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Refinement of Isoperimetric Inequality of Minkowski with the Account of Singularities in Boundaries of Intrinsic Parallel Bodies

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The following inequalities are proved:

$$S^n(A,B) \geq n^n \sum_{i=0}^{k-1} V(B_{A_i}) \left(V^{n-1}(A_i) - V^{n-1}(A_{i+1})\right) + S^n(A_{-T}(B), B),$$

$$S^n(A,B) \geq n^n \int\limits_0^T g(t) df(t) + S^n(A_{-T}(B), B),$$

$$S^n(A,B) \geq n^n \int\limits_0^q g(t) df(t) + S^n(A_{-q}(B), B),$$

where $V(A)$, $V(B)$ stand for the volumes of convex bodies A and B in $\mathbb{R}^n$ $(n \geq 2)$, $S(A,B)$ denotes the area of the surface of the body A relative to the body B, q is the capacity factor of the body B with respect to the body A, $A_i = A_{-t_i}(B) = A/(t_i B)$ is the inner body parallel to the body A with respect to the body B at a distance t_i, $0 = t_0 < t_1 < \ldots < t_i < \ldots < t_{k-1} < t_k = T < q$, B_{A_i} is a shape body of A_i relative to B, $g(t) = V(B_{A_{-t}(B)})$, $f(t) = -V^{n-1}(A_{-t}(B))$, $\int\limits_0^T g(t)df(t)$ is the Riemann–Stieltjes integral of the function $g(t)$ by the function $f(t)$, and $\int\limits_0^q g(t)df(t) = \lim\limits_{T \to q} \int\limits_0^T g(t)df(t)$.

Key words: convex body, isoperimetric inequality, Minkowski inequality.

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By a convex body in the n-dimensional Euclidean space $\mathbb{R}^n$ $(n \geq 2)$ we mean a convex compact set with non-empty interior.

Let A and B be convex bodies in $\mathbb{R}^n$, $A + \lambda B$ with $\lambda \geq 0$ stand for their linear combination in Minkowski's sense, E be a unit ball whose center coincides with the origin $\bar{o}$ of the coordinate system in $\mathbb{R}^n$, Ω denote the boundary sphere of the ball E.

The volume $V(A + \lambda E)$, $\lambda \geq 0$, is expressed by the Steiner formula

$$V(A + \lambda E) = \sum_{k=0}^{n} C_n^k V_{n-k}(A)\lambda^k,$$

where $V_{n-k}(A)$ is the $(n-k)$-th basic measure of the body A. In particular, $V_n(A) = V(A)$, whereas $nV_{n-1}(A) = S(A)$ is the area of the boundary surface of the body A [1, p. 176]. It follows from the Steiner formula that

$$S(A) = \lim_{\lambda \to +0} \frac{V(A + \lambda E) - V(A)}{\lambda}.$$

Minkowski [2, p. 57] obtained a generalization of the Steiner formula, which reads as follows:

$$V(A + \lambda B) = \sum_{k=0}^{n} C_n^k V_k(A, B)\lambda^k,$$

where $V_k(A, B)$ is the k-th mixed volume of the bodies A and B, $V_0(A, B) = V(A)$. It follows from the Minkowski formula that

$$nV_1(A, B) = \lim_{\lambda \to +0} \frac{V(A + \lambda B) - V(A)}{\lambda}.$$

Since $nV_1(A, E) = S(A)$, it is natural to call $nV_1(A, B)$ the area of the surface of the body A relative to the body B and denote it by $S(A, B)$.

If A is a polyhedron with k facets, then by Minkowski (see [3, p. 61]), its first mixed volume is expressed by the formula

$$V_1(A, B) = \frac{1}{n} \sum_{i=1}^{k} S(A_i) h_B(\hat{u}_i), \tag{1}$$

where $\bar{u}_i$ $(i = 1, 2, \ldots, k)$ are outside unit vectors normal to the facets of A, $S(A_i)$ is the area of the i-th facet of A, $h_B(\hat{u}_i)$ is the support value of the body B with respect to the vector $\bar{u}_i$.

