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A Remark on Condensation of Singularities

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Recently Alan D. Sokal in Amer. Math. Monthly **118** (2011), No. 5, 450–452, gave a very short and completely elementary proof of the uniform boundedness principle. The aim of this note is to point out that by using a similar technique one can give a short and simple proof of a stronger statement, namely a principle of condensation of singularities for certain double-sequences of non-linear operators on quasi-Banach spaces, which is a bit more general than a result of I. S. Gál from Duke Math. J. **20** (1953), No. 1, 27–35.

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Let us begin by recalling that a quasi-norm on a linear space X over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is a map $||\,.\,|| : X \to [0, \infty)$ such that

(i) $\|x\| = 0 \;\Rightarrow\; x = 0$,

(ii) $\|\lambda x\| = |\lambda|\,\|x\| \quad \forall x \in X, \forall \lambda \in \mathbb{K}$,

(iii) $\exists K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|) \quad \forall x, y \in X$.

The least constant $K \geq 1$ which fulfils (iii) is sometimes called the modulus of concavity of the quasi-norm, and the pair $(X, \|\,.\,\|)$ is called a quasi-normed space.

If $p \in (0, 1]$, then a map $\|\,.\,\| : X \to [0, \infty)$ is called a p-norm if it satisfies (i) and (ii) and the condition

(iv) $\|x + y\|^p \leq \|x\|^p + \|y\|^p \quad \forall x, y \in X$.

The pair $(X, \|\,.\,\|)$ is then called a p-normed space. Every p-norm is a quasi-norm with $K = 2^{1/p-1}$. The standard examples of p-normed spaces are of course the spaces $L^p(\mu)$ with the p-norm

$$\|f\|_p = \left(\int_S |f|^p \,\mathrm{d}\mu \right)^{1/p}$$

for any measure space $(S, \mathcal{A}, \mu)$, which includes in particular the sequence spaces ℓ^p.

The Aoki–Rolewicz-theorem (cf. [1, 11] or [9, Lemma 1.1]) states that if $\|\,.\,\|$ is a quasi-norm on X with modulus of concavity $\leq K$ and if $p \in (0,1]$ is defined by $(2K)^p = 2$, then for any $x_1, \ldots, x_n \in X$ the inequality

$$\left\| \sum_{i=1}^{n} x_i \right\|^p \leq 4 \sum_{i=1}^{n} \|x_i\|^p$$

holds.

Then an equivalent p-norm on X can be defined by

$$|||x||| = \inf\left\{ \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{1/p} : n \in \mathbb{N},\, x_1, \ldots, x_n \in X,\, x = \sum_{i=1}^{n} x_i \right\},$$

cf. [1, 11] or [9, Theorems 1.2 and 1.3].

The topology τ on X induced by the quasi-norm $\|\,.\,\|$ is defined by declaring a subset $O \subseteq X$ to be open if for every $x \in O$ there exists $\varepsilon > 0$ such that $\{y \in X : \|x - y\| < \varepsilon\} \subseteq O$, which is the same as the topology induced by the metric d that is defined by $d(x,y) = |||x - y|||^p$. The space (X, τ) is a topological vector space and $(X, \|\,.\,\|)$ is called a quasi-Banach space if (X, τ) is complete (or equivalently if X is complete with respect to the above metric d or any other translation-invariant metric on X that induces the topology τ).

A linear operator T between two quasi-normed spaces X and Y is continuous if and only if there is a constant $M \geq 0$ such that $\|Tx\| \leq M\|x\|$ for every $x \in X$. The space of continuous linear operators from X into Y is denoted by $L(X,Y)$. A quasi-norm on $L(X,Y)$ can be defined by $\|T\| = \sup\{\|Tx\| : x \in X,\, \|x\| \leq 1\}$. The modulus of concavity of this quasi-norm is less than or equal to the one of Y, and $L(X,Y)$ is complete if Y is complete, in particular the dual space $X^* = L(X, \mathbb{K})$ is always a Banach space.

