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A New Model of Quantum Dot Light Emitting-Absorbing Devices

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Motivated by the Jaynes–Cummings (JC) model, we consider here a quantum dot coupled simultaneously to a reservoir of photons and two electric leads (free-fermion reservoirs). This new Jaynes–Cummings-leads (JCL)-type model makes it possible that the fermion current through the dot creates a photon flux, which describes a light-emitting device. The same model also describes a transformation of the photon flux into the current of fermions, i.e., a quantum dot light-absorbing device. The key tool to obtain these results is the abstract Landauer–Büttiker formula.

Key words: Landauer–Büttiker formula, Jaynes–Cummings model, coupling to leads, light emission.

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Dedicated to the memory of Pierre Duclos

1. Introduction

The aim of the present paper is to analyze the fermion current going through a quantum dot as a function of: (1) the electro-chemical potentials on leads and (2) the contact with the external photon reservoir. Although the latter is the canonical JC-interaction, the coupling of the JC model with leads needs certain precautions if we want to remain in the framework of one-particle quantum mechanical Hamiltonian approach and the scattering theory. To this end we proposed a new Jaynes–Cummings-leads (JCL)-model [9]. It makes it possible to create a photon flux into the resonator by the fermion current through the

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dot, i.e., it describes a *light-emitting* device, as well as to transform the external photon flux into the current of fermions which corresponds to a quantum dot *light-absorbing* device.

We discuss the construction of our JCL-model in Secs. 2.1–2.5. For simplicity, for the leads Hamiltonian we choose the one-particle discrete Schrödinger operators with constant (electric) potentials of each of the leads. In Sec. 2.5, we show that our model fits into the framework of *trace-class scattering*. In Sec. 2.5, we verify the important property that the coupled Hamiltonian has no singular continuous spectrum. Our main tool for analyzing different currents is an abstract Landauer–Büttiker-type formula applied in Secs. 3.1 and 3.2 to the case of the JCL-model. It allows us to calculate the outgoing flux of photons induced by electric current through leads. This corresponds to a *light-emitting* device. We also found out that the pumping of the JCL quantum dot by the photon flux from the resonator may induce the current of fermions into leads. This reversing imitates a quantum *light-absorbing* cell device. These are the main properties of our model and the main application of the Landauer–Büttiker-type formula of Secs. 3.1 and 3.2. They are presented in Secs. 4 and 5, where we distinguish the *contact-induced* and the *photon-induced* fermion currents.

To describe the results of Secs. 4 and 5, we should note that in our setup the sample Hamiltonian is a *two-level* quantum dot *decoupled* from the one-mode photon resonator. Then the unperturbed Hamiltonian H_0 can be described as a collection of *four* totally decoupled subsystems: a sample, a resonator and two leads. The perturbed Hamiltonian H is a fully coupled system and the feature of our model is that it is *totally* (i.e., including the leads) *embedded* into the external electromagnetic field of the resonator. Hence, each electron can be interpreted as a fermion with *internal* harmonic degrees of freedom, or a Fermi-particle carrying an individual *photon cloud*.

Similarly to the "Black Box" system-leads (SL)-model $\{H_{SL}, H_0\}$ [1, 2], it turns out that the JCL-model also fits into the framework of the abstract Landauer–Büttiker formula and, in particular, is a trace-class scattering system $\{H_{JCL} = H, H_{SL}\}$. The current in the SL-model is called the *contact-induced* current J^c_{el}. It was a subject of numerous papers, see, e.g., [1, 3] or [2]. Note that the current J^c_{el} occurs due to the difference of electro-chemical potentials between two leads, but it may be *zero* even if this difference is not null [5, 6].

The fermion current of the JCL-model, which takes into account the effect of the electron-photon interaction under the assumption that the leads are already coupled, is called the *photon-induced* component J^{ph}_{el} of the total current. Up to our knowledge, the present paper is the first one where it is studied rigorously. We show that the total free-fermion current J_{el} in the JCL-model decomposes into a sum of the contact- and the photon-induced currents: $J_{el} := J^c_{el} + J^{ph}_{el}$. An extremal case is where the contact-induced current is zero, but the photon-

induced component is not, c.f. Sec. 5.1. In this case, the flux of photons J_{ph} out of the quantum dot (sample) is also non-zero, i.e., the dot serves as a light emitting device, c.f. Sec. 5.2. In general, $J_{ph} \neq 0$ only when the photon-induced component is not zero, i.e. $J_{el}^{ph} \neq 0$.

It turns out that when choosing the parameters of the model in a suitable manner, one gets either a photon emitting or a photon absorbing system. Hence the JCL-model can be used either as a light emission device or as a light-cell. The proofs of explicit formulas for the fermion and photon currents, J_{el} , J_{ph}, are the contents of Secs. 4 and 5.

Note that the JCL-model is called *mirror symmetric* if (roughly speaking) one can interchange the left and the right leads and the JCL-model remains unchanged. In Sec. 5, we discuss a surprising example of a mirror symmetric JCL-model in which the free-fermion current is *zero* but the model is photon emitting. This peculiarity is due to a specific choice of the *photon-electron* interaction which produces fermions with *internal harmonic* degrees of freedom.

2. Jaynes–Cummings Quantum Dot Coupled to Leads

2.1. Jaynes–Cummings model

The starting point for the construction of our JCL-model is the quantum optics Jaynes–Cummings Hamiltonian H_{JC}. Its simplest version is a *two-level* system (quantum dot) with the energy spacing ε defined by the Hamiltonian h_S on the Hilbert space $\mathfrak{h}_S = \mathbb{C}^2$, see, e.g., [7]. It is assumed that this system is "open" and interacts with the one-mode ω photon resonator with Hamiltonian h^{ph}.

Since mathematically h^{ph} coincides with the quantum harmonic oscillator, the Hilbert space of the resonator is the boson Fock space $\mathfrak{h}^{ph} = \mathfrak{F}_+(\mathbb{C})$ over $\mathbb{C}$ and

$$h^{ph} = \omega\, b^* b\ . \tag{2.1}$$

Here b^* and b are verifying the Canonical Commutation Relations (CCR) creation and annihilation operators with the domains in $\mathfrak{F}_+(\mathbb{C}) \simeq \ell^2(\mathbb{N}_0)$, here $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Operator (2.1) is self-adjoint on its domain

$$\mathrm{dom}(h^{ph}) = \left\{ (k_0, k_1, k_2, \ldots) \in \ell^2(\mathbb{N}_0) : \sum_{n\in\mathbb{N}_0} n^2 |k_n|^2 < \infty \right\}.$$

Note that the canonical basis $\{\phi_n := (0, 0, \ldots, k_n = 1, 0, \ldots)\}_{n\in\mathbb{N}_0}$ in $\ell^2(\mathbb{N}_0)$ consists of eigenvectors of the operator (2.1): $h^{ph}\phi_n = n\omega\, \phi_n$.

To model the *two-level* system with the energy spacing ε, one fixes in $\mathbb{C}^2$ two ortho-normal vectors $\{e_0^S, e_1^S\}$, for example, $e_0^S := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $e_1^S := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which

are eigenvectors of the Hamiltonian h_S with the eigenvalues $\{\lambda_0^S = 0,\ \lambda_1^S = \varepsilon\}$. To this end, we put

$$h_S := \varepsilon \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.2}$$

and we introduce two *ladder* operators:

$$\sigma^+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma^- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.3}$$

Then one gets $h_S = \varepsilon\ \sigma^+\sigma^-$ as well as $e_1^S = \sigma^+ e_0^S$, $e_0^S = \sigma^- e_1^S$ and $\sigma^- e_0^S = 0$. Thus, e_0^S is the ground state of the Hamiltonian h_S. Note that the *non-interacting* Jaynes–Cummings Hamiltonian H_0^{JC} lives in the space $\mathfrak{H}^{JC} = \mathfrak{h}_S \otimes \mathfrak{h}^{ph} = \mathbb{C}^2 \otimes \mathfrak{F}_+(\mathbb{C})$ and it is defined as the *matrix* operator

$$H_0^{JC} := h_S \otimes I_{\mathfrak{h}^{ph}} + I_{\mathfrak{h}_S} \otimes h^{ph}\ . \tag{2.4}$$

Here $I_{\mathfrak{h}^{ph}}$ denotes the identity operator in the Fock space $\mathfrak{h}^{ph}$, whereas $I_{\mathfrak{h}_S}$ stays for the identity matrix in the space $\mathfrak{h}_S$.

With operators (2.3), the interaction V_{Sb} between the quantum dot and photons (bosons) in the resonator is defined (in the rotating-wave approximation [7]) by the operator

$$V_{Sb} := g_{Sb}\ (\sigma^+ \otimes b + \sigma^- \otimes b^*)\ . \tag{2.5}$$

Operators (2.4) and (2.5) define the Jaynes-Cummings model Hamiltonian

$$H_{JC} := H_0^{JC} + V_{Sb}\ , \tag{2.6}$$

which is self-adjoint operator on the common domain $\mathrm{dom}(H_0^{JC}) \cap \mathrm{dom}(V_{Sb})$. The standard interpretation of H_{JC} is that (2.6) describes an "open" two-level system interacting with external one-mode electromagnetic field [7].

Since the one-mode resonator is able to absorb *infinitely* many bosons, this interpretation sounds reasonable, but one can see that the spectrum $\sigma(H^{JC})$ of the Jaynes–Cummings model is *discrete*. Note that the so-called number operator $\mathfrak{N}_{JC} := \sigma^+\sigma^- \otimes I_{\mathfrak{h}^{ph}} + I_{\mathfrak{h}_S} \otimes b^*b$ commutes with H_{JC}. Then, since for any $n \geq 0$,

$$\mathfrak{H}_{n>0}^{JC} := \{\zeta_0 e_0^S \otimes \phi_n + \zeta_1 e_1^S \otimes \phi_{n-1}\}_{\zeta_{0,1}\in\mathbb{C}}\ ,\ \mathfrak{H}_{n=0}^{JC} := \{\zeta_0 e_0^S \otimes \phi_0\}_{\zeta_0\in\mathbb{C}}$$

are eigenspaces of the operator $\mathfrak{N}_{JC}$, which reduce H_{JC}, i.e., $H_{JC} : \mathfrak{H}_n^{JC} \to \mathfrak{H}_n^{JC}$. Note also that $\mathfrak{H}^{JC} = \bigoplus_{n\geq 0} \mathfrak{H}_n^{JC}$, where each $\mathfrak{H}_n^{JC}$ is an invariant subspace of the operator (2.6). Therefore, it has the representation

$$H_{JC} = \bigoplus_{n\in\mathbb{N}_0} H_{JC}^{(n)}. \tag{2.7}$$

Here the operators $H_{JC}^{(n)}$ are the restrictions of H_{JC} to $\mathfrak{H}_n^{JC}$ such that $H_{JC}^{(0)} = 0$ and

$$\begin{aligned} &H_{JC}^{(n)}(\zeta_0\ e_0^S \otimes \phi_n + \zeta_1\ e_1^S \otimes \phi_{n-1}) \\ &= [\zeta_0 n\omega + \zeta_1 g_{Sb}\sqrt{n}]\ e_0^S \otimes \phi_n + [\zeta_1(\varepsilon + (n-1)\omega) + \zeta_0 g_{Sb}\sqrt{n}]\ e_1^S \otimes \phi_{n-1}\,. \end{aligned} \tag{2.8}$$

Hence the spectrum $\sigma(H_{JC}) = \bigcup_{n\geq 0}\sigma(H_{JC}^{(n)})$. By virtue of (2.8), the spectrum $\sigma(H_{JC}^{(n)})$ is defined for $n \geq 1$ by the eigenvalues $E(n)$ of the two-by-two matrix $\widehat{H}_{JC}^{(n)}$ acting on the coefficient space $\{\zeta_0, \zeta_1\}$:

$$\widehat{H}_{JC}^{(n)}\begin{pmatrix}\zeta_1\\ \zeta_0\end{pmatrix} = \begin{pmatrix}\varepsilon + (n-1)\,\omega & g_{Sb}\sqrt{n}\\ g_{Sb}\sqrt{n} & n\omega\end{pmatrix}\begin{pmatrix}\zeta_1\\ \zeta_0\end{pmatrix} = E(n)\begin{pmatrix}\zeta_1\\ \zeta_0\end{pmatrix},\ n \geq 1. \tag{2.9}$$

Then (2.7) and (2.9) imply that the spectrum of the Jaynes–Cummings model Hamiltonian H_{JC} is *pure point*:

$$\sigma(H_{JC}) = \sigma_{p.p.}(H_{JC}) = \{0\} \cup \bigcup_{n\in\mathbb{N}}\left\{n\omega + \frac{1}{2}(\varepsilon - \omega) \pm \sqrt{(\varepsilon-\omega)^2/4 + g_{Sb}^2 n}\right\}.$$

This property evidently persists for any system Hamiltonian h_S with discrete spectrum and linear interaction (2.5) with a finite mode photon resonator [7].

We resume the above observations concerning the Jaynes–Cummings model, which is our starting point, by the following remarks:

(a) The standard Hamiltonian (2.6) instead of the *flux* describes only the oscillations of photons between the resonator and the quantum dot, i.e., the system h_S is not "open" enough.

(b) Since one of our aims is to model a *light-emitting* device, the system h_S needs an *external* source of energy to pump it into the dot further to be transformed by interaction (2.5) into the outgoing *photon current* pumping the resonator.

(c) To reach this aim, we extend the standard Jaynes–Cummings model to our JCL-model by attaching to the quantum dot h_S (2.2) two *leads* which are (infinite) reservoirs of *free* fermions. Manipulating with *electro-chemical* potentials of fermions in these reservoirs, we can force one of them to inject fermions in the quantum dot, whereas another one to absorb the fermions out the quantum dot with the same rate. This current of fermions throughout the dot would pump it and produce the photon current according scenario (b).

(d) The most subtle point is to invent a *leads-dot* interaction V_{lS}, which ensures the above mechanism and which is simple enough that one should be able to treat this JCL-model using our extension of the Landauer–Büttiker formalism.

2.2. The JCL-model

First let us make some general remarks and formulate certain conditions indispensable when one follows the modeling (d).