In the general case, an expression for the first mixed volume, obtained by A.D. Aleksandrov [4, p. 39], has the form

$$V_1(A,B) = \frac{1}{n}\int\limits_{\Omega} h_B(\bar{u})F(A,d\omega),$$

where $F(A,\omega)$ is the surface function of the body A, ω denotes a domain in Ω.

Recall the first inequality of Minkowski for mixed volumes (see [3, p. 65]),

$$V_1^n(A,B) \geq V(B)V^{n-1}(A). \tag{2}$$

The equality holds if and only if A is positively homothetic to the body B, i.e., $A = kB$, $k > 0$.

The isoperimetric inequality of Minkowski,

$$S^n(A,B) = (nV_1(A,B))^n \geq n^n V(B)V^{n-1}(A), \tag{3}$$

is virtually the same as (2). The equality holds in (3) if and only if it holds in (2). Besides, if $B = E$, then (3) leads to the following classical isoperimetric inequality:

$$S^n(A) \geq n^n V(E)V^{n-1}(A). \tag{4}$$

Let us recall some notions. The Minkowski difference $D = A/B$ of the convex bodies A and B is defined as the set of all points $\bar{d} \in \mathbb{R}^n$ such that $\bar{d} + B \subset A$ (see [1, p. 83]). It is known that the body D is convex. Moreover, if the origin in $\mathbb{R}^n$ is moved, then D is subject to a parallel translation. The capacity factor $q = q(A,B)$ of the body B with respect to the body A is defined as the greatest number α such that the body αB can be placed inside A by a parallel translation [5, p. 100]. Given $0 \leq \sigma \leq q$, the body $A_{-\sigma}(B) = A/(\sigma B)$ is called the intrinsic body parallel to the body A relative to the body B at the distance σ.

In [4, p. 97], A.D. Alexandrov introduced the notion of a convex body with a given domain of definition of the support function. Namely, let Ω' be a closed subset of the unit sphere Ω which does not belong to any closed hemisphere of Ω. Let $H^*(\bar{u})$ be a continuous positive function defined in Ω'. Given a vector $\bar{u} \in \Omega'$, consider a hyperplane $T(\bar{u})$ in $\mathbb{R}^n$ orthogonal to $\bar{u}$ and lying at the distance $H^*(\bar{u})$ from the origin $\bar{o}$ in the direction of $\bar{u}$. Denote by $\overline{T(\bar{u})}$ the closed half-space in $\mathbb{R}^n$ bounded by $T(\bar{u})$ and containing the origin. By A.D. Alexandrov, the convex body $N = \cap_{\bar{u}\in\Omega'}\overline{T(\bar{u})}$ is called a convex body with the definition domain Ω' of the support function. In what follows, we will use the notation $N = (\Omega', H^*(\bar{u}))$.

When dealing with functionals invariant under parallel translations, the choice of the origin in $\mathbb{R}^n$ does not matter. Hence we will assume that $qB \subset A$ and that the origin $\bar{o}$ belongs to the interior of B. Let $H_A(\bar{u})$ denote the support function of the body A. Then $A = (\Omega, H_A(\bar{u}))$ since $H_A(\bar{u})$ is continuous and positive in

Ω. It is shown in [5, p. 107] that for A there exists a minimal domain $\Omega' = \Omega_A$ such that $A = (\Omega_A, H_A^*(\bar{u}))$, where $H_A^*(\bar{u})$ is the restriction to Ω_A of the support function $H_A(\bar{u})$ of the body A. For example, if A is a polyhedron in $\mathbb{R}^n$, then Ω_A is just a set of outward unit vectors normal to the facets of A. The body $B_A = (\Omega_A, H_B^*(\bar{u}))$ is called the shape body of the body A relative to the body B (see [5, p. 108]). Note that $B \subset B_A$. Remarkably, the body B_A takes into account the singularities in the boundary of the body A relative to the body B.