It should be mentioned that the Hahn–Banach-theorem fails in general for quasi-normed spaces, some of them do not even possess a separating dual, for example, it is a well-known result that for $p \in (0,1)$ the dual space of $L^p[0,1]$ contains only 0. However, the classical results on Banach spaces like the open mapping and the closed graph theorem, as well as the uniform boundedness principle, still hold for quasi-Banach spaces with analogous proofs using the Baire category theorem.*

The original proofs of the uniform boundedness principle by Banach and Hahn (cf. [2] and [7]) made no use of the Baire category theorem, but of a technique which has come to be known as "gliding hump" method. The Baire category argument was originally introduced into this theory by S. Saks.

*For more information on quasi-Banach spaces the reader is referred to [8] and references therein.

In 1927, Banach and Steinhaus proved a by now well-known generalization of the uniform boundedness principle, the so-called principle of condensation of singularities, which reads as follows.

Theorem. (Principle of condensation of singularities, [3]). *Let X be a Banach space and $(Y_n)_{n\in\mathbb{N}}$ a sequence of normed linear spaces. If $G_n \subseteq L(X, Y_n)$ is unbounded for every $n \in \mathbb{N}$, then there exists an element $x \in X$ such that*

$$\sup_{T\in G_n} \|Tx\| = \infty \quad \forall n \in \mathbb{N}.$$

The proof from [3] also uses the Baire theorem.

Over the years many generalizations of the uniform boundedness principle and the principle of condensation of singularities to suitable classes of non-linear operators on Banach and quasi-Banach spaces have been studied. In [5], I. S. Gál proved a principle of uniform boundedness for sequences of bounded homogenous operators (see below for definitions) on Banach spaces that satisfy certain conditions of "asymptotic subadditivity" and later in [6] he generalized his results to a principle of condensation of singularities for double-sequences of the same type. The authors of [4] considered quantitative versions of the principle of condensation of singularities for double-sequences of operators on quasi-Banach spaces that fulfil similar conditions. The proofs of these results use refinements of the usual gliding hump argument. A proof of Gál's results via the Baire category theorem can be found in [10] (it even works for some non-complete spaces, one only needs to assume that the space is of second category in itself).

In [12], Alan D. Sokal gave a very short and elegant proof of the classical uniform boundedness principle for linear operators on Banach spaces, which is also completely elementary (in particular, it does not use the Baire category theorem).

We wish to point out here that by using a suitable refinement of Sokal's idea we can give a proof of a theorem slightly more general than the aforementioned result of Gál that is shorter than the original proof from [6]. The technique we will use in the proof also resembles the one from [4] but since we are not interested in quantitative versions it is less complicated.

To formulate the result we first recall that a map $T : X \to Y$, where X and Y are of course quasi-normed spaces, is called homogenous if

$$\|T(\lambda x)\| = |\lambda|\|Tx\| \quad \forall x \in X, \forall \lambda \in \mathbb{K}$$

and it is called bounded if $\sup\{\|Tx\| : x \in X,\ \|x\| \leq 1\}$ is finite. For a bounded homogenous map we denote this supremum again by $\|T\|$. It follows that $\|Tx\| \leq \|T\|\,\|x\|$ for every $x \in X$ and $\|T\|$ is the best possible such constant.

The result then reads as follows.

Theorem. *Let X be a quasi-Banach space and $(Y_{nm})_{(n,m)\in\mathbb{N}^2}$ a double-sequence of quasi-normed spaces, as well as $(T_{nm} : X \to Y_{nm})_{(n,m)\in\mathbb{N}^2}$ a double-sequence of bounded homogenous operators satisfying*

$$\sup_{n\in\mathbb{N}} \|T_{nm}\| = \infty \quad \forall m \in \mathbb{N}. \tag{+}$$

Suppose further that there are two sequences $(C_m)_{m\in\mathbb{N}}$ and $(L_m)_{m\in\mathbb{N}}$ of positive real numbers, a sequence $(N_m)_{m\in\mathbb{N}}$ in $\mathbb{N}$ and two double-sequences $(c_{nm})_{(n,m)\in\mathbb{N}^2}$ and $(f_{nm})_{(n,m)\in\mathbb{N}^2}$ of functions from X into $[0,\infty)$ such that

