \setminus B_{n,k}) = 0$ and

$$f_\Gamma(t) = \frac{1}{\mu(A)} \int_A f\,d\mu = \frac{1}{\mu(A \cap B_{n,k})} \int_{A\cap B_{n,k}} f\,d\mu \in \overline{\mathrm{co}}\big(f(A \cap B_{n,k})\big) \subset \overline{\mathrm{co}}(D_{n,k}),$$

thanks to the Hahn–Banach separation theorem. Since $f(t) \in f(B_{n,k}) \subset D_{n,k}$ and $\mathrm{diam}(\overline{\mathrm{co}}(D_{n,k})) = \mathrm{diam}(D_{n,k}) < \varepsilon_n$, we get $\|f_\Gamma(t) - f(t)\| < \varepsilon_n$, as claimed.

The previous claim ensures us that $\lim_\Gamma f_\Gamma = f$ uniformly on $U := \bigcap_{n\in\mathbb{N}} V_n$, which belongs to Σ and satisfies

$$\mu(\Omega \setminus U) = \mu\Big(\bigcup_{n\in\mathbb{N}} \Omega \setminus V_n\Big) \le \sum_{n\in\mathbb{N}} \mu(\Omega \setminus V_n) < \sum_{n\in\mathbb{N}} \frac{\varepsilon}{2^n} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, a standard argument now implies that $\lim_\Gamma f_\Gamma = f$ μ-a.e.

■

We stress that, in particular, Lemma 3.7 is applicable to ordinary real-valued Lebesgue integrable functions.

Remark 3.8. *The conclusion of Lemma* 3.7 *can fail without the strong measurability assumption. In fact, for a Banach space X, a Pettis integrable function $f : \Omega \to X$ is strongly measurable if, and only if, we have*

(*a*) $\lim_\Gamma f_\Gamma = f$ *μ-a.e. and*

(*b*) *the set $\mathcal{R}(f) = \{\int_A f\, d\mu : A \in \Sigma\}$ is separable.*

Such separability condition is fulfilled automatically for any Pettis integrable function $f : \Omega \to X$ under mild assumptions on either μ or X, see, e.g., [22, *Sections* 9 *and* 10] *and* [25, *Chapters* 4 *and* 5]. *Also, if $f : \Omega \to X$ is Birkhoff or McShane integrable then $\mathcal{R}(f)$ is relatively norm compact* [2, 13], *hence $\mathcal{R}(f)$ is separable and therefore for such an f if $\lim_\Gamma f_\Gamma = f$ μ-a.e. then f is neccesarily strongly measurable.*

From now on we work with a fixed ultrafilter $\mathcal{U}$ on $\mho$ containing all subsets of $\mho$ of the form $\{\Gamma' \in \mho : \Gamma' \succ \Gamma\}$ where $\Gamma \in \mho$.

Definition 3.9. *Let $\{V_\Gamma : \Gamma \in \mho\}$ be a collection of subsets of X. We denote by $\mathrm{LIM}_\Gamma V_\Gamma$ the set (maybe empty) of all $x \in X$ for which there exist $v_\Gamma \in V_\Gamma$, $\Gamma \in \mho$, such that $\mathcal{U} - \lim_\Gamma v_\Gamma = x$.*

P r o o f of Theorem 3.1. For every $\Gamma \in \mho$, define $M_\Gamma : \Omega \to \mathrm{ck}(X)$ by

$$M_\Gamma := \sum_{A \in \Gamma} \frac{M(A)}{\mu(A)} \mathbf{1}_A.$$

Given any $t \in \Omega$, we have $M_\Gamma(t) \subset Q$ for all $\Gamma \in \mho$ and so the compactness of Q yields $\emptyset \neq \mathrm{LIM}_\Gamma M_\Gamma(t) \subset Q$. We can define a multifunction $F : \Omega \to \mathrm{ck}(X)$ by

$$F(t) := \overline{\mathrm{LIM}_\Gamma M_\Gamma(t)}. \tag{1}$$

Let us check that F satisfies the required properties.