G. Hadwiger [6, p. 368] obtained the following refinement of the classical isoperimetric inequality (4):

$$S^n(A) \geq n^n V(E_A) V^{n-1}(A), \tag{5}$$

where $E_A = (\Omega_A, H_E^*(\bar{u}))$. In [5, 108], inequality (5) was generalized by taking into account singularities in the boundary of the body A relative to the body B,

$$S^n(A, B) \geq n^n V(B_A) V^{n-1}(A). \tag{6}$$

In [7, p. 43], inequality (6) was refined by taking into account the non-degeneracy of $A_{-q}(B)$,

$$S^n(A, B) \geq n^n V(B_A) V^{n-1}(A) + S^n(A_{-q}(B), B). \tag{7}$$

Now let T be an arbitrary number satisfying $0 < T < q$. Divide the segment $[0, T]$ into k parts by points $0 = t_0 < t_1 < \ldots < t_i < \ldots t_k = T$. Denote $A_i = A_{-t_i}(B)$, $1 \leq i \leq k-1$, so $A_0 = A$, $A_k = A_{-T}(B)$.

The aim of the paper is to prove the following.

Theorem 1. *The following inequality holds true, which refines inequality (6) by taking into account singularities in the boundaries of the bodies A_1, A_2, $\ldots$, A_{k-1}:*

$$S^n(A, B) \geq n^n \sum_{i=0}^{k-1} V(B_{A_i}) \left(V^{n-1}(A_i) - V^{n-1}(A_{i+1})\right) + S^n(A_{-T}(B), B).$$

Theorem 2. *The following inequality holds true:*

$$S^n(A, B) \geq n^n \int_0^T g(t) df(t) + S^n(A_{-T}(B), B),$$

where $g(t) = V(B_{A_{-t}(B)})$, $f(t) = -V^{n-1}(A_{-t}(B))$, $\int_0^T g(t) df(t)$ is the Riemann–Stieltjes integral of $g(t)$ by $f(t)$.

Theorem 3. *The following inequality holds true:*

$$S^n(A,B) \geq n^n \int_0^q g(t)df(t) + S^n(A_{-q}(B),B), \tag{8}$$

where $\int_0^q g(t)df(t) = \lim_{T\to q} g(t)df(t)$.

P r o o f of Theorem 1. To prove Theorem 1, we will apply Lemma 1 obtained in [7, p. 43].

Lemma 1. *The following inequality holds true for any* $0 < \rho < q$*:*

$$V_1^n(A,B) - V(B_A)V^{n-1}(A) \geq V_1^n(A_{-\rho}(B),B) - V(B_A)V^{n-1}(A_{-\rho}(B)).$$

Putting $\rho = t_1$ in this inequality, we rewrite it as follows:

$$V_1^n(A,B) \geq V(B_A)\left(V^{n-1}(A) - V^{n-1}(A_1(B))\right) + V_1^n(A_1(B),B). \tag{9}$$

Next, replacing A by A_1 and A_1 by A_2, we have

$$V_1^n(A_1,B) \geq V(B_{A_1})\left(V^{n-1}(A_1) - V^{n-1}(A_2(B))\right) + V_1^n(A_2(B),B).$$

Substituting this lower bound for $V_1^n(A_1,B)$ in (9), we obtain the inequality

$$V_1^n(A,B) \geq V(B_A)\left(V^{n-1}(A) - V^{n-1}(A_1(B))\right) +$$

$$+V(B_{A_1})\left(V^{n-1}(A_1) - V^{n-1}(A_2(B))\right) + V_1^n(A_2(B),B). \tag{10}$$

Similarly, using (9), one can obtain lower bounds for $V_1^n(A_2,B)$, $V_1^n(A_3,B),\ldots,$ and substitute these bounds step by step into (10). After k steps, this iterative procedure results into the desired inequality given in Theorem 1, q.e.d.

The proofs of Theorems 2 and 3 will be preceded by five lemmas.