(1) Note that since the Landauer–Büttiker formalism [6] is essentially a scattering theory on a contact between two subsystems, it is developed only on a "one-particle" level. This allows us to study with this formalism only ideal (*non-interacting*) many-body systems. This condition is imposed on many-body fermion systems (electrons) in two leads. Thus, only direct interaction between different components of the system, dot-photons V_{Sb} and electron-dot V_{lS}, is allowed.

(2) It is well known that the fermion reservoirs are technically simpler to treat than the boson ones [6]. Moreover, in the framework of our model it is also very natural since we study the electric current although produced by "non-interacting electrons". So, below we will use fermions/electrons as synonymous.

(3) In spite of the precautions formulated above, the first difficulty in considering an ideal many-body system interacting with the quantized electromagnetic field (photons) is an induced *indirect* interaction. If electrons are able to emit and absorb photons, it is possible for one electron to emit a photon that another electron absorbs, thus creating indirect photon-mediated electron-electron interaction. This interaction makes impossible to develop the Landauer–Büttiker formula, which requires non-interacting framework.

Assumption 2.1. *To solve this difficulty, in our model we forbid the photon-mediated interaction. To this end, we suppose that every electron (in the leads and the dot) interacts with its own distinct copy of the electromagnetic field. So, we consider electrons together with their individual photon clouds as non-interacting "composed particles". This allows us to apply the Landauer–Büttiker approach. Formally it corresponds to the "one-electron" Hilbert space* $\mathfrak{h}^{el} \otimes \mathfrak{h}^{ph}$*, where* $\mathfrak{h}^{ph}$ *is the Hilbert space of the individual photon field. The fermion description of the composed-particles* $\mathfrak{h}^{el} \otimes \mathfrak{h}^{ph}$ *corresponds to the antisymmetric Fock space* $\mathfrak{F}_-(\mathfrak{h}^{el} \otimes \mathfrak{h}^{ph})$.

The composed-particle assumption 2.1 allows us to use the Landauer–Büttiker formalism developed for ideal many-body fermion systems. Now we have come closer to the formal description of our resonator.

Recall that the Hilbert space of the Jaynes–Cummings Hamiltonian with two energy levels is $\mathfrak{H}^{JC} = \mathbb{C}^2 \otimes \mathfrak{F}_+(\mathbb{C})$. The boson Fock space is constructed from a one-dimensional Hilbert space since we consider only photons of a single fixed frequency. We model the electrons in the leads as free fermions living on a discrete semi-infinite lattices. Thus,

$$\mathfrak{h}^{el} = \ell^2(\mathbb{N}) \oplus \mathbb{C}^2 \oplus \ell^2(\mathbb{N}) = \mathfrak{h}_l^{el} \oplus \mathfrak{h}_S \oplus \mathfrak{h}_r^{el} \tag{2.10}$$

is a one-particle Hilbert space for the electrons and the dot. Here, $\mathfrak{h}^{el}_\alpha$, $\alpha \in \{l, r\}$, are Hilbert spaces of the *left*, respectively the *right*, lead, and $\mathfrak{h}_S = \mathbb{C}^2$ is a Hilbert space of the quantum dot. We denote by $\{\delta^\alpha_n\}_{n\in\mathbb{N}}$ and $\{\delta^S_n\}^1_{j=0}$ the canonical basis consisting of individual lattice sites of $\mathfrak{h}^{el}_\alpha$, $\alpha \in \{l, r\}$, and of $\mathfrak{h}_S$, respectively. With the Hilbert space for photons, $\mathfrak{h}^{ph} = \mathfrak{F}_+(\mathbb{C}) \simeq \ell^2(\mathbb{N}_0)$, we define the Hilbert space of the *full* system, i.e., the quantum dot with leads and with the photon field, as

$$\mathfrak{H} = \mathfrak{h}^{el} \otimes \mathfrak{h}^{ph} = \big(\ell^2(\mathbb{N}) \oplus \mathbb{C}^2 \oplus \ell^2(\mathbb{N})\big) \otimes \ell^2(\mathbb{N}_0). \tag{2.11}$$

R e m a r k 2.2. Note that the structure of full space (2.11) takes into account condition 2.1 and produces composed fermions via the last tensor product. It also manifests that electrons *in the dot* as well as those *in the leads* are composed of photons. This makes difference with the picture imposed by the Jaynes–Cummings model, where *only the dot* is composed of photons:

$$\mathfrak{H} = \ell^2(\mathbb{N}) \oplus \mathbb{C}^2 \otimes \ell^2(\mathbb{N}_0) \oplus \ell^2(\mathbb{N}), \quad \mathfrak{H}^{JC} = \mathbb{C}^2 \otimes \ell^2(\mathbb{N}_0), \tag{2.12}$$

see (2.4), (2.5) and (2.6), where $\mathfrak{H}^{JC} = \mathfrak{h}_S \otimes \mathfrak{h}^{ph}$. The next step is a choice of interactions between the subsystems: dot-resonator-leads.

According to (2.10), the decoupled leads-dot Hamiltonian is the matrix operator

$$h^{el}_0 = \begin{pmatrix} h^{el}_l & 0 & 0 \\ 0 & h_S & 0 \\ 0 & 0 & h^{el}_r \end{pmatrix} \quad \text{on} \quad u = \begin{pmatrix} u_l \\ u_S \\ u_r \end{pmatrix}, \ \{u_\alpha \in \ell^2(\mathbb{N})\}_{\alpha\in\{l,r\}}, \ u_S \in \mathbb{C}^2,$$

where $h^{el}_\alpha = -\Delta^D + v_\alpha$ with a constant potential bias $v_\alpha \in \mathbb{R}$, $\alpha \in \{l, r\}$, and h_S can be any self-adjoint two-by-two matrix with eigenvalues $\{\lambda^S_0, \lambda^S_1 := \lambda^S_0 + \varepsilon\}$, $\varepsilon > 0$, and eigenvectors $\{e^S_0, e^S_1\}$, cf (2.2). Here, Δ^D denotes the discrete Laplacian on $\ell^2(\mathbb{N})$ with homogeneous Dirichlet boundary conditions given by

$$\begin{aligned} (\Delta^D f)(x) &:= f(x+1) - 2f(x) + f(x-1), \quad x \in \mathbb{N}, \\ \mathrm{dom}(\Delta^D) &:= \{f \in \ell^2(\mathbb{N}_0) : f(0) := 0\}, \end{aligned}$$

which is obviously a bounded self-adjoint operator. Notice that $\sigma(-\Delta^D) = [0, 4]$.

We define the *lead-dot interaction* for the coupling $g_{el} \in \mathbb{R}$ by the matrix operator acting in (2.10) as

$$v_{el} = g_{el} \begin{pmatrix} 0 & \langle\cdot, \delta^S_0\rangle\delta^l_1 & 0 \\ \langle\cdot, \delta^l_1\rangle\delta^S_0 & 0 & \langle\cdot, \delta^r_1\rangle\delta^S_1 \\ 0 & \langle\cdot, \delta^S_1\rangle\delta^r_1 & 0 \end{pmatrix}, \tag{2.13}$$

where non-trivial off-diagonal entries are *projection* operators in the Hilbert space (2.10) with the scalar product $u, v \mapsto \langle u, v\rangle$ for $u, v \in \mathfrak{h}^{el}$. Here, $\{\delta^S_0, \delta^S_1\}$ is

ortho-normal basis in $\mathfrak{h}_S^{el}$, which in general may be different from $\{e_0^S, e_1^S\}$. Thus interaction (2.13) describes quantum *tunneling* between the leads and the dot via the contact sites of the leads, which are supports of δ_1^l and δ_1^r.

Then the Hamiltonian for the system of interacting leads and the dot is defined as $h^{el} := h_0^{el} + v_{el}$. Here both h_0^{el} and h^{el} are bounded self-adjoint operators on $\mathfrak{h}^{el}$.

Recall that the photon Hamiltonian in the one-mode resonator is defined by the operator $h^{ph} = \omega b^* b$ with the domain in the Fock space $\mathfrak{F}_+(\mathbb{C}) \simeq \ell^2(\mathbb{N}_0)$, (2.1). We denote the canonical basis in $\ell^2(\mathbb{N}_0)$ by $\{\Upsilon_n\}_{n\in\mathbb{N}_0}$. Then for the spectrum of h^{ph}, one obviously gets $\sigma(h^{ph}) = \sigma_{pp}(h^{ph}) = \bigcup_{n\in\mathbb{N}_0}\{n\omega\}$.

We introduce the following decoupled Hamiltonian H_0, which describes the system where the leads are decoupled from the quantum dot and the electron does not interact with the photon field:

$$H_0 := H_0^{el} + H^{ph}, \tag{2.14}$$

where

$$H_0^{el} := h_0^{el} \otimes I_{\mathfrak{h}^{ph}} \quad \text{and} \quad H^{ph} := I_{\mathfrak{h}^{el}} \otimes h^{ph}.$$

The operator H_0 is self-adjoint on $\mathrm{dom}(H_0) = \mathrm{dom}(I_{\mathfrak{h}^{el}} \otimes h^{ph})$. Recall that h_0^{el} and h^{ph} are self-adjoint operators. Hence, H_0^{el} and H^{el} are semi-bounded from below, which yields that H_0 is semi-bounded from below.

The interaction of photons and electrons in the quantum dot is given by the coupling of dipole moment of electrons to the electromagnetic field in the rotating wave approximation. Namely,

$$V_{ph} = g_{ph}\left((\cdot, e_0^S)e_1^S \otimes b + (\cdot, e_1^S)e_0^S \otimes b^*\right)$$

for some coupling constant $g_{ph} \in \mathbb{R}$. The total Hamiltonian is given by

$$H := H^{el} + H^{ph} + V_{ph} = H_0 + V_{el} + V_{ph}, \tag{2.15}$$

where $H^{el} := h^{el} \otimes I_{\mathfrak{h}^{ph}}$ and $V_{el} := v_{el} \otimes I_{\mathfrak{h}^{ph}}$.

Further we will call $\mathcal{S} = \{H, H_0\}$ the Jaynes–Cummings-leads system, in short, the JCL-model, which we are going to analyze. In particular, we are interested in electron and photon currents for this system. The analysis will be based on the abstract Landauer–Büttiker formula, cf. [1, 6]. We note that the Hamiltonian H is self-adjoint and bounded from below. Moreover, $\mathrm{dom}(H) = \mathrm{dom}(H_0)$, cf. Lemma 2.3 of [9].

2.3. Time reversible symmetric systems

A system described by the Hamiltonian H is called time reversible symmetric if there is a conjugation Γ defined on $\mathfrak{H}$ such that $\Gamma H = H\Gamma$. Recall that Γ is a conjugation if the conditions $\Gamma^2 = I$ and $(\Gamma f, \Gamma g) = \overline{(f, g)}$, $f, g \in \mathfrak{H}$ are satisfied.

Let $\mathfrak{h}^{ph}_n$, $n \in \mathbb{N}_0$, be the subspace spanned by the eigenvector Υ_n in $\mathfrak{h}^{ph}$. We set

$$\mathfrak{H}_{n_\alpha} := \mathfrak{h}^{el}_\alpha \otimes \mathfrak{h}^{ph}_n, \quad n \in \mathbb{N}_0, \quad \alpha \in \{l, r\}. \tag{2.16}$$

Notice that

$$\mathfrak{H} = \bigoplus_{n\in\mathbb{N}_0,\alpha\in\{l,r\}} \mathfrak{H}_{n_\alpha}.$$

Definition 2.3. *The JCL-model is called time reversible symmetric if there is a conjugation* Γ *acting on* $\mathfrak{H}$ *such that* H *and* H_0 *are time reversible symmetric and the subspaces* $\mathfrak{H}_{n_\alpha}$, $n \in \mathbb{N}_0$, $\alpha \in \{l, r\}$, *reduce* Γ.

E x a m p l e 2.4. Let γ^{el}_α and γ^{el}_S be the conjugations defined by

$$\gamma^{el}_\alpha f_\alpha := \overline{f_\alpha} := \{\overline{f_\alpha(k)}\}_{k\in\mathbb{N}}, \quad f_\alpha \in \mathfrak{h}^{el}_\alpha, \quad \alpha \in \{l, r\},$$

and

$$\gamma^{el}_S f_S = \gamma^{el}_S \begin{pmatrix} f_S(0) \\ f_S(1) \end{pmatrix} := \begin{pmatrix} \overline{f_S(0)} \\ \overline{f_S(1)} \end{pmatrix}.$$

We set $\gamma^{el} := \gamma^{el}_l \oplus \gamma^{el}_S \oplus \gamma^{el}_r$. Further, we set

$$\gamma^{ph}\psi := \overline{\psi} = \{\overline{\psi(n)}\}_{n\in\mathbb{N}_0}, \quad \psi \in \mathfrak{h}^{ph}.$$

We also set $\Gamma := \gamma^{el} \otimes \gamma^{ph}$. One can easily check that Γ is a conjugation on $\mathfrak{H} = \mathfrak{h}^{el} \otimes \mathfrak{h}^{ph}$.

Lemma 2.5. *Let* γ^{el}_α, $\alpha \in \{S, l, r\}$, *and* γ^{ph} *be given by Example 2.4.*

(i) *If the conditions* $\gamma^{el}_S e^S_0 = e^S_0$ *and* $\gamma^{el}_S e^S_1 = e^S_1$ *are satisfied, then* H_0 *is time reversible symmetric with respect to* Γ *and, moreover, the subspaces* $\mathfrak{H}_{n_\alpha}$, $n \in \mathbb{N}_0$, $\alpha \in \{l, r\}$, *reduce* Γ.