(i) *for every $x \in X$, each $y \in X$ with $\|y\| \le 1$ and all $n, m \in \mathbb{N}$ with $n \ge N_m$ we have*

$$\|T_{nm}(x+y)\| \le C_m(\|T_{nm}x\| + \|T_{nm}\|\,\|y\| + f_{nm}(x)),$$

(ii) *for every $x \in X$ we have $f_{nm}(x) = O(1)$ as $n \to \infty$ uniformly in $m \in \mathbb{N}$,*

(iii) *for every $x \in X$, each $y \in X$ with $\|y\| \le 1$ and all $n, m \in \mathbb{N}$ the inequality*

$$\|T_{nm}y\| \le L_m(\|T_{nm}(x+y)\| + \|T_{nm}x\| + c_{nm}(x)\|T_{nm}\|)$$

holds,

(iv) *for every $x \in X$ and every $m \in \mathbb{N}$ we have $c_{nm}(x) \to 0$ as $n \to \infty$.*

Then there is an element $x \in X$ such that

$$\sup_{n\in\mathbb{N}} \|T_{nm}x\| = \infty \quad \forall m \in \mathbb{N}.$$

As we said, the conditions (i)–(iv) include in particular the case of Gál's asymptotically subadditive double-sequences (essentially only the functions f_{nm} are new here). For some examples, where such conditions of asymptotic subadditivity occur naturally (e. g., the so-called metric-mean interpolations), we refer the reader to [5] and [6].

Before we can come to the main proof, we need an easy lemma.

Lemma. *For every sequence $(\alpha_n)_{n\in\mathbb{N}}$ of positive real numbers and every $\beta > 0$ there is a sequence $(\beta_n)_{n\in\mathbb{N}}$ of positive real numbers such that*

$$\sum_{i=n+1}^{\infty} \beta_i < \alpha_n \beta_n \text{ and } \beta_n \le \beta \quad \forall n \in \mathbb{N}.$$

P r o o f. Just choose inductively $0 < \beta_n \leq \beta$ such that

$$\sum_{i=m+1}^{n} \beta_i < \frac{\alpha_m \beta_m}{2} \quad \forall m \in \{1, \dots, n-1\}.$$

■

P r o o f of the Theorem. By the Aoki–Rolewicz-theorem we may assume without loss of generality that X is p-normed for some $p \in (0, 1]$ (actually this is not necessary for the proof but it is more convenient).

Define $\psi : \mathbb{N} \to \mathbb{N}$ by $(\psi(1), \psi(2), \dots) = (1, 1, 2, 1, 2, 3, 1, 2, 3, 4, \dots)$.

It easily follows from (iii) that for any $r \in (0, 1]$, each $x \in X$ and all $n \geq N_m$ we have

$$\begin{aligned} \frac{r\|T_{nm}\|}{L_m} &\leq \sup_{y \in B_r(x)} \|T_{nm}y\| + \|T_{nm}x\| + c_{nm}(x)\|T_{nm}\| \\ &\leq 2 \sup_{y \in B_r(x)} \|T_{nm}y\| + c_{nm}(x)\|T_{nm}\|, \end{aligned} \tag{1}$$

where $B_r(x) = \{y \in X : \|x - y\| \leq r\}$ (this is an analogue of the lemma from [12]).

According to the above lemma, we can find a sequence $(\tilde{\beta}_n)_{n \in \mathbb{N}}$ in $(0, 1]$ such that

$$\sum_{i=n+1}^{\infty} \tilde{\beta}_i < \frac{\tilde{\beta}_n}{8^p L_{\psi(n)}^p C_{\psi(n)}^p} \quad \forall n \in \mathbb{N}. \tag{2}$$

For every $n \in \mathbb{N}$ we put $\beta_n = \tilde{\beta}_n^{1/p}$ and

$$\gamma_n = \frac{\beta_n}{8L_{\psi(n)}C_{\psi(n)}} - \left(\sum_{i=n+1}^{\infty} \beta_i^p \right)^{1/p}, \tag{3}$$

which by (2) is strictly positive. Let us also put $x_0 = 0$.