Lemma 2 **[7].** *If the inequality*

$$H_L(\bar{u}) \leq H^*(\bar{u})$$

holds true for any $\bar{u} \in \Omega'$*, then* $L \subset \bar{L} \subset N$.

Next, let $C = A_{-\sigma}(B) = (\Omega', H_C^*(\bar{u}))$, where $\sigma \in [0,q]$ is a fixed number. Consider the body $(\Omega', H_B^*(\bar{u}))$ and the function $H_t^*(\bar{u}) = H_C^*(\bar{u}) - tH_B^*(\bar{u})$, $\bar{u} \in \Omega'$, $t \in (0, q-\sigma)$. Since the support function of a convex body is continuous in Ω, the function $H_t^*(\bar{u})$ is continuous in $\bar{\omega} \in \Omega'$. Moreover, the inclusion $qB \subset A$, the choice of the origin o and the inequality $\sigma + t < q$ together imply that

$H_t^*(\bar{u}) = H_C^*(\bar{u}) - th_B^*(\bar{u}) > 0$ holds for any $\bar{u} \in \Omega'$. Therefore the function $H_t^*(\bar{u})$ generates a convex body $N_t = (\Omega', H_t^*(\bar{u}))$.

Lemma 3. *Given the convex bodies C and B, the following equality holds true for any $0 < t < q - \sigma$:*

$$N_t = C_{-t}(\bar{B}) = C_{-t}(B).$$

P r o o f. Let us show that $N_t \subset C_{-t}(\bar{B}) = C/(t\bar{B})$. Let $\bar{a}$ be a point of N_t. Then $H_{\bar{a}}(\bar{u}) \leq H_t^*(\bar{u})$ holds for any $\bar{u} \in \Omega'$. The support function $H_{\bar{a}+t\bar{B}}(\bar{u})$ of the body $\bar{a} + t\bar{B}$ satisfies the inequality

$$H_{\bar{a}+t\bar{B}}(\bar{u}) = H_{\bar{a}}(\bar{u}) + tH_{\bar{B}}(\bar{u}) \leq H^*t(\bar{u}) + tH_{\bar{B}}(\bar{u}) = H_C^*(\bar{u}), \quad \bar{u} \in \Omega'.$$

Then it follows from Lemma 2 that $\bar{a} + t\bar{B} \subset C$. Therefore, $\bar{a} \in C/(t\bar{B})$. Thus, $N_t \subset C_{-t}(\bar{B}) = C/(t\bar{B})$.

Now let us show that $C_{-t}(B) \subset N_t$. Let $\bar{b}$ be a point not belonging to N_t. Then there exists $\bar{u}_0 \in \Omega'$ such that $H_{\bar{b}}(\bar{u}_0) > H_t^*(\bar{u}_0)$. Consequently, $H_{\bar{b}}(\bar{u}_0) + tH_B(\bar{u}_0) = H_{\bar{b}}(\bar{u}_0) + tH_B^*(\bar{u}_0) > H_C^*(\bar{u}_0)$. This means that the body $\bar{b} + tB$ does not belong to C. Hence $\bar{b}$ does not belong to $C_{-t}(B) = C/(tB)$. Thus, $C_{-t}(B) \subset N_t$.

The inclusions $N_t \subset C_{-t}(\bar{B}) \subset C_{-t}(B) \subset N_t$ result in the desired equalities $N_t = C_{-t}(\bar{B}) = C_{-t}(B)$, q.e.d.

Lemma 4. *The value of $V\left(B_{A_{-\sigma}(B)}\right)$ is finite positive for any $0 \leq \sigma < q$.*

P r o o f. Consider the shape body $B_{A_{-\sigma}(B)}$ of the body $A_{-\sigma}(B)$ relative to the body B. The support function of this shape body is defined in the domain $\Omega_{A_{-\sigma}(B)}$, which is the minimal domain of definition of the support function of the body $A_{-\sigma}(B)$. The minimal domain of definition Ω_A for the support function of the convex body A contains the set of all outward unit vectors normal to the support planes of the body A at regular points of the surface of A. Recall that a point in the boundary of A is called regular if there exists a unique support plane of A passing through this point. In [6, p. 368], it is shown that Ω_A can not belong to any closed hemisphere of the unit sphere Ω. Hence the shape body of the convex body A relative to the convex body B is a convex body and it has a finite positive volume, q.e.d.