(ii) *If in addition the conditions* $\gamma^{el}_S \delta^S_0 = \delta^S_0$ *and* $\gamma^{el}_S \delta^S_1 = \delta^S_1$ *are satisfied, then the JCL-model is time reversible symmetric.*

P r o o f. (i) Obviously we have

$$\gamma^{el}_\alpha h^{el}_\alpha = h^{el}_\alpha \gamma^{el}_\alpha, \quad \alpha \in \{l, r\}, \quad \text{and} \quad \gamma^{ph} h^{ph} = h^{ph} \gamma^{ph}.$$

If $\gamma^{el}_S e^S_0 = e^S_0$ and $\gamma^{el}_S e^S_1 = e^S_1$ are satisfied, then $\gamma^{el}_S h^{el}_S = h^{el}_S \gamma^{el}_S$, which yields $\gamma^{el} h^{el}_0 = h^{el}_0 \gamma^{el}$, and hence $\Gamma H_0 = H_0 \Gamma$. Since $\gamma^{el}\mathfrak{h}^{el}_\alpha = \mathfrak{h}^{el}_\alpha$ and $\gamma^{ph}\mathfrak{h}^{ph} = \mathfrak{h}^{ph}$, one gets $\Gamma\mathfrak{H}_{n_\alpha} = \mathfrak{H}_{n_\alpha}$, which shows that $\mathfrak{H}_{n_\alpha}$ reduces Γ.

(ii) Notice that $\gamma^{el}_\alpha \delta^\alpha_1 = \delta^\alpha_1$, $\alpha \in \{l, r\}$. If in addition the conditions $\gamma^{el}_S \delta^S_0 = \delta^S_0$ and $\gamma^{el}_S \delta^S_1 = \delta^S_1$ are satisfied, then $\gamma^{el} v_{el} = v_{el} \gamma^{el}$ is valid, which yields $\gamma^{el} h^{el} = h^{el} \gamma^{el}$. Therefore, $\Gamma H = H\Gamma$. Together with (i) this proves that the JCL-model is time reversible symmetric. ∎

Choosing

$$e_0^S := \begin{pmatrix}1\\0\end{pmatrix}, \quad e_1^S := \begin{pmatrix}0\\1\end{pmatrix}, \quad \delta_0^S := \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \quad \delta_1^S := \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix},$$

one satisfies the conditions $\gamma_S^{el} e_0^S = e_0^S$ and $\gamma_S^{el} e_1^S = e_1^S$ as well as $\gamma_S^{el}\delta_0^S = \delta_0^S$ and $\gamma_S^{el}\delta_1^S = \delta_1^S$.

2.4. Mirror symmetric systems

A unitary operator U acting on $\mathfrak{H}$ is called a mirror symmetry if the conditions

$$U\mathfrak{H}_{n_\alpha} = \mathfrak{H}_{n_{\alpha'}}, \quad \alpha, \alpha' \in \{l, r\}, \quad \alpha \neq \alpha',$$

are satisfied. In particular, this yields $U\mathfrak{H}^{JC} = \mathfrak{H}^{JC}$, $\mathfrak{H}^{JC} := \mathfrak{h}_S^{el} \otimes \mathfrak{h}^{ph}$.

Definition 2.6. The JCL-model is called mirror symmetric if there is a mirror symmetry commuting with H_0 and H.

One easily verifies that if H_0 is mirror symmetric, then

$$H_{n_{\alpha'}} U = U H_{n_\alpha}, \qquad n \in \mathbb{N}_0, \quad \alpha, \alpha' \in \{l, r\}, \quad \alpha \neq \alpha',$$

where

$$H_{n_\alpha} := h_\alpha^{el} \otimes I_{\mathfrak{h}_n^{ph}} + I_{\mathfrak{h}_\alpha^{el}} \otimes h_n^{ph} = h_\alpha^{el} + n\omega, \quad n \in \mathbb{N}_0, \quad \alpha, \alpha' \in \{l, r\}, \quad \alpha \neq \alpha'.$$

In particular, this yields that $v_\alpha = v_{\alpha'}$. Moreover, one gets $UH_S = H_S U$, where $H_S := h_S^{el} \otimes I_{\mathfrak{h}^{ph}} + I_{\mathfrak{h}^{el}} \otimes h^{ph}$.

Notice that if H and H_0 commute with the same mirror symmetry U, then the operator $H_c := h^{el} \otimes I_{\mathfrak{h}^{ph}} + I_{\mathfrak{h}^{el}} \otimes h^{ph}$ also commutes with U, i.e., is mirror symmetric.

E x a m p l e 2.7. Let $\mathcal{S} = \{H, H_0\}$ be the JCL-model. Let $v_l = v_r$ and let e_0^S and e_1^S as well as δ_0^S and δ_1^S be given by (2.3.). We set

$$u_S^{el} e_0^S := e_0^S \quad \text{and} \quad u_S^{el} e_1^S = -e_1^S \tag{2.17}$$

as well as $u^{ph}\Upsilon_n = e^{-in\pi}\Upsilon_n, \quad n \in \mathbb{N}_0$. Obviously, $U_S := u_S^{el} \otimes u^{ph}$ defines a unitary operator on $\mathfrak{H}^{JC}$. A straightforward computation shows that $U_S H_S = H_S U_S$ and $U_S V_{ph} = V_{ph} U_S$. Furthermore, we set $u_{rl}^{el}\delta_n^l := \delta_n^r$ and $u_{lr}^{el}\delta_n^r = \delta_n^l$, $n \in \mathbb{N}$, and

$$u^{el} := \begin{pmatrix} 0 & 0 & u_{lr}^{el} \\ 0 & u_S^{el} & 0 \\ u_{rl}^{el} & 0 & 0 \end{pmatrix}.$$

We have

$$v_{el}\,u^{el}\begin{pmatrix} f_l \\ f_S \\ f_r \end{pmatrix} = \begin{pmatrix} < f_S, (u_S^{el})^* \delta_0^S > \delta_1^l \\ < f_r, (u_{lr}^{el})^* \delta_1^l > \delta_0^S + < f_l, (u_{rl}^{el})^* \delta_1^r > \delta_1^S \\ < f_S, (u_S^{el})^* \delta_1^S > \delta_1^r \end{pmatrix}. \tag{2.18}$$

Since $\delta_0^S := \frac{1}{\sqrt{2}}(e_0^S + e_1^S)$ and $\delta_1^S := \frac{1}{\sqrt{2}}(e_0^S - e_1^S)$, from (2.17) we get

$$(u_S^{el})^* \delta_0^S = \delta_1^S \quad \text{and} \quad (u_S^{el})^* \delta_1^S = \delta_0^S. \tag{2.19}$$

Obviously we have

$$(u_{lr}^{el})^* \delta_1^l = \delta_1^r \qquad (u_{rl}^{el})^* \delta_1^r = \delta_1^l. \tag{2.20}$$

Inserting (2.19) and (2.20) into (2.18), we find

$$v_{el}\,u^{el}\begin{pmatrix} f_l \\ f_S \\ f_r \end{pmatrix} = \begin{pmatrix} < f_S, \delta_1^S > \delta_1^l \\ < f_r, \delta_1^r > \delta_0^S + < f_l, \delta_1^l > \delta_1^S \\ < f_S, \delta_0^S > \delta_1^r \end{pmatrix}. \tag{2.21}$$

Further, we have

$$u^{el}v_{el}\begin{pmatrix} f_l \\ f_S \\ f_r \end{pmatrix} = \begin{pmatrix} < f_S, \delta_1^S > \delta_1^l \\ < f_l, \delta_1^l > \delta_1^S + < f_r, \delta_1^r > \delta_0^S \\ < f_s, \delta_0^S > \delta_1^r \end{pmatrix}. \tag{2.22}$$

Comparing (2.21) and (2.22), we get $u^{el}v_{el} = v_{el}u^{el}$. Setting $U := u^{el} \otimes u^{ph}$, one immediately proves that $UH_0 = H_0U$ and $UH = HU$. Since $U\mathfrak{H}_{n_\alpha} = \mathfrak{H}_{n_{g\alpha'}}$ is satisfied, $\mathcal{S}$ is mirror symmetric.

Notice that in addition Example 2.7 $\mathcal{S}$ is time reversible symmetric.

2.5. Spectral properties of H and spectral representation

In the following, our goal is to apply the Landauer–Büttiker formula to the JCL-model. By $\mathfrak{L}_p(\mathfrak{H})$, $1 \le p \le \infty$, we denote below the Schatten–von Neumann ideals.

Proposition 2.8. ([9, Proposition 2.9]) *If $\mathcal{S} = \{H, H_0\}$ is the JCL-model, then $(H+i)^{-1} - (H_0+i)^{-1} \in \mathfrak{L}_1(\mathfrak{H})$. In particular, the absolutely continuous parts H^{ac} and H_0^{ac} are unitarily equivalent.*

Thus, the JCL-model $\mathcal{S} = \{H, H_0\}$ is a $\mathfrak{L}_1$-scattering system. Let us recall that $h_\alpha^{el} = -\Delta^D + v_\alpha$, $\alpha \in \{l, r\}$, on $\mathfrak{h}_l^{el} = \mathfrak{h}_r^{el} = \ell^2(\mathbb{N})$.

Lemma 2.9. ([9, Lemma 2.10]) *Let $\alpha \in \{l, r\}$. We have $\sigma(h_\alpha^{el}) = \sigma_{ac}(h_\alpha^{el}) = [v_\alpha, 4 + v_\alpha]$. The normalized generalized eigenfunctions of h_α^{el} are given by*

$$g_\alpha(x, \lambda) = \pi^{-\frac{1}{2}}(1 - (-\lambda + 2 + v_\alpha)^2/4)^{-\frac{1}{4}} \sin\big(\arccos((-\lambda + 2 + v_\alpha)/2)x\big)$$

for $x \in \mathbb{N}$, $\lambda \in (v_\alpha, 4 + v_\alpha)$.

From these two lemmas we obtain the following corollary that gives us the spectral properties of H_0.

Proposition 2.10. ([9, Proposition 2.11]) *Let $\mathcal{S} = \{H, H_0\}$ be the JCL-model. Then $\sigma(H_0) = \sigma_{ac}(H_0) \cup \sigma_{pp}(H_0)$, where*

$$\sigma_{ac}(H_0) = \bigcup_{n\in\mathbb{N}_0} [v_l + n\omega, v_l + 4 + n\omega] \cup [v_r + n\omega, v_r + 4 + n\omega]$$

and $\sigma_{pp}(H_0) = \bigcup_{n\in\mathbb{N}_0}\{\lambda_j^S + n\omega : j = 0, 1\}$. The eigenvectors are given by $\widetilde{g}(m, n) = e_m^S \otimes \Upsilon_n$, $m = 0, 1$, $n \in \mathbb{N}_0$. The generalized eigenfunctions are given by $\widetilde{g}_\alpha(\cdot, \lambda, n) = g_\alpha(\cdot, \lambda - n\omega) \otimes \Upsilon_n$ for $\lambda \in \sigma_{ac}(H_0)$, $n \in \mathbb{N}_0$, $\alpha \in \{l, r\}$.

For the convenience of the reader, we define here what we mean under a spectral representation of the absolutely continuous part K_0^{ac} of a self-adjoint operator K_0 on a separable Hilbert space $\mathfrak{k}$. Let $\mathfrak{k}$ be an auxiliary separable Hilbert space. We consider the Hilbert space $L^2(\mathbb{R}, d\lambda, \mathfrak{k})$. By $\mathcal{M}$, we define the multiplication operator induced by the independent variable λ in $L^2(\mathbb{R}, d\lambda, \mathfrak{k})$. Let $\Phi : \mathfrak{K}^{ac}(K_0) \longrightarrow L^2(\mathbb{R}, d\lambda, \mathfrak{k})$ be an isometry acting from $\mathfrak{K}^{ac}(K_0)$ into $L^2(\mathbb{R}, d\lambda, \mathfrak{k})$ such that $\Phi\mathrm{dom}(K_0^{ac}) \subseteq \mathrm{dom}(\mathcal{M})$ and

$$\mathcal{M}\Phi f = \Phi K_0^{ac} f, \quad f \in \mathrm{dom}(K_0^{ac}).$$

Obviously, the orthogonal projection $P := \Phi\Phi^*$ commutes with $\mathcal{M}$, which yields the existence of a measurable family $\{P(\lambda)\}_{\lambda\in\mathbb{R}}$ such that $(P\,\widehat{f}\,)(\lambda) = P(\lambda)\,\widehat{f}\,(\lambda)$, $\widehat{f} \in L^2(\mathbb{R}, \lambda, \mathfrak{k})$. We set $L^2(\mathbb{R}, d\lambda, \mathfrak{k}(\lambda)) := PL^2(\mathbb{R}, \lambda, \mathfrak{k})$, $\mathfrak{k}(\lambda) := P(\lambda)\mathfrak{k}$. $\Pi(K_0^{ac}) := \{L^2(\mathbb{R}, d\lambda, \mathfrak{k}(\lambda)), \mathcal{M}, \Phi\}$ is called a spectral representation of K_0^{ac}. If $\{L^2(\mathbb{R}, d\lambda, \mathfrak{k}(\lambda)), \mathcal{M}, \Phi\}$ is a spectral representation of K^{ac}, then K^{ac} is unitarily equivalent $\mathcal{M}_0 := \mathcal{M} \restriction L^2(\mathbb{R}, d\lambda, \mathfrak{k}(\lambda))$. Indeed, one has $\Phi K_0^{ac}\Phi^* = \mathcal{M}_0$. The function $\xi_{K_0}^{ac}(\lambda) := \dim(\mathfrak{k}(\lambda))$, $\lambda \in \mathbb{R}$, is called the spectral multiplicity function of K_0^{ac}. Notice that $0 \le \xi_{K_0}^{ac}(\lambda) \le \infty$ for $\lambda \in \mathbb{R}$.