In order to prove (i), let m be a countably additive selector of M. For each $\Gamma \in \mho$, define $m_\Gamma : \Omega \to Q \subset X$ by $m_\Gamma := \sum_{A \in \Gamma} \frac{m(A)}{\mu(A)} \mathbf{1}_A$. Then we can define a function $f : \Omega \to X$ by $f(t) := \mathcal{U} - \lim_\Gamma m_\Gamma(t)$. Observe that f is a selector of F, because m_Γ is a selector of M_Γ for every $\Gamma \in \mho$.

Fix $x^* \in X^*$. Clearly, $x^* \circ m$ is countably additive. Moreover, the inequalities

$$\min(x^*(Q))\mu(A) \leq x^*(m(A)) \leq \max(x^*(Q))\mu(A) \quad \text{for all } A \in \Sigma$$

imply that $x^* \circ m$ has bounded variation and $x^* \circ m \ll \mu$. Let h be the Radon–Nikodým derivative of $x^* \circ m$ with respect to μ. Note that for each $\Gamma \in \mho$ we have

$$h_\Gamma = \sum_{A\in\Gamma} \Big(\frac{1}{\mu(A)} \int\limits_A h\, d\mu\Big) \mathbf{1}_A = \sum_{A\in\Gamma} \frac{(x^* \circ m)(A)}{\mu(A)} \mathbf{1}_A = x^* \circ m_\Gamma.$$

By Lemma 3.7 applied to h, we have $\lim_\Gamma x^* \circ m_\Gamma = h$ μ-a.e. On the other hand, by the definition of f we have $\mathcal{U} - \lim_\Gamma (x^* \circ m_\Gamma)(t) = (x^* \circ f)(t)$ for every $t \in \Omega$. It follows that $x^* \circ f = h$ μ-a.e. Hence $x^* \circ f$ is integrable and satisfies

$$\int\limits_A x^* \circ f\, d\mu = \int\limits_A h\, d\mu = x^*(m(A)) \quad \text{for all } A \in \Sigma.$$

As $x^* \in X^*$ is arbitrary, f is a Pettis integrable Radon–Nikodým derivative of m.

We now turn to the proof of (ii). Fix $x^* \in X^*$. The finitely additive measure $\nu : \Sigma \to \mathbb{R}$ defined by the formula $\nu(A) := \delta^*(x^*, M(A))$ satisfies

$$\min(x^*(Q))\mu(A) \le \nu(A) \le \max(x^*(Q))\mu(A) \quad \text{for all } A \in \Sigma.$$

Hence ν is countably additive, it has bounded variation and $\nu \ll \mu$. Let g be the Radon–Nikodým derivative of ν with respect to μ. For every $\Gamma \in \mho$ we have

$$g_\Gamma = \sum_{A\in\Gamma} \Big(\frac{1}{\mu(A)} \int\limits_A g\, d\mu\Big) \mathbf{1}_A = \sum_{A\in\Gamma} \frac{\delta^*(x^*, M(A))}{\mu(A)} \mathbf{1}_A = \delta^*(x^*, M_\Gamma).$$

Lemma 3.7 applied to g ensures us that $\lim_\Gamma \delta^*(x^*, M_\Gamma) = g$ μ-a.e; bearing in mind now the equality (1) it follows that

$$\delta^*(x^*, F) \le \mathcal{U} - \lim_\Gamma \delta^*(x^*, M_\Gamma) = g \quad \mu\text{-a.e.} \tag{2}$$