Lemma 5. *Let $0 \leq \sigma_1 < \sigma_2 < q$. Then the following inclusions hold true: $\Omega_{A_{-\sigma_1}(B)} \supset \Omega_{A_{-\sigma_2}(B)}$, $B_{A_{-\sigma_1}(B)} \subset B_{A_{-\sigma_2}(B)}$. Moreover, the function $g(\sigma) = V(B_{A_{-\sigma}(B)})$ is increasing for any $\sigma \in [0, q)$.*

P r o o f. Let Ω' be a domain of definition for the support function of the body $A_{-\sigma_1}(B)$. It follows from Lemma 3 that Ω' is a domain of definition for the support function of the body $A_{-\sigma_2}(B)$ since $\sigma_2 = \sigma_1 + t$, where $t = \sigma_2 - \sigma_1 > 0$.

Replace Ω' by the minimal domain of definition $\Omega_{A_{-\sigma_1}(B)}$ of the support function of the body $A_{-\sigma_1}(B)$. It follows from Lemma 3 that $\Omega_{A_{-\sigma_1}(B)}$ is a domain of definition for the support function of the body $A_{-\sigma_2}(B)$. Therefore, $\Omega_{A_{-\sigma_1}(B)} \supset \Omega_{A_{-\sigma_2}(B)}$.

Let us prove the inclusion $B_{A_{-\sigma_1}(B)} \subset B_{A_{-\sigma_2}(B)}$. Actually, $B_{A_{-\sigma_1}(B)}$ (respectively, $B_{A_{-\sigma_2}(B)}$) is the intersection of the closed half-spaces supporting the body B whose outward unite normal vectors, translated to $\bar{o}$, belong to $\Omega_{A_{-\sigma_1}(B)}$ (respectively, to $\Omega_{A_{-\sigma_2}(B)}$). Since $\Omega_{A_{-\sigma_2}(B)} \subset \Omega_{A_{-\sigma_1}(B)}$, then $B_{A_{-\sigma_1}(B)} \subset B_{A_{-\sigma_2}(B)}$.

Moreover, $V(B_{A_{-\sigma_1}(B)}) \leq V(B_{A_{-\sigma_2}(B)})$. Therefore, the function $g(\sigma) = V(B_{A_{-\sigma}(B)})$, defined for $\sigma \in [0, q)$, is increasing, q.e.d.

Set $f_1(\sigma) = V(A_{-\sigma}(B))$.

Lemma 6. *The function $f_1(\sigma)$, defined for $\sigma \in [0, q)$, is continuous and decreasing everywhere in $[0, q)$.*

P r o o f. Along with the body $N = (\Omega', H^*(\bar{u}))$, let us consider a family of bodies $N_t = (\Omega', H^*(\bar{u}) + t\delta H^*(\bar{u}))$, where $\delta H^*(\bar{u})$ is a continuous function defined for $\bar{u} \in \Omega'$. A.D. Alexandrov proved that the first variation of the volume $V(N)$, i.e.,

$$\delta V(N) = \lim_{t \to 0} \frac{V(N_t) - V(N)}{t},$$

is equal to

$$\delta V(N) = \int_{\Omega'} \delta H^*(\bar{u}) F(N, d\omega),$$

where $F(N, d\omega)$ is a surface measure of the body N which satisfies $F(N, \Omega - \Omega') = 0$ (see [4, p. 100–101]). It is shown in [7, p. 44] that $N_\sigma = (\Omega', H^*_\sigma(\bar{u})) = A_{-\sigma}(B)$. Applying the above to the body N_σ and to the family of bodies $(N_\sigma)_t = \big(\Omega', H^*_{\sigma+t}(\bar{u})\big) = (\Omega', H^*_\sigma(\bar{u}) - tH^*_B(\bar{u}))$, we get

$$\frac{dV(A_{-\sigma}(B))}{d\sigma} = -\int_{\Omega'} H^*_B(\bar{u}) F(A_{-\sigma}(B), d\omega) = -nV_1(A_{-\sigma}(B), B), \quad \sigma \in [0, q).$$