For $\alpha \in \{l, r\}$, the generalized eigenfunctions of h_α^{el} define the generalized Fourier transforms by $\phi_\alpha^{el} : \mathfrak{h}_\alpha^{el} = \mathfrak{h}_\alpha^{el,ac}(h_\alpha^{el}) \to L^2([v_\alpha, v_\alpha + 4])$ and

$$(\phi_\alpha^{el} f_\alpha)(\lambda) = \sum_{x\in\mathbb{N}_0} g_\alpha(x, \lambda) f_\alpha(x), \quad f_\alpha \in \mathfrak{h}_\alpha^{el}.$$

Setting

$$\mathfrak{h}^{el}_\alpha(\lambda) := \begin{cases} \mathbb{C} & \lambda \in [v_\alpha, v_\alpha + 4] \\ 0 & \lambda \in \mathbb{R} \setminus [v_\alpha, v_\alpha + 4], \end{cases}$$

one easily verifies that $\Pi(h^{el}_\alpha) = \{L^2(\mathbb{R}, d\lambda, \mathfrak{h}^{el}_\alpha(\lambda)), \mathcal{M}, \phi^{el}_\alpha\}$ is a spectral representation of $h^{el}_\alpha = h^{el,ac}_\alpha$, $\alpha = l, r$, where we always assumed implicitly that $(\phi^{el}_\alpha f_\alpha)(\lambda) = 0$ for $\lambda \in \mathbb{R} \setminus [v_\alpha, v_\alpha + 4]$. Setting

$$\mathfrak{h}^{el}(\lambda) := \begin{matrix} \mathfrak{h}^{el}_l(\lambda) \\ \oplus \\ \mathfrak{h}^{el}_r(\lambda) \end{matrix} \subseteq \mathbb{C}^2, \quad \lambda \in \mathbb{R},$$

and introducing the map

$$\phi^{el} : \mathfrak{h}^{el,ac}(h^{el}_0) = \begin{matrix} \mathfrak{h}^{el}_l \\ \oplus \\ \mathfrak{h}^{el}_r \end{matrix} \longrightarrow L^2(\mathbb{R}, d\lambda, \mathfrak{h}^{el}(\lambda))$$

defined by

$$\phi^{el} f := \begin{pmatrix} \phi^{el}_l f_l \\ \phi^{el}_r f_r \end{pmatrix}, \quad \text{where} \quad f := \begin{pmatrix} f_l \\ f_r \end{pmatrix},$$

we obtain a spectral representation $\Pi(h^{el,ac}_0) = \{L^2(\mathbb{R}, d\lambda, \mathfrak{h}^{el}(\lambda)), \mathcal{M}, \phi^{el}\}$ of the absolutely continuous part $h^{el,ac}_0 = h^{el}_l \oplus h^{el}_r$ of h^{el}_0. One easily verifies that $0 \le \xi^{ac}_{h^{el}_0}(\lambda) \le 2$ for $\lambda \in \mathbb{R}$. Introducing

$$\lambda^{el}_{\min} := \min\{v_l, v_r\} \quad \text{and} \quad \lambda^{el}_{\max} := \max\{v_l + 4, v_r + 4\}, \tag{2.23}$$

one easily verifies that $\xi^{ac}_{h^{el}_0}(\lambda) = 0$ for $\lambda \in \mathbb{R} \setminus [\lambda^{el}_{\min}, \lambda^{el}_{\max}]$. Notice, if $v_r + 4 \le v_l$, then

$$\mathfrak{h}^{el}(\lambda) = \begin{cases} \mathbb{C}, & \lambda \in [v_r, v_r + 4] \cup [v_l, v_l + 4], \\ \{0\}, & \text{otherwise} \end{cases}$$

which shows that h^{el}_0 has a simple spectrum. In particular, it holds $\xi^{ac}_{h^{el}_0}(\lambda) = 1$ for $\lambda \in [v_r, v_r + 4] \cup [v_l, v_l + 4]$ and otherwise $\xi^{ac}_{h^{el}_0}(\lambda) = 0$.

Let us introduce the Hilbert space $\mathfrak{h} := l^2(\mathbb{N}_0, \mathbb{C}^2) = \bigoplus_{n\in\mathbb{N}_0} \mathfrak{h}_n$, $\mathfrak{h}_n := \mathbb{C}^2$, $n \in \mathbb{N}_0$. Regarding $\mathfrak{h}^{el}(\lambda - n\omega)$ as a subspace of $\mathfrak{h}_n$, one regards

$$\mathfrak{h}(\lambda) := \bigoplus_{n\in\mathbb{N}_0} \mathfrak{h}_n(\lambda), \quad \mathfrak{h}_n(\lambda) := \mathfrak{h}^{el}(\lambda - n\omega), \quad \lambda \in \mathbb{R}, \tag{2.24}$$

as a measurable family of subspaces in $\mathfrak{h}$. Notice that $0 \le \dim(\mathfrak{h}(\lambda)) < \infty$, $\lambda \in \mathbb{R}$. We consider the Hilbert space $L^2(\mathbb{R}, d\lambda, \mathfrak{h}(\lambda))$.

Furthermore, we introduce the isometric map $\Phi : \mathfrak{H}(H_0^{ac}) \longrightarrow L^2(\mathbb{R}, d\lambda, \mathfrak{h}(\lambda))$ defined by

$$(\Phi f)(\lambda) = \bigoplus_{n\in\mathbb{N}_0} \begin{pmatrix} (\phi_l^{el} f_l(n))(\lambda - n\omega) \\ (\phi_r^{el} f_r(n))(\lambda - n\omega) \end{pmatrix}, \quad \lambda \in \mathbb{R},$$

where

$$\bigoplus_{n\in\mathbb{N}_0} \begin{pmatrix} f_l(n) \\ f_r(n) \end{pmatrix} \in \bigoplus_{n\in\mathbb{N}_0} \mathfrak{h}^{el,ac}(h_0^{el}) \otimes \mathfrak{h}_n^{ph} = \bigoplus_{n\in\mathbb{N}} \begin{pmatrix} \mathfrak{h}_l^{el} \otimes h_n^{ph} \\ \oplus \\ \mathfrak{h}_r^{el} \otimes h_n^{ph} \end{pmatrix},$$

where $\mathfrak{h}_{ph} = \bigoplus_{n\in\mathbb{N}_0} \mathfrak{h}_n^{ph}$, and $\mathfrak{h}_n^{ph}$ is the subspace spanned by the eigenvectors Υ_n of h^{ph}. One easily verifies that Φ is an isometry acting from $\mathfrak{H}^{ac}(H_0^{ac})$ onto $L^2(\mathbb{R}, d\lambda, \mathfrak{h}(\lambda))$.

Lemma 2.11. ([9, Lemma 2.12]) *The triplet* $\{L^2(\mathbb{R}, d\lambda, \mathfrak{h}(\lambda)), \mathcal{M}, \Phi\}$ *forms a spectral representation of* H_0^{ac}*, that is,* $\Pi(H_0^{ac}) = \{L^2(\mathbb{R}, d\lambda, \mathfrak{h}(\lambda)), \mathcal{M}, \Phi\}$ *such that* $0 \le \xi_{H_0}^{ac}(\lambda) \le 2d_{\max}$ *for* $\lambda \in \mathbb{R}$ *where* $d_{\max} := \frac{\lambda_{\max}^{el} - \lambda_{\min}^{el}}{\omega}$ *and* $\lambda_{\max}^{el}$ *and* $\lambda_{\min}^{el}$ *are given by* (2.23)*.*

In the following, we denote the orthogonal projection from $\mathfrak{h}(\lambda)$ onto $\mathfrak{h}_n(\lambda)$ by $P_n(\lambda)$, $\lambda \in \mathbb{R}$, cf (2.24). Since $\mathfrak{h}(\lambda) = \bigoplus_{n\in\mathbb{N}_0} \mathfrak{h}_n(\lambda)$, we have $I_{\mathfrak{h}(\lambda)} = \sum_{n\in\mathbb{N}_0} P_n(\lambda)$, $\lambda \in \mathbb{R}$. Further, we introduce the subspaces $\mathfrak{h}_{n_\alpha}(\lambda) := \mathfrak{h}_\alpha^{el}(\lambda - n\omega)$, $\lambda \in \mathbb{R}$, $n \in \mathbb{N}_0$. Notice that

$$\mathfrak{h}_n(\lambda) = \bigoplus_{\alpha\in\{l,r\}} \mathfrak{h}_{n_\alpha}(\lambda), \quad \lambda \in \mathbb{R}, \quad n \in \mathbb{N}_0.$$

By $P_{n_\alpha}(\lambda)$, we denote the orthogonal projection from $\mathfrak{h}(\lambda)$ onto $\mathfrak{h}_{n_\alpha}(\lambda)$, $\lambda \in \mathbb{R}$. Obviously we have $P_n(\lambda) = \sum_{\alpha\in\{l,r\}} P_{n_\alpha}(\lambda)$, $\lambda \in \mathbb{R}$.

Since we have full information on the spectral properties of H_0, we can use it to show that H has no singular continuous spectrum.

Proposition 2.12. ([9, Proposition 2.16]) *The Hamiltonian* H *defined by* (2.15) *has no singular continuous spectrum, that is,* $\sigma_{sc}(H) = \emptyset$*.*

Let Z be a bounded operator acting on $\mathfrak{H}^{ac}(H_0)$ and commuting with H_0^{ac}. Since Z commutes with H_0^{ac}, there is a measurable family $\{Z(\lambda)\}_{\lambda\in\mathbb{R}}$ of bounded operators acting on $\mathfrak{h}(\lambda)$ such that Z is unitarily equivalent to the multiplication operator induced by $\{Z(\lambda)\}_{\lambda\in\mathbb{R}}$ in $\Pi(H_0^{ac})$. We set

$$Z_{m_\alpha n_\varkappa}(\lambda) := P_{m_\alpha}(\lambda) Z(\lambda) \restriction \mathfrak{h}_{n_\varkappa}(\lambda), \quad \lambda \in \mathbb{R}, \qquad m, n \in \mathbb{N}_0, \quad \alpha, \varkappa \in \{l, r\}.$$

Let $Z_{m_\alpha n_\varkappa} := P_{m_\alpha} Z P_{n_\varkappa}$, where P_{m_α} is the orthogonal projection from $\mathfrak{H}$ onto the subspace $\mathfrak{H}_{m_\alpha} \subseteq \mathfrak{H}^{ac}(H_0)$, cf. (2.16). Therefore, the multiplication operator induced by $\{Z_{m_\alpha n_\varkappa}(\lambda)\}_{\lambda\in\mathbb{R}}$ in $\Pi(H_0^{ac})$ is unitarily equivalent to $Z_{m_\alpha n_\varkappa}$.

Since by Lemma 2.11 $\mathfrak{h}(\lambda)$ is a finite dimensional space, the operators $Z(\lambda)$ are finite dimensional ones and we can introduce the quantity

$$\sigma_{m_\alpha n_\varkappa}(\lambda) = \mathrm{tr}(Z_{m_\alpha n_\varkappa}(\lambda)^* Z_{m_\alpha n_\varkappa}(\lambda)), \quad \lambda \in \mathbb{R}, \qquad m, n \in \mathbb{N}_0, \quad \alpha, \varkappa \in \{l, r\}.$$

Lemma 2.13. ([9, Lemma 2.14]) *Let H_0 be the self-adjoint operator defined by* (2.14) *on $\mathfrak{H}$. Further, let Z be a bounded operator on $\mathfrak{H}^{ac}(H_0)$ commuting with H_0^{ac}*

(i) *Let Γ be a conjugation on $\mathfrak{H}$, cf. Sec. 2.3. If Γ commutes with H_0 and P_{n_α}, $n \in \mathbb{N}_0$, $\alpha \in \{l, r\}$, and $\Gamma Z \Gamma = Z^*$ holds, then $\sigma_{m_\alpha n_\varkappa}(\lambda) = \sigma_{n_\varkappa m_\alpha}(\lambda)$, $\lambda \in \mathbb{R}$.*

(ii) *Let U be a mirror symmetry on $\mathfrak{H}$. If U commutes with H_0 and Z, then $\sigma_{m_\alpha n_\varkappa}(\lambda) = \sigma_{m_{\alpha'} n_{\varkappa'}}(\lambda)$, $\lambda \in \mathbb{R}$, $m, n \in \mathbb{N}_0$, $\alpha, \alpha', \varkappa, \varkappa' \in \{l, r\}$, $\alpha \neq \alpha'$, $\varkappa \neq \varkappa'$.*

3. Landauer–Büttiker Formula and Applications

3.1. Landauer–Büttiker formula

The abstract Landauer–Büttiker formula can be used to calculate flows in devices. Usually one considers a pair $\mathcal{S} = \{K, K_0\}$ to be a self-adjoint operator where the unperturbed Hamiltonian K_0 describes a totally decoupled system, which means that the inner system is closed and the leads are decoupled from it, while the perturbed Hamiltonian K describes the system where the leads are coupled to the inner system. An important ingredient is the system $\mathcal{S} = \{K, K_0\}$, which is a complete scattering or even a trace class scattering system.

In [1], an abstract Landauer–Büttiker formula was derived in the framework of a trace class scattering theory for semi-bounded self-adjoint operators which allows us to reproduce the results of [8] and [4] rigorously. In [6], the results of [1] were generalized to non-semi-bounded operators. Following [1], we consider a trace class scattering system $\mathcal{S} = \{K, K_0\}$. We recall that $\mathcal{S} = \{K, K_0\}$ is called a trace class scattering system if the resolvent difference of K and K_0 belongs to the trace class. If $\mathcal{S} = \{K, K_0\}$ is a trace class scattering system, then the wave operators $W_\pm(K, K_0)$ exist and are complete. The scattering operator is defined by $S(K, K_0) := W_+(K, K_0)^* W_-(K, K_0)$. The main ingredients besides the trace class scattering system $\mathcal{S} = \{K, K_0\}$ are the density and charge operators ρ and Q, respectively.

Let K_0 be a self-adjoint operator on the separable Hilbert space $\mathfrak{k}$. We call ρ a density operator for K_0 if ρ is a bounded non-negative self-adjoint operator commuting with K_0. Since ρ commutes with K_0, one gets that ρ leaves invariant the subspace $\mathfrak{k}^{ac}(K_0)$. We set $\rho_{ac} := \rho \restriction \mathfrak{K}^{ac}(K_0)$, call ρ_{ac} the *ac*-density part of ρ.

A bounded self-adjoint operator Q commuting with K_0 is called a charge. If Q is a charge, then $Q_{ac} := Q \restriction \mathfrak{K}^{ac}(K_0)$ is called its *ac*-charge part.