♦ *Claim. The function $\delta^*(x^*, F)$ is integrable.*

Indeed, observe first that the mapping $M|^{x^*} : \Sigma \to \mathrm{ck}(X)$ is a strong multimeasure (by Lemma 3.5) and has bounded variation because $M|^{x^*}(A) \subset \mu(A)Q$ for every $A \in \Sigma$. Lemma 3.4 provides us with a countably additive selector m of $M|^{x^*}$, that is also a selector of M that satisfies

$$x^*(m(A)) = \delta^*(x^*, M(A)) = \int\limits_A g\, d\mu \quad \text{for all } A \in \Sigma.$$

By (ii), there is a Pettis integrable selector f of F such that for every $A \in \Sigma$ we have $\int_A f\, d\mu = m(A)$ and so $\int_A x^* \circ f\, d\mu = x^*(m(A)) = \int_A g\, d\mu$. Therefore $x^* \circ f = g$ μ-a.e. On the other hand, in view of (2) we also have $x^* \circ f \le$

$\delta^*(x^*, F) \leq g$ μ-a.e. It follows that $\delta^*(x^*, F) = x^* \circ f$ μ-a.e., hence $\delta^*(x^*, F)$ is integrable, as claimed.

Moreover, for every $A \in \Sigma$ we have

$$\int_A \delta^*(x^*, F)\, d\mu = \int_A x^* \circ f\, d\mu = x^*(m(A)) = \delta^*(x^*, M(A)).$$

Since $x^* \in X^*$ is arbitrary, F is Pettis integrable and $M(A) = \int_A F\, d\mu$ for all $A \in \Sigma$.

To finish the proof of (ii), fix $A \in \Sigma$ and note that the Hahn–Banach separation theorem implies

$$M(A) \supset \left\{ \int_A f\, d\mu : f \text{ is a Pettis integrable selector of } F \right\} =: S(A).$$

In order to prove the converse inclusion, take any $x \in M(A)$. Lemma 3.4 applied to the restriction of M to the trace σ-algebra $\Sigma_A := \{B \cap A : B \in \Sigma\}$ guarantees the existence of a countably additive selector $m_1 : \Sigma_A \to X$ of $M|_{\Sigma_A}$ such that $x = m_1(A)$. Now let $m : \Sigma \to X$ be any countably additive selector of M. Then the formula

$$\tilde{m_1}(B) := m_1(B \cap A) + m(B \setminus A)$$

defines a countably additive selector $\tilde{m_1} : \Sigma \to X$ of M extending m_1. By part (i) applied to $\tilde{m_1}$, there is a Pettis integrable selector f of F such that $\tilde{m_1}(B) = \int_B f\, d\mu$ for all $B \in \Sigma$. So, $x = \tilde{m_1}(A) = \int_A f\, d\mu \in S(A)$. We have finally established that $M(A) = S(A)$. The proof is over. ∎

In the last result of this section $\mathrm{cw^*k}(X^*)$ denotes the family of all non-empty convex w^*-compact subsets of the dual X^* of a Banach space X; w^* is the weak* topology of X^* and B_{X^*} stands for the closed dual unit ball. For the concept and properties of Gel'fand integral for multifunctions we refer to the paper [5].

Proposition 3.10. *Let X be a Banach space and let $M : \Sigma \to \mathrm{cw^*k}(X^*)$ be a μ-continuous strong multimeasure for the dual norm with bounded norm variation $|M|$. Then there exists a Gel'fand integrable multifunction $F : \Omega \to \mathrm{cw^*k}(X^*)$ such that for every $A \in \Sigma$ we have*

$$M(A) = \int_A F\, d\mu = \left\{ \int_A f\, d\mu : f \textit{ is a Gel'fand integrable selector of } F \right\}.$$

P r o o f. Let g be the Radon–Nikodým derivative of $|M|$ with respect to μ. For each $n \in \mathbb{N}$ let us define $A_n := \{t \in \Omega : n - 1 \leq g(t) < n\} \in \Sigma$ and write