Therefore, the function $f_1(\sigma)$ has a finite derivative at the interval $(0, q)$ and a finite right-hand side derivative at $\sigma = 0$, which is actually equal to $nV_1(A, B)$. This leads to the continuity of $f_1(\sigma)$ in $[0, q)$.

It follows from (2) that the first mixed volume $V_1(A_{-\sigma}(B), B)$ of $A_{-\sigma}(B)$ and B is positive. Hence, $\frac{dV(A_{-\sigma}(B))}{d\sigma} < 0$ for $\sigma \in [0, q)$, q.e.d.

P r o o f of Theorem 2. Let us show that the integral $I = \int\limits_0^T g(t)df(t)$, where $0 < T < q$, $g(t) = V(B_{A_{-t}(B)})$, $f(t) = -V^{n-1}(A_{-t}(B))$, i.e., the Riemann–Stieltjes integral of $g(t)$ relative to $f(t)$ (see [8, p. 201]), exists.

By Lemma 5, the function $g(t) = V(B_{A_{-t}(B)})$ is increasing in the interval $t \in [0, T]$. Therefore, $g(t)$ is a function of finite variation in $[0, T]$. Besides, in view of Lemma 6, the function $f_1(t) = V(A_{-t}(B))$ is continuous in the interval $t \in [0, q)$. Thus the function $-V^{n-1}(A_{-t}(B))$ is continuous in $[0, T]$.

In [8, p. 204], it is proved that any function of finite variation is integrable with respect to any continuous function. Therefore the integral $I = \int\limits_0^T g(t)df(t)$ exists.

Consider a Riemann sum of the integral in question which corresponds to a partition of the segment $[0, T]$ by points $0 = t_0 < t_1 < \ldots < T_{k-1} < t_k = T$ with $\xi_i = t_i$. This Riemann sum has the form

$$\sigma = \sum_{i=0}^{k-1} V(B_{A_i}) \left(-V^{n-1}(A_{i+1}) - (-V^{n-1}(A_i))\right).$$

Hence the statement of Theorem 1 can be rewritten as follows:

$$S^n(A, B) \geq n^n \sigma + S^n(A_{-T}(B), B).$$

Applying an appropriate choice both of a sufficiently fine partition of the segment $[0, T]$ into segments $[t_i, t_{i+1}]$ and of points $\xi_i \in [t_i, t_{i+1}]$, one can get $|I - \sigma| < \varepsilon$ for any arbitrary $\varepsilon > 0$. Thus Theorem 2 is proved.

P r o o f of Theorem 3. Let $\varphi(T) = \int\limits_0^T g(t)df(t)$, $\psi(T) = S^n(A_{-T}(B), B)$. Then the inequality in Theorem 2 can be rewritten as follows:

$$S^n(A, B) \geq n^n \varphi(T) + \psi(T), \quad 0 \leq T < q. \tag{11}$$

Given $0 \leq t_1 < t_2 \leq q$, we have $A_{-t_2}(B) \subset A_{-t_1}(B)$. Indeed, if a point a_2 belongs to $A_{-t_2}(B)$, then $a_2 + t_2 B \subset A$. Moreover, since $t_1 B \subset t_2 B$, then $a_2 + t_1 B \subset A$. Therefore, a_2 belongs to $A_{-t_1}(B)$, so $A_{-t_2}(B) \subset A_{-t_1}(B)$.