Let $\Pi(K_0^{ac}) = \{L^2(\mathbb{R}, d\lambda, \mathfrak{k}(\lambda)), \mathcal{M}, \Phi\}$ be a spectral representation of K_0^{ac}. If ρ is a density operator, then there is a measurable family $\{\rho_{ac}(\lambda)\}_{\lambda \in \mathbb{R}}$ of bounded

self-adjoint operators such that the multiplication operator

$$(\mathcal{M}_{\rho_{ac}}\widehat{f})(\lambda) := \rho_{ac}(\lambda)\,\widehat{f}\,(\lambda), \quad \widehat{f} \in \mathrm{dom}(M_{\rho_{ac}}) := L^2(\mathbb{R}, d\lambda, \mathfrak{k}(\lambda)),$$

is unitarily equivalent to the *ac*-part ρ_{ac}, that is, $\mathcal{M}_{\rho_{ac}} = \Phi\rho_{ac}\Phi^*$. In particular, this yields that ess-sup $_{\lambda\in\mathbb{R}}\|\rho_{ac}(\lambda)\|_{\mathcal{B}(\mathfrak{k}(\lambda)} = \|\rho_{ac}\|_{\mathcal{B}(\mathfrak{K}^{ac}(K_0))}$. In the following, we call $\{\rho_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ the density matrix of ρ_{ac}.

Similarly, one gets that if Q is a charge, then there is a measurable family $\{Q_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ of bounded self-adjoint operators such that the multiplication operator

$$\begin{aligned}(\mathcal{M}_{Q_{ac}}\widehat{f})(\lambda) &:= Q_{ac}(\lambda)\,\widehat{f}\,(\lambda),\\ \widehat{f} \in \mathrm{dom}(Q_{ac}) &:= \{f \in L^2(\mathbb{R}, d\lambda, \mathfrak{k}(\lambda)) : Q_{ac}(\lambda)\,\widehat{f}\,(\lambda) \in L^2(\mathbb{R}, d\lambda, \mathfrak{k}(\lambda))\},\end{aligned}$$

is unitarily equivalent to Q_{ac}, i.e., $\mathcal{M}_{Q_{ac}} = \Phi Q_{ac}\Phi^*$. In particular, ess-sup $_{\lambda\in\mathbb{R}}\|Q_{ac}(\lambda)\| = \|Q_{ac}\|$. If Q is a charge, then the family $\{Q_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ is called the charge matrix of the *ac*-part of Q.

Let $\mathcal{S} = \{K, K_0\}$ be a trace scattering system. By $\{S(\lambda)\}_{\lambda\in\mathbb{R}}$, we denote the scattering matrix which corresponds to the scattering operator $S(K, K_0)$ with respect to the spectral representation $\Pi(K_0^{ac})$. The operator $T := S(K, K_0) - P^{ac}(K_0)$ is called the transmission operator. By $\{T(\lambda)\}_{\lambda\in\mathbb{R}}$, we denote the transmission matrix which is related to the transmission operator. The scattering and transmission matrices are related by $S(\lambda) = I_{\mathfrak{k}(\lambda)} + T(\lambda)$ for a.e. $\lambda \in \mathbb{R}$. Notice that $T(\lambda)$ belongs to the trace class a.e. $\lambda \in \mathbb{R}$.

Theorem 3.1. ([6, Theorem 3.1]) *Let $\mathcal{S} := \{K, K_0\}$ be a trace class scattering system and let $\{S(\lambda)\}_{\lambda\in\mathbb{R}}$ be the scattering matrix of $\mathcal{S}$ with respect to the spectral representation $\Pi(K_0^{ac})$. Further, let ρ and Q be the density and the charge operators and let $\{\rho_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ and $\{Q_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ be the density and the charge matrices with respect to $\Pi(K_0^{ac})$ of the ac-parts ρ_{ac} and Q_{ac}, respectively. If $(I + K_0^2)\rho$ is bounded, then the current $J^{\mathcal{S}}_{\rho,Q}$ admits the representation*

$$J^{\mathcal{S}}_{\rho,Q} = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{tr}\big(\rho_{ac}(\lambda)(Q_{ac}(\lambda) - S^*(\lambda)Q_{ac}(\lambda)S(\lambda))\big)d\lambda, \tag{3.1}$$

where the integrand on the right-hand side and the current $J^{\mathcal{S}}_{\rho,Q}$ satisfy the following estimates:

$$|\mathrm{tr}\,(\rho_{ac}(\lambda)(Q_{ac}(\lambda) - S^*(\lambda)Q_{ac}(\lambda)S(\lambda)))| \le 4\|\rho_{ac}(\lambda)\|\|T(\lambda)\|_{\mathfrak{L}_1(\mathfrak{k}(\lambda))}\|Q_{ac}(\lambda)\| \tag{3.2}$$

for a.e. $\lambda \in \mathbb{R}$ and

$$|J^{\mathcal{S}}_{\rho,Q}| \le C_0\|(H + i)^{-1} - (H_0 + i)^{-1}\|_{\mathfrak{L}_1(\mathfrak{K})}, \tag{3.3}$$

where $C_0 := \frac{2}{\pi}\|(1 + H_0^2)\rho\|_{\mathfrak{L}(\mathfrak{K})}||Q||$.

In applications not every charge Q is a bounded operator. We say the self-adjoint operator Q commuting with K_0 is a p-tempered charge if $Q(H_0-i)^{-p}$ is a bounded operator for $p \in \mathbb{N}_0$. As above, we can introduce $Q_{ac} := Q \restriction \mathrm{dom}(Q) \cap \mathfrak{K}^{ac}(K_0)$. It turns out that $QE_{K_0}(\Delta)$ is a bounded operator for any bounded Borel set Δ. This yields that the corresponding charge matrix $\{Q_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ is a measurable family of bounded self-adjoint operators such that the condition $\text{ess-sup}_{\lambda\in\mathbb{R}}(1+\lambda^2)^{-p/2}\|Q_{ac}(\lambda)\| < \infty$ is satisfied. It turns out that formula (3.1) remains valid for p-tempered charges.

Corollary 3.2. ([9, Corollary 3.2]) *Let the assumptions of Theorem 3.1 be satisfied. If for some $p \in \mathbb{N}_0$ the operator $(H_0 - i)^{p+2}\rho$ is bounded and Q is a p-tempered charge for K_0, then representation* (3.1) *and estimate* (3.2) *remain valid. Moreover, estimate* (3.3) *holds with C_0 replaced by $C_p := \frac{2}{\pi}\|(1 + H_0^2)^{p+2/2}\rho\| \, \|Q(I + H_0^2)^{-p/2}\|$.*

At first glance, formula (3.1) is not very similar to the original Landauer–Büttiker formula of [4, 8]. To make the formula more convenient, we recall that a standard application example for the Landauer–Büttiker formula is the so-called black-box model, cf. [1]. In this case, the Hilbert space $\mathfrak{K}$ is given by $\mathfrak{K} = \mathfrak{K}_S \oplus \bigoplus_{j=1}^N \mathfrak{K}_j$, $2 \le N < \infty$, and K_0 by $K_0 = K_S \oplus \bigoplus_{j=1}^N K_j$, $2 \le N < \infty$. The Hilbert space $\mathfrak{K}_S$ is called the sample or dot and K_S is the sample or dot Hamiltonian. The Hilbert spaces $\mathfrak{K}_j$ are called reservoirs or leads and K_j are the reservoir or lead Hamiltonians. For simplicity, we assume that the reservoir Hamiltonians K_j are absolutely continuous and the sample Hamiltonian K_S has a point spectrum. A typical choice for the density operator is

$$\rho = f_S(K_S) \oplus \bigoplus_{j=1}^N f_j(K_j),$$

where $f_S(\cdot)$ and $f_j(\cdot)$ are non-negative bounded Borel functions, and for the charge,

$$Q = g_S(H_s) \oplus \bigoplus_{j=1}^N g_j(H_j),$$

where $g_S(\cdot)$ and $g_j(\cdot)$ a bounded Borel functions. Making this choice, the Landauer–Büttiker formula (3.1) takes the form

$$J^{\mathcal{S}}_{\rho,Q} = \frac{1}{2\pi} \sum_{j,k=1}^N \int_{\mathbb{R}} (f_j(\lambda) - f_k(\lambda)) g_j(\lambda) \sigma_{jk}(\lambda) d\lambda, \tag{3.4}$$

where $\sigma_{jk}(\lambda) := \mathrm{tr}(T_{jk}(\lambda)^* T_{jk}(\lambda))$, $j,k = 1,\dots,N$, $\lambda \in \mathbb{R}$, are called the *total transmission probability* from the reservoir k to the reservoir j, cf. [1]. We call it

the *cross-section* of the scattering process going from the channel k to the channel j at energy $\lambda \in \mathbb{R}$. $\{T_{jk}(\lambda)\}_{\lambda\in\mathbb{R}}$ is called the transmission matrix from the channel k to the channel j at energy $\lambda \in \mathbb{R}$ with respect to the spectral representation $\Pi(K_0^{ac})$. We note that $\{T_{jk}(\lambda)\}_{\lambda\in\mathbb{R}}$ corresponds to the transmission operator

$$T_{jk} := P_j T(K, K_0) P_k, \quad T(K, K_0) := S(K, K_0) - P^{ac}(K_0),$$

acting from the reservoir k to the reservoir j, where $T(K, K_0)$ is called the transmission operator. Let $\{T(\lambda)\}_{\lambda\in\mathbb{R}}$ be the transmission matrix. Following [1], the current $J^{\mathcal{S}}_{\rho,Q}$ given by (3.4) is directed from the reservoirs into the sample.

The quantity $\|T(\lambda)\|_{\mathfrak{L}_2} = \mathrm{tr}(T(\lambda)^* T(\lambda))$ is well-defined and is called the cross-section of the scattering system $\mathcal{S}$ at energy $\lambda \in \mathbb{R}$. Notice that

$$\sigma(\lambda) = \|T(\lambda)\|_{\mathfrak{L}_2} = \mathrm{tr}(T(\lambda)^* T(\lambda)) = \sum_{j,k=1}^{N} \sigma_{jk}(\lambda), \quad \lambda \in \mathbb{R}.$$

We point out that the channel cross-sections $\sigma_{jk}(\lambda)$ admit the property

$$\sum_{j=1}^{N} \sigma_{jk}(\lambda) = \sum_{j=1}^{N} \sigma_{kj}(\lambda), \quad \lambda \in \mathbb{R}, \tag{3.5}$$

which is a consequence of the unitarity of the scattering matrix. Moreover, if there is a conjugation J such that $KJ = JK$ and $K_0 J = JK_0$ hold, that is, if the scattering system $\mathcal{S}$ is time reversible symmetric, then we have even more, namely, it holds $\sigma_{jk}(\lambda) = \sigma_{kj}(\lambda)$, $\lambda \in \mathbb{R}$.

Usually the Landauer–Büttiker formula (3.4) is used to calculate the electron current entering the reservoir j from the sample. In this case one has to choose $Q := Q_j^{el} := -\mathfrak{e} P_j$, where P_j is the orthogonal projection form $\mathfrak{K}$ onto $\mathfrak{K}_j$ and $\mathfrak{e} > 0$ is the magnitude of the elementary charge. It is equivalent to choose $g_j(\lambda) = -\mathfrak{e}$ and $g_k(\lambda) = 0$ for $k \neq j$, $\lambda \in \mathbb{R}$. Doing so, we get that the Landauer–Büttiker formula simplifies to

$$J^{\mathcal{S}}_{\rho,Q_j^{el}} = -\frac{\mathfrak{e}}{2\pi} \sum_{k=1}^{N} \int_{\mathbb{R}} (f_j(\lambda) - f_k(\lambda)) \sigma_{jk}(\lambda) d\lambda.$$

To restore the original Landauer–Büttiker formula, one sets $f_j(\lambda) = f(\lambda - \mu_j)$, $\lambda \in \mathbb{R}$, where μ_j is the chemical potential of the reservoir $\mathfrak{K}_j$, and $f(\cdot)$ is a bounded non-negative Borel function called the distribution function. This gives to the formula

$$J^{\mathcal{S}}_{\rho,Q_j^{el}} = -\frac{\mathfrak{e}}{2\pi} \sum_{k=1}^{N} \int_{\mathbb{R}} (f(\lambda - \mu_j) - f(\lambda - \mu_k)) \sigma_{jk}(\lambda) d\lambda. \tag{3.6}$$

In particular, one chooses

$$f(\lambda) := f_{FD}(\lambda) := \frac{1}{1+e^{\beta\lambda}}, \quad \beta > 0, \quad \lambda \in \mathbb{R}, \tag{3.7}$$

where $f_{FD}(\cdot)$ is the so-called Fermi–Dirac distribution function. If we have only two reservoirs and $f(\lambda) = f_{FD}(\lambda)$, $\lambda \in \mathbb{R}$, then

$$J^{\mathcal{S}}_{\rho,Q^{el}_l} = -\frac{\mathfrak{e}}{2\pi}\int_{\mathbb{R}} (f_{FD}(\lambda-\mu_l) - f_{FD}(\lambda-\mu_r))\sigma_{lr}(\lambda)d\lambda.$$

One can easily check that $J^{\mathcal{S}}_{\rho,Q_l} \leq 0$ if $\mu_l \geq \mu_r$. It means that current is leaving the left reservoir and is entering the right one, which is in accordance with physical intuition.

E x a m p l e 3.3. Notice that $s_c := \{h^{el}, h^{el}_0\}$ is a $\mathfrak{L}_1$-scattering system. The Hamiltonian h^{el} takes into account the effect of coupling of reservoirs or leads $\mathfrak{h}_l := l^2(\mathbb{N})$ and $\mathfrak{h}_r := l^2(\mathbb{N})$ to the sample $\mathfrak{h}_S = \mathbb{C}^2$, which is also called the quantum dot. The lead Hamiltonians are given by $h^{el}_\alpha = -\Delta^D + v_\alpha$, $\alpha = l, r$. The sample or quantum dot Hamiltonian is given by h^{el}_S. The wave operators are given by

$$w_\pm(h^{el}, h^{el}_0) := s\text{-}\lim_{t\to\infty} e^{ith^{el}} e^{-ith^{el}_0} P^{ac}(h^{el}_0). \tag{3.8}$$

The scattering operator is given by $s_c := w_+(h^{el}, h^{el}_0)^* w_-(h^{el}, h^{el}_0)$. Let $\Pi(h^{el,ac}_0)$ be the spectral representation of $h^{el,ac}_0$ introduced in Sec. 2.5. If ρ^{el} and q^{el} are the density and the charge operators for h^{el}_0, then the Landauer–Büttiker formula takes the form

$$J^{s_c}_{\rho^{el},q^{el}} = \frac{1}{2\pi}\int_{\mathbb{R}} \operatorname{tr}\left(\rho^{el}_{ac}(\lambda)\left(q^{el}_{ac} - s_c(\lambda)^* q^{el}_{ac}(\lambda) s_c(\lambda)\right)\right), \tag{3.9}$$

where $\{s_c(\lambda)\}_{\lambda\in\mathbb{R}}$, $\{q^{el}(\lambda)\}_{\lambda\in\mathbb{R}}$ and $\{\rho^{el}(\lambda)\}_{\lambda\in\mathbb{R}}$ are the scattering, the charge and the density matrices with respect to $\Pi(h^{el,ac}_0)$, respectively. The condition that $((h^{el}_0)^2 + I_{\mathfrak{h}^{el}})\rho^{el}$ is a bounded operator is superfluous because h^{el}_0 is a bounded operator. For the same reason, we have that every p-tempered charge q^{el} is in fact a charge, i.e., q^{el} is a bounded self-adjoint operator.