$\Sigma_n := \{A \cap A_n : A \in \Sigma\}$. The restriction $M|_{\Sigma_n} : \Sigma_n \to \mathrm{cw^*k}(X^*)$ satisfies that for every $A \in \Sigma_n$ we have $M(A) \subset \mu(A)\big(nB_{X^*}\big)$. According to Theorem 3.1 applied to the locally convex space (X^*, w^*) there is a Gel'fand integrable multifunction $F_n : A_n \to \mathrm{cw^*k}(X^*)$ such that

$$M(A) = \int_A F_n \, d\mu \quad \text{for every } A \in \Sigma_n. \tag{3}$$

Note that $\{A_n\}_{n \in \mathbb{N}}$ is a partition of Ω in Σ, therefore the multifunction

$$F : \Omega \to \mathrm{cw^*k}(X^*), \quad F(t) := F_n(t) \text{ whenever } t \in A_n,$$

clearly satisfies that $\delta^*(x, F)$ is measurable for every $x \in X$. On the other hand for every $n \in \mathbb{N}$, every $A \in \Sigma_n$ and every $x \in X$, we have that

$$\begin{aligned}\int_A \delta^*(x, F) \, d\mu &= \int_A \delta^*(x, F_n) \, d\mu = \delta^*(x, M(A)) \\ &\leq \|x\| \|M(A)\| \leq \|x\| |M|(A) = \|x\| \int_A g \, d\mu.\end{aligned}$$

Therefore, for every $x \in X$ we have $\delta^*(x, F(t)) \leq \|x\| g(t)$ for μ-a.e. $t \in \Omega$. Hence, for every $x \in X$ we also have

$$-\delta^*(x, F(t)) = \inf\{-x^*(x) : x^* \in F(t)\} \leq \delta^*(-x, F(t)) \leq \|-x\| g(t) = \|x\| g(t)$$

for μ-a.e. $t \in \Omega$ and consequently $\delta^*(x, F)$ is integrable. So F is Gel'fand integrable.

To finish we will establish that for every $A \in \Sigma$ we have $M(A) = \int_A F \, d\mu$, which is equivalent to proving that $\delta^*(x, M(A)) = \delta^*(x, \int_A F \, d\mu)$ for every $x \in X$, because both sets are convex and w^*-compact (see [5, Theorem 4.5] for the latter). On one hand, for each $x \in X$ the measure $\nu_x : \Sigma \to \mathbb{R}$ given by $\nu_x(A) := \delta^*(x, M(A))$ is countably additive. On the other hand, since F is Gel'fand integrable, for each $x \in X$ the measure $\sigma_x : \Sigma \to \mathbb{R}$ defined by $\sigma_x(A) := \delta^*(x, \int_A F \, d\mu) = \int_A \delta^*(x, F) \, d\mu$ is also countably additive. The formula (3) implies in particular that $\nu_x|_{\Sigma_n} = \sigma_x|_{\Sigma_n}$ for every $n \in \mathbb{N}$, and finally the countable additivity of ν_x and σ_x leads to $\nu_x = \sigma_x$ in Σ. The proof is over. ■

4. Set-Valued Derivatives in Banach Spaces with the RNP

Throughout this section X is assumed to be a Banach space. Our aim here is to demonstrate the following statement:

Theorem 4.1. *Suppose X has the RNP. Let $M : \Sigma \to \mathrm{bcc}(X)$ be a strong multimeasure of bounded variation with $M \ll \mu$. Then there is a Pettis integrable multifunction $F : \Omega \to \mathrm{bcc}(X)$ such that:*

(*i*) *For every countably additive selector* m *of* M *there is a Bochner integrable selector of* F *such that* $m(A) = \int_A f\,d\mu$ *for all* $A \in \Sigma$.