Because the mixed volume is monotone with respect to any of its arguments and non-negative [2, p. 49], the function $\psi(T)$ is decreasing for $0 \leq T < q$, and its minimal value is equal to $S(A_{-q}(B), B)$.

From Lemma 4, it follows that $g(t) = V(B_{A_{-t}(B)})$ is a finite positive number for any $t \in [0, T]$. Moreover, it follows from Lemma 6 that $f(t) = -V^{n-1}(A_{-t}(B)$ is increasing for any $t \in [0, T]$. Therefore, the integral sum $\varphi(T)$ is non-negative and it increases if T increases. Because the summands in the right-hand side of

(11) are non-negative, any of them is less than or equal to the left-hand side of (11). Therefore, $S^n(A,B) \geq n^n \varphi(T)$, and hence $\varphi(T)$ has a limit value at $t \to q$. Denote this limit value by $\lim\limits_{T\to q} \varphi(T) = \int\limits_0^q g(t)df(t)$. Then, passing to the limit in the right-hand side of (11), we get (8), q.e.d.

Let us give an example when $\int\limits_0^q g(t)df(t)$ exists and (8) refines (7).

E x a m p l e. We will consider convex polygons in the plane (see Fig. 1) where $n = 2$. Let B be an isosceles right triangle, $k = 3$, assuming that the origin $\bar{o}$ belongs to B. Let A be a hexagon bounded by the broken line $abcdef$. Clearly, $A = A_0(B) = A/(0B) = A/\bar{o}$. Moreover, A and B are chosen in such a way that $q = 4$.

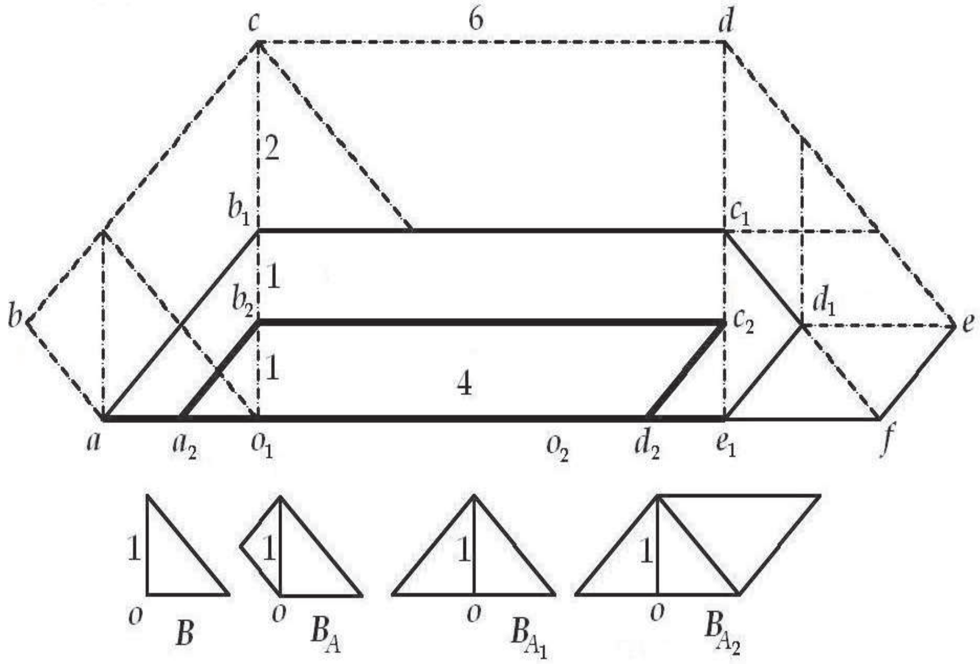

Fig. 1. The isosceles right triangle B, hexagon A, and the sequence of shape bodies.