The scattering system s_c is a black-box model with the reservoirs $\mathfrak{h}^{el}_l$ and $\mathfrak{h}^{el}_r$. Choosing $\rho^{el} = f_l(h^{el}) \oplus f_S(h^{el}_S) \oplus f_r(h^{el}_r)$, where $f_\alpha(\cdot)$, $\alpha = l, r$, are bounded non-negative Borel functions and $q^{el} = g_l(h^{el}_l) \oplus g_S(h^{el}_S) \oplus g_r(h^{el}_r)$, where $g_\alpha(\cdot)$, $\alpha \in \{l, r\}$, are locally bounded Borel functions, then from (3.4) it follows that

$$J^{s_c}_{\rho^{el},q^{el}} = \frac{1}{2\pi}\sum_{\substack{\alpha,\varkappa\in\{l,r\}\\ \alpha\neq\varkappa}} \int_{\mathbb{R}} (f_\alpha(\lambda) - f_\varkappa(\lambda))g_\alpha(\lambda)\sigma_c(\lambda)d\lambda,$$

where $\{\sigma_c(\lambda)\}_{\lambda\in\mathbb{R}}$ is the channel cross-section from left to right and vice versa. Indeed, let $\{t_c(\lambda)\}_{\lambda\in\mathbb{R}}$ be the transition matrix which corresponds to the transition operator $t_c := s_c - I_{\mathfrak{h}^{el}}$. Obviously, one has $t_c(\lambda) = s_c(\lambda) - I_{\mathfrak{h}(\lambda)}$, $\lambda \in \mathbb{R}$. Let $\{p^{el}_\alpha(\lambda)\}_{\lambda\in\mathbb{R}}$ be the matrix which corresponds to the orthogonal projection p^{el}_α from $\mathfrak{h}^{el}$ onto $\mathfrak{h}^{el}_\alpha$. Further, let $t^c_{rl}(\lambda) := p^{el}_r(\lambda)t_c(\lambda)p^{el}_l$ and $t^c_{lr} := p^{el}_l(\lambda)t_c(\lambda)p^{el}_r$. Notice that both quantities are in fact scalar functions. Obviously, the channel cross-sections $\sigma^c_{lr}(\lambda)$ and $\sigma^c_{rl}(\lambda)$ at energy $\lambda \in \mathbb{R}$ are given by $\sigma_c(\lambda) := \sigma^c_{lr}(\lambda) = |t^c_{lr}(\lambda)|^2 = |t^c_{rl}(\lambda)|^2 = \sigma^c_{rl}(\lambda)$, $\lambda \in \mathbb{R}$. In particular, if $g_l(\lambda) = 1$ and $g_r = 0$, then

$$J^{s_c}_{\rho^{el},q^{el}_l} = \frac{1}{2\pi}\int_{\mathbb{R}}(f_l(\lambda) - f_r(\lambda))\sigma_c(\lambda)d\lambda, \tag{3.10}$$

and $q^{el}_l := p^{el}_l$. Following [1], $J^{s_c}_{\rho^{el},q^{el}_l}$ denotes the current entering the quantum dot from the left lead.

3.2. Application to the JCL-model

Let $\mathcal{S} = \{H, H_0\}$ be now the JCL-model. Further, let ρ and Q be a density operator and a charge for H_0, respectively. Let us introduce the intermediate scattering system $\mathcal{S}_c := \{H, H_c\}$ where $H_c := h^{el} \otimes I_{\mathfrak{h}^{ph}} + I_{\mathfrak{h}^{el}} \otimes h^{ph} = H_0 + V_{el}$. The Hamiltonian H_c describes the coupling of the leads to a quantum dot but under the assumption that the photon interaction is not switched on.

Obviously, $\mathcal{S}_{ph} := \{H, H_c\}$ and $\mathcal{S}_c := \{H_c, H_0\}$ are $\mathfrak{L}_1$-scattering systems. The corresponding scattering operators are denoted by S_{ph} and S_c, respectively. Let us assume that $\Pi(H^{ac}_c) = \{L^2(\mathbb{R}, d\lambda, \mathfrak{h}_c(\lambda)), \mathcal{M}, \Phi_c\}$ is a spectral representation of H^{ac}_c. The scattering matrix of the scattering system $\{H, H_c\}$ with respect to $\Pi(H^{ac}_c)$ is denoted by $\{S_{ph}(\lambda)\}_{\lambda\in\mathbb{R}}$. The scattering matrix of the scattering system $\{H_c, H_0\}$ with respect to $\Pi(H^{ac}_0) = \{L^2(\mathbb{R}, d\lambda, \mathfrak{h}_0(\lambda)), \mathcal{M}, \Phi_0\}$ is denoted by $\{S_c(\lambda)\}_{\lambda\in\mathbb{R}}$.

Since $\mathcal{S}_c$ is a $\mathfrak{L}_1$-scattering system, the wave operators $W_\pm(H_c, H_0)$ exist and are complete, and since $\Phi_c W_\pm(H_c, H_0)\Phi^*_0$ commute with $\mathcal{M}$, there exist the measurable families $\{W_\pm(\lambda)\}_{\lambda\in\mathbb{R}}$ of isometries acting from $\mathfrak{h}_0(\lambda)$ onto $\mathfrak{h}_c(\lambda)$ for a.e. $\lambda \in \mathbb{R}$ such that

$$(\Phi_c W_\pm(H_c, H_0)\Phi^*_0\,\widehat{f}\,)(\lambda) = W_\pm(\lambda)\,\widehat{f}\,(\lambda), \quad \lambda \in \mathbb{R}, \quad \widehat{f}\, \in L^2(\mathbb{R}, d\lambda, \mathfrak{h}_0(\lambda)).$$

The families $\{W_\pm(\lambda)\}_{\lambda\in\mathbb{R}}$ are called the wave matrices.
A straightforward computation shows that $\widehat{S}_{ph} := W_+(H_c, H_0)^* S_{ph} W_+(H_c, H_0)$ commutes with H_0. Hence, with respect to the spectral representation $\Pi(H^{ac}_0)$, the operator $\widehat{S}_{ph}$ is unitarily equivalent to a multiplication induced by a measurable family $\{\widehat{S}_{ph}(\lambda)\}_{\lambda\in\mathbb{R}}$ of unitary operators in $\mathfrak{h}_0(\lambda)$. A straightforward

computation shows that

$$\widehat{S}_{ph}(\lambda) = W_+(\lambda)^* S_{ph}(\lambda) W_+(\lambda) \tag{3.11}$$

for a.e. $\lambda \in \mathbb{R}$. Roughly speaking, $\{\widehat{S}_{ph}(\lambda)\}_{\lambda\in\mathbb{R}}$ is the scattering matrix of S_{ph} with respect to the spectral representation $\Pi(H_0^{ac})$.

Furthermore, let

$$\rho^c := W_-(H_c, H_0)\rho W_-(H_c, H_0)^* \tag{3.12}$$

and

$$Q^c := W_+(H_c, H_0) Q W_+(H_c, H_0)^*. \tag{3.13}$$

The operators ρ^c and Q^c are the density and the tempered charge operators for the scattering system $\mathcal{S}_{ph}$. Indeed, one easily verifies that ρ^c and Q^c commute with H_c. Moreover, ρ^c is non-negative. Furthermore, if Q is a charge, then Q^c is a charge, too. If Q is a p-tempered charge and $(H_0 - i)^{p+2}\rho$ is a bounded operator, then one easily checks that Q^c is a p-tempered charge and $(H_c - i)^{p+2}\rho^c$ is a bounded operator. Finally, we note that the corresponding matrices $\{\rho^c_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ and $\{Q^c_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ are related to the matrices $\{\rho_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ and $\{Q_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ by $\rho^c_{ac}(\lambda) = W_-(\lambda)\rho_{ac}(\lambda)W_-(\lambda)^*$ and $Q^c_{ac}(\lambda) = W_+(\lambda)Q_{ac}(\lambda)W_+(\lambda)^*$ for a.e. $\lambda \in \mathbb{R}$.

Proposition 3.4. ([9, Proposition 3.4]) *Let $\mathcal{S} = \{H, H_0\}$ be the JCL-model. Further, let ρ and Q be a density operator and a p-tempered charge for H_0, respectively. Moreover, let $\{S_c(\lambda)\}_{\lambda\in\mathbb{R}}$, $\{\rho_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ and $\{Q_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ be scattering, density and charge matrices of S_c, ρ_{ac} and Q_{ac} with respect to $\Pi(H_0^{ac})$. Furthermore, let $\{S_{ph}(\lambda)\}_{\lambda\in\mathbb{R}}$, $\{\rho^c_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ and $\{Q^c_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ be scattering, density and charge matrices of the scattering operator S_{ph}, the density operator ρ^c_{ac}, cf.* (3.12)*, and the charge operator Q^c_{ac}, cf.* (3.13)*, with respect to the spectral representation $\Pi(H_c^{ac})$. If $(H_0 - i)^{p+2}\rho$ is a bounded operator, then the decomposition*

$$J^{\mathcal{S}}_{\rho,Q} = J^c_{\rho,Q} + J^{ph}_{\rho,Q} \tag{3.14}$$

holds, where

$$\begin{aligned} J^c_{\rho,Q} &:= \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{tr}(\rho_{ac}(\lambda)(Q_{ac}(\lambda) - S_c(\lambda)^* Q_{ac}(\lambda) S_c(\lambda)))d\lambda, \\ J^{ph}_{\rho,Q} &:= \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{tr}(\rho^c_{ac}(\lambda)(Q^c_{ac}(\lambda) - S_{ph}(\lambda)^* Q^c_{ac}(\lambda) S_{ph}(\lambda)))d\lambda. \end{aligned}$$

R e m a r k 3.5. (i) The current $J^c_{\rho,Q}$ occurs due to the coupling of the leads to the quantum dot and is therefore called the *contact-induced current*.

(ii) The current $J^{ph}_{\rho,Q}$ occurs due to the interaction of photons with electrons and is therefore called the *photon-induced current*. Notice that this current is calculated under the assumption that the leads have already contacted to the dot.

Corollary 3.6. ([9, Corollary 3.6]) *Let the assumptions of Proposition 3.4 be satisfied. With respect to the spectral representation* $\Pi(H_0^{ac})$ *of* H_0^{ac}*, the photon induced current* $J^{ph}_{\rho,Q}$ *can be represented by*

$$J^{ph}_{\rho,Q} := \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{tr}(\,\widehat{\rho}_{ac}(\lambda)\,(Q_{ac}(\lambda) - \,\widehat{S}_{ph}(\lambda)^*\,Q_{ac}(\lambda)\,\widehat{S}_{ph}\,(\lambda)))d\lambda, \tag{3.15}$$

where the measurable families $\{\,\widehat{S}_{ph}(\lambda)\,\}_{\lambda\in\mathbb{R}}$ *and* $\{\,\widehat{\rho}_{ac}(\lambda)\,\}_{\lambda\in\mathbb{R}}$ *are given by* (3.11) *and*

$$\widehat{\rho}_{ac}(\lambda) := S_c(\lambda)\rho_{ac}(\lambda)S_c(\lambda)^* \quad \lambda\in\mathbb{R}, \tag{3.16}$$

respectively.

R e m a r k 3.7. In the following, we call $\{\,\widehat{\rho}_{ac}(\lambda)\,\}_{\lambda\in\mathbb{R}}$, cf. (3.16), the photon modified electron density matrix. Notice that $\{\,\widehat{\rho}_{ac}(\lambda)\,\}_{\lambda\in\mathbb{R}}$ might be non-diagonal even if the electron density matrix $\{\rho_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ is diagonal.

4. Analysis of Currents

In the following, we analyze the currents $J^c_{\rho,Q}$ and $J^{ph}_{\rho,Q}$ under the assumption that ρ and Q have the tensor product structure

$$\rho = \rho^{el}\otimes\rho^{ph} \quad \text{and} \quad Q = q^{el}\otimes q^{ph}, \tag{4.1}$$

where ρ^{el} and ρ^{ph} as well as q^{el} and q^{ph} are the density operators and the (tempered) charges for h_0^{el} and h^{ph}, respectively. Since ρ^{ph} commutes with h^{ph}, which is discrete, the operator ρ^{ph} has the form $\rho^{ph} = \sum_{n\in\mathbb{N}_0}\rho^{ph}(n)(\cdot,\Upsilon_n)\Upsilon_n$, where $\rho^{ph}(n)$ are non-negative numbers. Similarly, q^{ph} can be represented by $q^{ph} = \sum_{n\in\mathbb{N}_0} q^{ph}(n)(\cdot,\Upsilon_n)\Upsilon_n$, where $q^{ph}(n)$ are real numbers.

Lemma 4.1. ([9, Lemma 4.1]) *Let* $\mathcal{S} = \{H, H_0\}$ *be the JCL-model. Assume that* $\rho\neq 0$ *and* Q *have the structure* (4.1)*, where* ρ^{el} *is a density operator and* q^{el} *is a charge for* h_0^{el}*.*

(i) *The operator* $(H_0 - i)^{p+2}\rho$*,* $p\in\mathbb{N}_0$*, is bounded if and only if the condition*

$$\sup_{n\in\mathbb{N}_0}\rho^{ph}(n)n^{p+2} < \infty \tag{4.2}$$

is satisfied.