(*ii*) *For every* $A \in \Sigma$ *the following equalities hold*:

$$M(A) = \int_A F\,d\mu = \overline{\left\{ \int_A f\,d\mu : f \text{ is a Bochner integrable selector of } F \right\}}.$$

This result generalizes Theorem 2 of [7] proved for separable spaces. The anonymous referee kindly communicated to us that in the case of strongly compact values the theorem reduces to the separable case, but already for weakly compact values there is no such a reduction. Before offering a proof for Theorem 4.1 we need a lemma:

Lemma 4.2. *Suppose* X *has the RNP. Let* $M : \Sigma \to \mathrm{bcc}(X)$ *be a strong multimeasure of bounded variation. Then there exist a norm dense set* $W \subset X^*$ *and a family* $\{m_{x^*}\}_{x^* \in W}$ *of countably additive selectors of* M *such that*

$$x^*(m_{x^*}(A)) = \delta^*(x^*, M(A)) \quad \text{for every } x^* \in W \text{ and every } A \in \Sigma.$$

P r o o f. By the Bishop–Phelps theorem (see, e.g., [10, p. 3]), the set $W := \mathrm{att}(M(\Omega))$ is norm dense in X^*. Fix $x^* \in W$. Then Lemma 3.5 ensures that $M|^{x^*} : \Sigma \to \mathrm{bcc}(X)$ is a strong multimeasure. Since $M|^{x^*}(A) \subset M(A)$ for all $A \in \Sigma$ and M has bounded variation, $M|^{x^*}$ has bounded variation as well. Since $M|^{x^*}(\Omega) \in \mathrm{bcc}(X)$ and X has the RNP, we have $\mathrm{ext}(M|^{x^*}(\Omega)) \neq \emptyset$ (see, e.g., [10, Theorem 1, p. 231]). Therefore, Lemma 3.4 applied to $M|^{x^*}$ guarantees the existence of a countably additive selector m_{x^*} of $M|^{x^*}$. Of course, m_{x^*} is a selector of M and we have $x^*(m_{x^*}(A)) = \delta^*(x^*, M(A))$ for all $A \in \Sigma$. ■

P r o o f of Theorem 4.1. As in the proof of Theorem 3.1, for each $\Gamma \in \mho$ we consider the multifunction $M_\Gamma : \Omega \to \mathrm{bcc}(X)$ given by

$$M_\Gamma := \sum_{A \in \Gamma} \frac{M(A)}{\mu(A)} \mathbf{1}_A$$

and, for each $t \in \Omega$, we define

$$G(t) := \overline{\mathrm{LIM}_\Gamma\, M_\Gamma(t)}$$

(with respect to the norm topology). We shall prove first that

$$\emptyset \neq G(t) \in \mathrm{bcc}(X) \quad \text{for } \mu\text{-a.e. } t \in \Omega. \tag{4}$$

Obviously, $G(t)$ is convex for all $t \in \Omega$. To check that $G(t) \neq \emptyset$ for μ-a.e. $t \in \Omega$, let m be any countably additive selector of M (apply Lemma 4.2). Observe that m has bounded variation and $m \ll \mu$. Since X has the RNP, there is a Bochner integrable function $f : \Omega \to X$ such that $m(A) = \int_A f\, d\mu$ for all $A \in \Sigma$. In particular, for every $\Gamma \in \mho$ and every $t \in \Omega$ we have

$$f_\Gamma(t) = \sum_{A \in \Gamma} \Big(\frac{1}{\mu(A)} \int\limits_A f\, d\mu \Big) \mathbf{1}_A(t) = \sum_{A \in \Gamma} \frac{m(A)}{\mu(A)} \mathbf{1}_A(t) \in M_\Gamma(t).$$