Because of (iv) and our assumption (+), we can find $n_1 \geq N_{\psi(1)}$ such that $c_{n_1\psi(1)}(x_0) \leq \beta_1 / 2L_{\psi(1)}$ and $\|T_{n_1\psi(1)}\| \geq 1/\gamma_1$.

It follows from (1) that

$$\frac{\beta_1}{4L_{\psi(1)}} \|T_{n_1\psi(1)}\| \leq \sup_{y \in B_{\beta_1}(x_0)} \|T_{n_1\psi(1)}y\|$$

and hence there is some $x_1 \in X$ such that $\|x_1 - x_0\| \leq \beta_1$ and $\|T_{n_1\psi(1)}x_1\| \geq \beta_1\|T_{n_1\psi(1)}\| / 8L_{\psi(1)}$.

Next we use (iv) and (+) to choose an index $n_2 > \max\{n_1, N_{\psi(2)}\}$ such that $c_{n_2\psi(2)}(x_1) \leq \beta_2 / 2L_{\psi(2)}$ and $\|T_{n_2\psi(2)}\| \geq 2/\gamma_2$ and then find (using (1)) an element $x_2 \in X$ with $\|x_2 - x_1\| \leq \beta_2$ and $\|T_{n_2\psi(2)}x_2\| \geq \beta_2\|T_{n_2\psi(2)}\| / 8L_{\psi(2)}$.

Continuing in this way, we obtain a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ and a sequence $(x_k)_{k\in\mathbb{N}}$ in X such that for every $k \in \mathbb{N}$ we have $n_k \geq N_{\psi(k)}$ and

$$\|x_k - x_{k-1}\| \leq \beta_k, \tag{4}$$

$$\|T_{n_k\psi(k)}x_k\| \geq \frac{\beta_k}{8L_{\psi(k)}}\|T_{n_k\psi(k)}\|, \tag{5}$$

$$\|T_{n_k\psi(k)}\| \geq \frac{k}{\gamma_k}. \tag{6}$$

From (4) it follows that $d(x_n, x_m) = \|x_n - x_m\|^p \leq \sum_{i=m+1}^{n} \beta_i^p$ for every $n > m$ and hence $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence. By completeness, the limit $x = \lim_{k\to\infty} x_k$ exists and it follows that

$$\|x - x_m\|^p \leq \sum_{i=m+1}^{\infty} \beta_i^p \quad \forall m \in \mathbb{N}. \tag{7}$$

Now for sufficiently large $k \in \mathbb{N}$ we have by (i), (5), (7), (3) and (6)

$$f_{n_k\psi(k)}(x) + \|T_{n_k\psi(k)}x\| \geq \frac{\|T_{n_k\psi(k)}x_k\|}{C_{\psi(k)}} - \|T_{n_k\psi(k)}\|\,\|x_k - x\|$$

$$\geq \|T_{n_k\psi(k)}\|\left(\frac{\beta_k}{8L_{\psi(k)}C_{\psi(k)}} - \left(\sum_{i=k+1}^{\infty}\beta_i^p\right)^{1/p}\right) = \|T_{n_k\psi(k)}\|\gamma_k \geq k.$$

Together with (ii) this implies $\|T_{n_k\psi(k)}x\| \to \infty$ for $k \to \infty$. Since for every $m \in \mathbb{N}$ the set $\psi^{-1}(\{m\})$ is infinite, it follows that $\sup_{n\in\mathbb{N}} \|T_{nm}x\| = \infty$ for every $m \in \mathbb{N}$. ∎

Let us remark that if we only wish to prove the usual principle of condensation of singularities for linear operators as it is stated on page 450, then all we have to do is to replace "$T_n \in \mathcal{F}$" by "$T_n \in G_{\psi(n)}$" at the beginning of Sokal's original proof from [12] (where ψ is defined as in our preceding proof) and the rest of the argument remains exactly the same.

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