To provide a partition of the segment $[0, q]$, we chose $t_1 = 2$, $t_2 = 3$, $t_3 = 4$. Then $A_1 = A_{-2}(B) = A/(2B) = A/(2B_A)$ is a pentagon bounded by the broken line $ab_1c_1d_1e_1$. Next, $A_2 = A_{-3}(B) = A/(3B) = A_1/B = A_1/B_{A_1}$ is a parallelogram bounded by a broken line $a_2b_2c_2d_2$. Finally, $A_3 = A_{-4}(B) = A/(4B) = A_2/B = A_2/B_{A_2}$ is a segment o_1o_2.

Let us describe the minimal domains of definition for support functions of the planar convex polygons in question. All these domains belong to the unite circle Ω centered at the origin o. We have the following.

Ω_A consists of six points in Ω, which are the end-points of outward normal unit vectors to the sides of the hexagon $abcdef$.

Ω_{A_1} consists of five points in Ω, which are the end-points of outward normal unit vectors to the sides of the pentagon $ab_1c_1d_1e_1$. Ω_{A_1} is the same as Ω_A excluding the end-point of the outward normal unit vector to the side ab.

Ω_{A_2} differs from Ω_{A_1} by one point, the end-point of the outward normal unit vector to the side c_1d_1.

$\Omega_{o_1o_2}$ consists of two points in Ω, which are the end-points of outward normal unit vectors to the segment o_1o_2.

$\Omega_{A_{-t}(B)} = \Omega_A$ for any $t \in [0,2)$. Hence the shape body for any $\Omega_{A_{-t}(B)}$, $t \in [0,2)$, is B_A.

$\Omega_{A_{-t}(B)} = \Omega_{A_1}$ for any $t \in [2,3)$. Hence the shape body for any $\Omega_{A_{-t}(B)}$, $t \in [2,3)$, is B_{A_1}.

$\Omega_{A_{-t}(B)} = \Omega_{A_2}$ for any $t \in [3,4)$. Hence the shape body for any $\Omega_{A_{-t}(B)}$, $t \in [3,4)$, is B_{A_2}.

Since $n = 2$, replace S by l and V by S in (8) to get

$$S(A) = 38,\ S(A_1) = 15,\ S(A_2) = 6,$$
$$S(B_A) = \tfrac{3}{4},\ S(B_{A_1}) = 1,\ S(B_{A_2}) = 2,\ |o_1o_2| = 4.$$

Applying (1) and the equality $S(A,B) = nV_1(A,B)$, we have

$$S(A,B) = 2\,V_1(A,B) = 2\left(\frac{1}{2}\sum_{I=1}^{6} a_i h_B(\bar{u}_i)\right)$$
$$= |ab|\cdot 0 + |bc|\cdot\frac{\sqrt{2}}{2} + |cd|\cdot 1 + |de|\cdot\frac{\sqrt{2}}{2} + |ef|\cdot\frac{\sqrt{2}}{2} + |fa|\cdot 0 = 13,$$

$$S(A_{-q}(B),B) = |o_1o_2|\cdot 1 + |o_2o_1|\cdot 0 = 4.$$

Therefore, the left-hand side of (8) is equal to $l^2(A,B) = 13^2 = 169$.

On the other hand, $g(t)$ is a step-like function,

$$g(t) = \begin{cases} S(B_A) = \frac{3}{4}, & t \in [0,2); \\ S(B_{A_1}) = 1, & t \in [2,3); \\ S(B_{A_2}) = 2, & t \in [3,4). \end{cases}$$

Hence the right-hand side of (8) is equal to

$$4\int_0^4 g(t)df(t) + S^2(A_{-q}(B), B)$$

$$= 4\left(\int_0^2 g(t)df(t) + \int_2^3 g(t)df(t) + \int_4^4 g(t)df(t)\right) + S^2(A_{-q}(B), B)$$

$$= 4\left(\frac{3}{4}(38-15) + 1\cdot(15-6) + 2\cdot 6\right) + 4^2 = 169.$$

The left-hand side of (7) also equals 169, whereas the right-hand side of (7) is equal to 130. Hence (8) refines (7).

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