According to Lemma 3.7, we have $\lim_\Gamma f_\Gamma = f$ μ-a.e., hence $f(t) \in G(t) \neq \emptyset$ for μ-a.e. $t \in \Omega$. On the other hand, $|M|$ is a countably additive finite measure (Proposition 3.3) with $|M| \ll \mu$ and we can consider its Radon–Nikodým derivative g with respect to μ. For every $\Gamma \in \mho$ and every $t \in \Omega$ we have

$$\begin{aligned} \|M_\Gamma(t)\| &= \sum_{A \in \Gamma} \frac{\|M(A)\|}{\mu(A)} \mathbf{1}_A(t) \leq \sum_{A \in \Gamma} \frac{|M|(A)}{\mu(A)} \mathbf{1}_A(t) \\ &= \sum_{A \in \Gamma} \Big(\frac{1}{\mu(A)} \int\limits_A g\, d\mu \Big) \mathbf{1}_A(t) = g_\Gamma(t). \end{aligned}$$

Bearing in mind that $\lim_\Gamma g_\Gamma = g$ μ-a.e. (by Lemma 3.7), it follows from the equality above that $\|G(t)\| \leq g(t) < \infty$ for μ-a.e. $t \in \Omega$. This finishes the proof of (4).

Now let $F : \Omega \to \mathrm{bcc}(X)$ be any multifunction such that $F(t) = G(t)$ for μ-a.e. $t \in \Omega$. We shall check that F satisfies the required properties. Observe that (i) has already been obtained in the proof of (4).

By Lemma 4.2, there exist a norm dense set $W \subset X^*$ and a family $\{m_{x^*}\}_{x^* \in W}$ of countably additive selectors of M such that

$$x^*(m_{x^*}(A)) = \delta^*(x^*, M(A)) \quad \text{for every } x^* \in W \text{ and every } A \in \Sigma.$$

Thus, (i) applied to each m_{x^*} ensures us of the existence of a family $\{f_{x^*}\}_{x^* \in W}$ of Bochner integrable selectors of F such that

$$\int\limits_A (x^* \circ f_{x^*})\, d\mu = x^* \left(\int\limits_A f_{x^*}\, d\mu \right) = \delta^*(x^*, M(A)) \tag{5}$$

for every $x^* \in W$ and every $A \in \Sigma$.

Fix $x^* \in W$. Given $\Gamma \in \mho$ and $t \in \Omega$, we have

$$\begin{aligned} \delta^*(x^*, M_\Gamma(t)) &\overset{t \in A,\, A \in \Gamma}{=} \sup \left\{ x^*(x) : x \in \frac{M(A)}{\mu(A)} \right\} = \frac{\delta^*(x^*, M(A))}{\mu(A)} \\ &\overset{(5)}{=} \frac{1}{\mu(A)} \int\limits_A (x^* \circ f_{x^*})\, d\mu = (x^* \circ f_{x^*})_\Gamma(t). \end{aligned}$$

Last equality and Lemma 3.7 (applied to $x^* \circ f_{x^*}$) yield $\delta^*(x^*, F) \le x^* \circ f_{x^*}$ μ-a.e. Since f_{x^*} is a selector of F we have $\delta^*(x^*, F) = x^* \circ f_{x^*}$ μ-a.e. Hence $\delta^*(x^*, F)$ is integrable and (5) says that

$$\int_A \delta^*(x^*, F)\, d\mu = \delta^*(x^*, M(A)) \quad \text{for all } A \in \Sigma. \tag{6}$$

Let us consider now an arbitrary $x^* \in X^*$. Since W is norm dense in X^*, we can find a sequence (x_n^*) in W such that $\|x_n^* - x^*\| \to 0$. Since F takes bounded values, we have $\delta^*(x_n^*, F) \to \delta^*(x^*, F)$ pointwise. Moreover, for each $n \in \mathbb{N}$ we have

$$|\delta^*(x_n^*, F(t))| \le \|x_n^*\|\|F(t)\| \le Cg(t) \text{ for } \mu\text{-a.e. } t \in \Omega,$$

where $C := \sup_{k\in\mathbb{N}} \|x_k^*\|$ and g is the Radon–Nikodým derivative of $|M|$ with respect to μ (see the proof of (4) above). An appeal to the Dominated Convergence Theorem assures that $\delta^*(x^*, F)$ is integrable and that