(ii) *The charge* Q *is p-tempered if and only if*

$$\sup_{n\in\mathbb{N}}|q^{ph}(n)|n^{-p} < \infty \tag{4.3}$$

is valid.

4.1. Contact-induced current

Let us recall that $\mathcal{S}_c = \{H_c, H_0\}$ is a $\mathfrak{L}_1$-scattering system. Obvious computations show that $W_\pm(H_c, H_0) = w_\pm(h^{el}, h_0^{el}) \otimes I_{\mathfrak{h}^{ph}}$, where $w_\pm(h^{el}, h_0^{el})$ is given by (3.8). Hence, $S_c = s_c \otimes I_{\mathfrak{h}^{ph}}$, where $s_c := w_+(h_c^{el}, h_0^{el})^* w_-(h_c^{el}, h_0^{el})$.

Proposition 4.2. ([9, Proposition 4.2]) *Let $\mathcal{S} = \{H, H_0\}$ be the JCL-model. Assume that ρ and Q are given by* (4.1)*, where ρ^{el} and q^{el} are the density and the charge operators for h_0^{el} and ρ^{ph} and q^{ph} for h^{ph}, respectively. If for some $p \in \mathbb{N}_0$ the conditions* (4.2) *and* (4.3) *are satisfied, then the current $J^c_{\rho,Q}$ is well-defined and admits the representation $J^c_{\rho,Q} = \gamma J^{s_c}_{\rho^{el},q^{el}}$, $\gamma := \sum_{n\in\mathbb{N}_0} q^{ph}(n)\rho^{ph}(n)$, where $J^{s_c}_{\rho^{el},q^{el}}$ is given by* (3.10)*. If $\mathrm{tr}(\rho^{ph}) = 1$ and $q^{ph} = I_{\mathfrak{h}^{ph}}$, then $J^c_{\rho,Q} = J^{s_c}_{\rho^{el},q^{el}}$.*

4.2. Photon-induced current

To calculate the current $J^{ph}_{\rho,Q}$, we used representation (3.15). We set

$$\widehat{S}^{ph}_{mn}(\lambda) := P_m(\lambda)\,\widehat{S}_{ph}(\lambda) \restriction \mathfrak{h}_n(\lambda), \quad \lambda \in \mathbb{R},$$

where $\{\widehat{S}_{ph}(\lambda)\}_{\lambda\in\mathbb{R}}$ is defined by (3.11) and $P_m(\lambda)$ is the orthogonal projection from $\mathfrak{h}(\lambda)$, cf. (2.24), onto $\mathfrak{h}_m(\lambda) := \mathfrak{h}^{el}(\lambda - m\omega)$, $\lambda \in \mathbb{R}$.

Proposition 4.3. ([9, Proposition 4.3]) *Let $\mathcal{S} = \{H, H_0\}$ be the JCL-model. Assume that ρ and Q are given by* (4.1)*, where ρ^{el} and q^{el} are the density and the charge operators for h_0^{el} and ρ^{ph} and q^{ph} for h^{ph}, respectively. If for some $p \in \mathbb{N}_0$ the conditions* (4.2) *and* (4.3) *are satisfied, then the current $J^{ph}_{\rho,Q}$ is well-defined and admits the representation*

$$J^{ph}_{\rho,Q} = \sum_{m\in\mathbb{N}_0} \rho^{ph}(m) \sum_{n\in\mathbb{N}_0} q^{ph}(n) \times \frac{1}{2\pi}\int_{\mathbb{R}} d\lambda\, \mathrm{tr}\Big(\widehat{\rho}^{el}_{ac}(\lambda - m\omega)\,\Big(q^{el}_{ac}(\lambda - n\omega)\delta_{mn} - \widehat{S}^{ph}_{nm}(\lambda)^* q^{el}_{ac}(\lambda - n\omega)\,\widehat{S}^{ph}_{nm}(\lambda)\Big)\Big), \tag{4.4}$$

where $\{\widehat{\rho}^{el}_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ is the photon-modified electron density cf. (3.16)*, which takes the form*

$$\widehat{\rho}^{el}_{ac}(\lambda) = s_c(\lambda)\rho^{el}(\lambda)s_c(\lambda)^*, \quad \lambda \in \mathbb{R}. \tag{4.5}$$

Corollary 4.4. ([9, Corollary 4.4]) *Let $\mathcal{S} = \{H, H_0\}$ be the JCL-model. Assume that ρ and Q are given by* (4.1)*, where ρ^{el} and q^{el} are the density and the charge operators for h_0^{el} and ρ^{ph} and q^{ph} for h^{ph}, respectively. If ρ^{el} is an*

equilibrium state, i.e., $\rho^{el} = f^{el}(h_0^{el})$*, then*

$$J^{ph}_{\rho,Q} = \sum_{m,n\in\mathbb{N}_0} q^{ph}(n)$$
$$\times\frac{1}{2\pi}\int_{\mathbb{R}} \Big(\rho^{ph}(n) f^{el}(\lambda - n\omega) - \rho^{ph}(m) f^{el}(\lambda - m\omega)\Big)$$
$$\times\mathrm{tr}\left(\widehat{S}^{ph}_{nm}(\lambda)^* q^{el}_{ac}(\lambda - n\omega)\,\widehat{S}^{ph}_{nm}(\lambda)\right) d\lambda.$$

5. Electron and Photon Currents

5.1. Electron current

To calculate the electron current induced by the contacts and a photons contact, we make the following choice throughout this section. We set

$$Q^{el}_\alpha := q^{el}_\alpha \otimes q^{ph}, \quad q^{el}_\alpha := -\mathfrak{e} p^{el}_\alpha \quad \text{and} \quad q^{ph} := I_{\mathfrak{h}^{ph}}, \quad \alpha \in \{l, r\}, \tag{5.1}$$

where p^{el}_α denotes the orthogonal projection from $\mathfrak{h}^{el}$ onto $\mathfrak{h}^{el}_\alpha$. By $\mathfrak{e} > 0$, we denote the magnitude of the elementary charge. Since p^{el}_α commutes with h^{el}_α, one easily verifies that Q^{el}_α commutes with H_0 which shows that Q^{el}_α is a charge. Following [1], the flux related to Q^{el}_α gives us the electron current $J^{\mathcal{S}}_{\rho,Q^{el}_\alpha}$ entering the lead α from the sample. Notice that $Q^{el}_\alpha = \mathfrak{e}P_\alpha$, where P_α is the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{H}_\alpha := \mathfrak{h}^{el}_\alpha \otimes \mathfrak{h}^{ph}$. Since $q^{ph} = I_{\mathfrak{h}^{ph}}$, the condition (4.3) is immediately satisfied for any $p \geq 0$.

Let $f(\cdot) : \mathbb{R} \longrightarrow \mathbb{R}$ be a non-negative bounded measurable function. We set

$$\rho^{el} = \rho^{el}_l \oplus \rho^{el}_S \oplus \rho^{el}_r, \quad \rho^{el}_\alpha := f(h^{el}_\alpha - \mu_\alpha), \quad \alpha \in \{l, r\}, \tag{5.2}$$

and $\rho = \rho^{el} \otimes \rho^{ph}$. The chemical potential of the lead α is denoted by μ_α. In applications one sets $f(\lambda) := f_{FD}(\lambda)$, $\lambda \in \mathbb{R}$, where $f_{FD}(\lambda)$ is the so-called Fermi–Dirac distribution given by (3.7). If $\beta = \infty$, then $f_{FD}(\lambda) := \chi_{\mathbb{R}_-}(\lambda)$, $\lambda \in \mathbb{R}$. Notice that $[\rho^{el}, p^{el}] = 0$. For ρ^{ph}, we choose the Gibbs state

$$\rho^{ph} := \frac{1}{Z} e^{-\beta h^{ph}}, \quad Z = \mathrm{tr}(e^{-\beta h^{ph}}) = \frac{1}{1 - e^{-\beta\omega}}. \tag{5.3}$$

Hence, $\rho^{ph} = (1 - e^{-\beta\omega}) e^{-\beta h^{ph}}$. If $\beta = \infty$, then $\rho^{ph} := (\cdot, \Upsilon_0)\Upsilon_0$. Obviously, $\mathrm{tr}(\rho^{ph}) = 1$. We note that $\rho^{ph}(n) = (1 - e^{-\beta\omega}) e^{-n\beta\omega}$, $n \in \mathbb{N}_0$, satisfies condition (4.2) for any $p \geq 0$. Obviously, $\rho_0 = \rho^{el} \otimes \rho^{ph}$ is a density operator for H_0.

Definition 5.1. *Let* $\mathcal{S} = \{H, H_0\}$ *be the JCL-model. If* $Q := Q^{el}_\alpha$*, where* Q^{el}_α *is given by* (5.1)*, and* $\rho := \rho_0 := \rho^{el} \otimes \rho^{ph}$*, where* ρ^{el} *and* ρ^{ph} *are given by* (5.2) *and* (5.3)*, then* $J^{el}_{\rho_0,Q^{el}_\alpha} := J^{\mathcal{S}}_{\rho_0,Q^{el}_\alpha}$ *is called the electron current entering the lead* α*. The currents* $J^{c}_{\rho_0,Q^{el}_\alpha}$ *and* $J^{ph}_{\rho_0,Q^{el}_\alpha}$ *are called the contact-induced and the photon-induced electron currents.*

5.1.1. Contact-induced electron current. The following proposition immediately follows from Proposition 4.2.

Proposition 5.2. *Let $\mathcal{S} = \{H, H_0\}$ be the JCL-model. Then the contact-induced electron current $J^c_{\rho_0,Q^{el}_\alpha}$, $\alpha \in \{l,r\}$, is given by $J^c_{\rho_0,Q^{el}_\alpha} = J^{s_c}_{\rho^{el},q^{el}_\alpha}$. In particular, one has*

$$J^c_{\rho_0,Q^{el}_\alpha} = -\frac{\mathfrak{e}}{2\pi}\int_{\mathbb{R}}(f(\lambda-\mu_\alpha) - f(\lambda-\mu_\varkappa)\sigma_c(\lambda)d\lambda, \quad \alpha,\varkappa \in \{l,r\}, \quad \alpha \neq \varkappa, \quad (5.4)$$

where $\{\sigma_c(\lambda)\}_{\lambda\in\mathbb{R}}$ is the channel cross-section from left to right of the scattering system $s_c = \{h^{el}, h^{el}_0\}$, cf. Example 3.3.

P r o o f. Since $\mathrm{tr}(\rho^{ph}) = 1$, it follows from Proposition 4.2 that $J^c_{\rho_0,Q^{el}_\alpha} = J^{s_c}_{\rho^{el},q^{el}_\alpha}$. From (3.10), cf. Example 3.3, we find (5.4). ∎

If $\mu_l > \mu_r$ and $f(\cdot)$ is decreasing, then $J^c_{\rho_0,Q^{el}_l} < 0$. Hence the electron contact current goes from the left lead to the right one, which is in accordance with physical intuition. In particular, this is valid for the Fermi–Dirac distribution.

Proposition 5.3. *Let $\mathcal{S} = \{H, H_0\}$ be the JCL-model. Further, let ρ^{el} and ρ^{ph} be given by* (5.2) *and* (5.3)*, respectively. If the charge Q^{el}_α is given by* (5.1)*, then the following holds:*

(E) *If $\mu_l = \mu_r$, then $J^c_{\rho_0,Q^{el}_\alpha} = 0$, $\alpha \in \{l,r\}$.*

(S) *If $v_l \geq v_r + 4$, then $J^c_{\rho_0,Q^{el}_\alpha} = 0$, $\alpha \in \{l,r\}$, even if $\mu_l \neq \mu_l$.*

(C) *If $e^S_0 = \delta^S_0$ and $e^S_1 = \delta^S_1$, then $J^c_{\rho_0,Q^{el}_\alpha} = 0$, $\alpha \in \{l,r\}$, even if $\mu_l \neq \mu_r$.*

P r o o f. (E) If $\mu_l = \mu_r$, then $f(\lambda-\mu_l) = f(\lambda-\mu_r)$. Applying formula (5.4), we obtain $J^c_{\rho_0,Q^{el}_\alpha} = 0$.

(S) If $v_l \geq v_r + 4$, then $h^{el,ac}_0$ has a simple spectrum. Hence the scattering matrix $\{s_c(\lambda)\}_{\lambda\in\mathbb{R}}$ of the scattering system $s_c = \{h^{el}, h^{el}_0\}$ is a scalar function, which immediately yields $\sigma_c(\lambda) = 0$, $\lambda \in \mathbb{R}$, which yields $J^c_{\rho_0,Q^{el}_\alpha} = 0$.

(C) In this case, the Hamiltonian h^{el} decomposes into a direct sum of two Hamiltonians which do not interact. Hence the scattering matrix of $\{s_c(\lambda)\}_{\lambda\in\mathbb{R}}$ of the scattering system $s_c = \{h^{el}, h^{el}_0\}$ is diagonal, which immediately yields $J^c_{\rho_0,Q^{el}_\alpha} = 0$. ∎

5.1.2. Photon-induced electron current. To analyze (4.4) is hopeless if we make no assumptions concerning ρ^{el} and the scattering operator s_c. The simplest assumption is that ρ^{el} and s_c commute. In this case, we get $\widehat{\rho}^{el}(\lambda) = \rho^{el}(\lambda)$, $\lambda \in \mathbb{R}$.