$$\int_A \delta^*(x^*, F)\, d\mu = \lim_{n\to\infty} \int_A \delta^*(x_n^*, F)\, d\mu \overset{(6)}{=} \lim_{n\to\infty} \delta^*(x_n^*, M(A)) = \delta^*(x^*, M(A))$$

for every $A \in \Sigma$. This proves that F is Pettis integrable and

$$M(A) = \int_A F\, d\mu \quad \text{for every } A \in \Sigma.$$

Fix $A \in \Sigma$. Observe that the inclusion

$$M(A) = \int_A F\, d\mu \supset \overline{\left\{ \int_A f\, d\mu : f \text{ is a Bochner integrable selector of } F \right\}} =: S(A)$$

follows directly from the Hahn–Banach separation theorem. To prove the converse inclusion, take any $x \in \text{ext}(M(A))$. By Lemma 3.4 applied to $M|_{\Sigma_A}$, there is a countably additive selector $m_1 : \Sigma_A \to X$ of $M|_{\Sigma_A}$ such that $x = m_1(A)$. Let $\tilde{m}_1 : \Sigma \to X$ be any countably additive selector of M extending m_1 (see the proof of (ii) in Theorem 3.1). By (i) applied to $\tilde{m}_1$, there is a Bochner integrable selector f of F such that $\tilde{m}_1(B) = \int_B f\, d\mu$ for all $B \in \Sigma$. Thus, $x = \tilde{m}_1(A) = \int_A f\, d\mu \in S(A)$. This shows that $\text{ext}(M(A)) \subset S(A)$ and so we have $\text{co}(\text{ext}(M(A))) \subset S(A)$ (because $S(A)$ is convex). Since

$$M(A) = \overline{\text{co}}\big(\text{ext}(M(A))\big)$$

thanks to the RNP (see, e.g., [10, Theorem 1, p. 231]), we conclude that $M(A) = S(A)$. The proof is over. ∎

Remark 4.3. *If we additionally assume that X^* has the RNP and M takes values in* $\mathrm{cwk}(X)$ *in Theorem 4.1, then*

$$S_F := \{f : \Omega \to X : f \text{ is Bochner integrable selector of } F\}$$

is relatively weakly compact in $L^1(\mu, X)$.

P r o o f. For any $f \in S_F$, let $m_f : \Sigma \to X$ be the countably additive selector of M defined by $m_f(A) := \int_A f\, d\mu$. Observe that

$$\int\limits_A \|f\|\, d\mu = |m_f|(A) \leq |M|(A) \quad \text{for all } A \in \Sigma.$$

From the previous inequality and the fact that $|M|$ is a μ-continuous countably additive finite measure (Proposition 3.3) it follows that S_F is uniformly integrable. Moreover, for each $A \in \Sigma$, the set

$$\left\{ \int\limits_A f\, d\mu : \; f \in S_F \right\} \subset M(A)$$

is relatively weakly compact in X. Since X^* has the RNP, we infer that S_F is relatively weakly compact in $L^1(\mu, X)$ (see, e.g., [11, Theorem 1, p. 101]). ■

If X is *separable* and $F : \Omega \to 2^X$ is an Effros measurable multifunction taking closed non-empty values, then the relative weak compactness of S_F in $L^1(\mu, X)$ implies that $F(t)$ is weakly compact in X for μ-a.e. $t \in \Omega$, see [18, Theorem 3.6]. So the answer to the following natural question is affirmative when X is separable [7, Theorem 3]:

Question. Under the assumptions of Theorem 4.1, suppose further that X^* has the RNP and M takes values in $\mathrm{cwk}(X)$. Is it possible to construct F in such a way that $F(t) \in \mathrm{cwk}(X)$ for μ-a.e. $t \in \Omega$? Does our construction give F with this additional property?

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