Lemma 5.4. *Let $\mathcal{S} = \{H, H_0\}$ be the JCL-model. Further, let ρ^{el} be given by* (5.2)*. If one of the cases* (*E*)*,* (*S*) *or* (*C*) *of Proposition 5.3 is realized, then ρ^{el} and s_c commute.*

P r o o f. If (E) holds, then $\rho^{el} = f(h_0^{el})$, which yields $[\rho^{el}, s_c] = 0$. If (S) is valid, then the scattering matrix $\{s_c(\lambda)\}_{\lambda\in\mathbb{R}}$ is a scalar function which shows $[\rho^{el}, s_c] = 0$. Finally, if (C) is realized, then the scattering matrix $\{s_c(\lambda)\}_{\lambda\in\mathbb{R}}$ is diagonal. Since ρ^{el} is given by (5.2), we get $[\rho^{el}, s_c] = 0$. ∎

We are going to calculate the current $J^{ph}_{\rho_0,Q^{el}_\alpha}$, see (4.4). Obviously we have $P_\alpha(\lambda) = \sum_{n\in\mathbb{N}_0} p^{el}_\alpha(\lambda - n\omega)$ and $I_{\mathfrak{h}(\lambda)} = P_l(\lambda) + P_r(\lambda)$, $\lambda \in \mathbb{R}$. We set

$$P_{n_\alpha}(\lambda) := P_\alpha(\lambda)P_n(\lambda) = P_n(\lambda)P_\alpha(\lambda) = p^{el}_\alpha(\lambda - n\omega),\ \alpha \in \{l, r\},$$

$n \in \mathbb{N}_0$, $\lambda \in \mathbb{R}$. In the following, we use the notation $\widehat{T}_{ph}(\lambda) = \widehat{S}_{ph}(\lambda) - I_{\mathfrak{h}(\lambda)}$, $\lambda \in \mathbb{R}$, where $\{\widehat{T}_{ph}(\lambda)\}_{\lambda\in\mathbb{R}}$ is called the transition matrix, and $\{\widehat{S}_{ph}(\lambda)\}_{\lambda\in\mathbb{R}}$ is given by (3.11). We set

$$\widehat{T}^{ph}_{k_\alpha m_\varkappa}(\lambda) := P_{k_\alpha}(\lambda)\,\widehat{T}_{ph}(\lambda)P_{m_\varkappa}(\lambda), \quad \lambda \in \mathbb{R}, \quad \alpha, \varkappa \in \{l, r\}, \quad k, m \in \mathbb{N}_0,$$

and

$$\widehat{\sigma}^{ph}_{k_\alpha m_\varkappa}(\lambda) = \mathrm{tr}(\,\widehat{T}^{ph}_{k_\alpha m_\varkappa}(\lambda)^*\,\widehat{T}^{ph}_{k_\alpha m_\varkappa}(\lambda)\,), \quad \lambda \in \mathbb{R}, \tag{5.5}$$

which is the cross-section between the channels k_α and $m_\varkappa$.

Proposition 5.5. *Let $\mathcal{S} = \{H, H_0\}$ be the JCL-model.*

(i) *If ρ^{el} commutes with the scattering operators s_c and q^{el}, then*

$$J^{ph}_{\rho_0,Q^{el}_\alpha} = -\sum_{\substack{m,n\in\mathbb{N}_0\\ \varkappa\in\{l,r\}}} \frac{\mathfrak{e}}{2\pi}\int_{\mathbb{R}} \Big(\rho^{ph}(n)f(\lambda - \mu_\alpha - n\omega) - \rho^{ph}(m)f(\lambda - \mu_\varkappa - m\omega)\Big)\,\widehat{\sigma}^{ph}_{n_\alpha m_\varkappa}(\lambda)\,d\lambda. \tag{5.6}$$

(ii) *If in addition $\mathcal{S} = \{H, H_0\}$ is time reversible symmetric, then*

$$J^{ph}_{\rho_0,Q^{el}_\alpha} = -\sum_{m,n\in\mathbb{N}_0} \frac{\mathfrak{e}}{2\pi}\int_{\mathbb{R}} \Big(\rho^{ph}(n)f(\lambda - \mu_\alpha - n\omega) - \rho^{ph}(m)f(\lambda - \mu_{\alpha'} - m\omega)\Big)\,\widehat{\sigma}^{ph}_{n_\alpha m_{\alpha'}}(\lambda)\,d\lambda, \tag{5.7}$$

$\alpha, \alpha' \in \{l, r\}$, $\alpha \neq \alpha'$.

P r o o f. (i) Let us assume that $q^{el} = \sum_{\varkappa\in\{l,r\}} g_\varkappa(h^{el}_\varkappa)$. Notice that

$$q^{el}_{ac}(\lambda) = \sum_{\varkappa\in\{l,r\}} g_\varkappa(\lambda) p^{el}_\varkappa(\lambda), \quad \lambda\in\mathbb{R}. \tag{5.8}$$

Inserting (5.8) into (4.4) and using $q^{ph} = I_{\mathfrak{h}^{ph}}$, we get

$$\begin{aligned} J^{ph}_{\rho_0,Q} = \sum_{\substack{m\in\mathbb{N}_0\\ \alpha\in\{l,r\}}} \rho^{ph}(m) \sum_{\substack{n\in\mathbb{N}_0\\ \varkappa\in\{l,r\}}} \frac{1}{2\pi}\int_{\mathbb{R}} d\lambda\ \phi_\alpha(\lambda-m\omega) g_\varkappa(\lambda-n\omega) \\ \times\mathrm{tr}\left(p^{el}_\alpha(\lambda-m\omega)\left(p^{el}_\varkappa(\lambda-n\omega)\delta_{mn} - \widehat{S}^{ph}_{nm}(\lambda)^* p^{el}_\varkappa(\lambda-n\omega)\,\widehat{S}^{ph}_{nm}(\lambda)\right)\right), \end{aligned}$$

where for simplicity we have set $\phi_\alpha(\lambda) := f(\lambda-\mu_\alpha)$, $\lambda\in\mathbb{R}$, $n\in\mathbb{N}_0$, $\alpha\in\{l,r\}$. Therefore, we have

$$\begin{aligned} J^{ph}_{\rho_0,Q} = \sum_{\substack{n\in\mathbb{N}_0\\ \varkappa\in\{l,r\}}} \rho^{ph}(n)\frac{1}{2\pi}\int_{\mathbb{R}} d\lambda\ \phi_\varkappa(\lambda-n\omega) g_\varkappa(\lambda-n\omega)\mathrm{tr}\left(p^{el}_\varkappa(\lambda-n\omega)\right) \\ - \sum_{\substack{n\in\mathbb{N}_0\\ \varkappa\in\{l,r\}}}\sum_{\substack{m\in\mathbb{N}_0\\ \alpha\in\{l,r\}}} \rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}} d\lambda\ \phi_\alpha(\lambda-m\omega) g_\varkappa(\lambda-n\omega) \\ \times\mathrm{tr}\left(p^{el}_\alpha(\lambda-m\omega)\,\widehat{S}^{ph}_{nm}(\lambda)^* p^{el}_\varkappa(\lambda-n\omega)\,\widehat{S}^{ph}_{nm}(\lambda) p^{el}_\alpha(\lambda-m\omega)\right). \end{aligned} \tag{5.9}$$

Since the scattering matrix $\{\widehat{S}^{ph}(\lambda)\}_{\lambda\in\mathbb{R}}$ is unitary, we have

$$p^{el}_\varkappa(\lambda-n\omega) = \sum_{\substack{m\in\mathbb{N}_0\\ \alpha\in\{l,r\}}} p^{el}_\varkappa(\lambda-n\omega)\,\widehat{S}^{ph}_{mn}(\lambda)^* p^{el}_\alpha(\lambda-m\omega)\,\widehat{S}^{ph}_{mn}(\lambda) p^{el}_\varkappa(\lambda-n\omega) \tag{5.10}$$

for $n\in\mathbb{N}_0$ and $\varkappa\in\{l,r\}$. Inserting (5.10) into (5.9), we find

$$\begin{aligned} J^{ph}_{\rho_0,Q} = \sum_{\substack{n\in\mathbb{N}_0\\ \varkappa\in\{l,r\}}}\sum_{\substack{m\in\mathbb{N}_0\\ \alpha\in\{l,r\}}} \rho^{ph}(n)\frac{1}{2\pi}\int_{\mathbb{R}} d\lambda\ \phi_\varkappa(\lambda-n\omega) g_\varkappa(\lambda-n\omega) \\ \times\mathrm{tr}\left(p^{el}_\varkappa(\lambda-n\omega)\,\widehat{S}^{ph}_{nm}(\lambda)^* p^{el}_\alpha(\lambda-m\omega)\,\widehat{S}^{ph}_{mn}(\lambda) p^{el}_\varkappa(\lambda-n\omega)\right) \\ - \sum_{\substack{n\in\mathbb{N}_0\\ \varkappa\in\{l,r\}}}\sum_{\substack{m\in\mathbb{N}_0\\ \alpha\in\{l,r\}}} \rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}} d\lambda\ \phi_\alpha(\lambda-m\omega) g_\varkappa(\lambda-n\omega) \\ \times\mathrm{tr}\left(p^{el}_\alpha(\lambda-m\omega)\,\widehat{S}^{ph}_{nm}(\lambda)^* p^{el}_\varkappa(\lambda-n\omega)\,\widehat{S}^{ph}_{nm}(\lambda) p^{el}_\alpha(\lambda-m\omega)\right). \end{aligned}$$

Using the notation (5.5), we find

$$J^{ph}_{\rho_0,Q} = \sum_{\substack{n\in\mathbb{N}_0\\ \varkappa\in\{l,r\}}}\sum_{\substack{m\in\mathbb{N}_0\\ \alpha\in\{l,r\}}} \rho^{ph}(n)\frac{1}{2\pi}\int\limits_{\mathbb{R}} d\lambda\ \phi_{\varkappa}(\lambda-n\omega)g_{\varkappa}(\lambda-n\omega)\,\widehat{\sigma}^{ph}_{m_\alpha n_\varkappa}(\lambda)$$
$$- \sum_{\substack{n\in\mathbb{N}_0\\ \varkappa\in\{l,r\}}}\sum_{\substack{m\in\mathbb{N}_0\\ \alpha\in\{l,r\}}} \rho^{ph}(m)\frac{1}{2\pi}\int\limits_{\mathbb{R}} d\lambda\ \phi_{\alpha}(\lambda-m\omega)g_{\varkappa}(\lambda-n\omega)\,\widehat{\sigma}^{ph}_{n_\varkappa m_\alpha}(\lambda)\,.$$

By (3.5), we find

$$\sum_{\substack{m\in\mathbb{N}_0\\ \alpha\in\{l,r\}}} \widehat{\sigma}^{ph}_{m_\alpha n_\varkappa}(\lambda) \;=\; \sum_{\substack{m\in\mathbb{N}_0\\ \alpha\in\{l,r\}}} \widehat{\sigma}^{ph}_{n_\varkappa m_\alpha}(\lambda)\,,\quad \lambda\in\mathbb{R}.$$

Using that, we get

$$J^{ph}_{\rho_0,Q} =$$
$$\sum_{\substack{m,n\in\mathbb{N}_0\\ \alpha,\varkappa\in\{l,r\}}} \frac{1}{2\pi}\int\limits_{\mathbb{R}}\Big(\rho^{ph}(n)\phi_{\varkappa}(\lambda-n\omega)-\rho^{ph}(m)\phi_{\alpha}(\lambda-m\omega)\Big)\,g_{\varkappa}(\lambda-n\omega)\,\widehat{\sigma}^{ph}_{n_\varkappa m_\alpha}(\lambda)\,d\lambda.$$

Setting $g_\alpha(\lambda)=-\mathfrak{e}$ and $g_\varkappa(\lambda)\equiv 0$, $\varkappa\neq\alpha$, we obtain (5.6).

(ii) A straightforward computation shows that

$$\sum_{n,m\in\mathbb{N}_0}\int\limits_{\mathbb{R}}\Big(\rho^{ph}(n)f(\lambda-\mu_\alpha-n\omega)-\rho^{ph}(m)f(\lambda-\mu_\alpha-m\omega)\Big)\,\widehat{\sigma}^{ph}_{n_\alpha m_\alpha}(\lambda)\,d\lambda$$
$$=\sum_{n,m\in\mathbb{N}_0}\int\limits_{\mathbb{R}}\Big(\rho^{ph}(m)f(\lambda-\mu_\alpha-m\omega)-\rho^{ph}(n)f(\lambda-\mu_\alpha-n\omega)\Big)\,\widehat{\sigma}^{ph}_{m_\alpha n_\alpha}(\lambda)\,d\lambda.$$

Since $\sigma^{ph}_{m_\alpha n_\alpha}(\lambda)=\sigma^{ph}_{n_\alpha m_\alpha}(\lambda)$, $\lambda\in\mathbb{R}$, we get

$$\sum_{n,m\in\mathbb{N}_0}\int\limits_{\mathbb{R}}\Big(\rho^{ph}(n)f(\lambda-\mu_\alpha-n\omega)-\rho^{ph}(m)f(\lambda-\mu_\alpha-m\omega)\Big)\,\widehat{\sigma}^{ph}_{n_\alpha m_\alpha}(\lambda)\,d\lambda$$
$$=-\sum_{n,m\in\mathbb{N}_0}\int\limits_{\mathbb{R}}\Big(\rho^{ph}(n)f(\lambda-\mu_\alpha-n\omega)-\rho^{ph}(m)f(\lambda-\mu_\alpha-m\omega)\Big)\,\widehat{\sigma}^{ph}_{n_\alpha m_\alpha}(\lambda)\,d\lambda,$$

which yields

$$\sum_{n,m\in\mathbb{N}_0}\int\limits_{\mathbb{R}}\Big(\rho^{ph}(n)f(\lambda-\mu_\alpha-n\omega)-\rho^{ph}(m)f(\lambda-\mu_\alpha-m\omega)\Big)\,\widehat{\sigma}^{ph}_{n_\alpha m_\alpha}(\lambda)\,d\lambda=0.$$

Using that, we get immediately representation (5.7) from (5.6). ∎

Corollary 5.6. *Let $\mathcal{S} = \{H, H_0\}$ be the JCL-model.*

(i) *If the cases (E), (S) or (C) of Proposition 5.3 are realized, then* (5.6) *holds.*

(ii) *If the case (E) of Proposition 5.3 is realized and the system $\mathcal{S} = \{H, H_0\}$ is time reversible symmetric, then*

$$J^{ph}_{\rho_0, Q^{el}_\alpha} = - \sum_{m,n \in \mathbb{N}_0} \frac{\mathfrak{e}}{2\pi} \int_{\mathbb{R}} (\rho^{ph}(n) f(\lambda - \mu - n\omega) - \rho^{ph}(m) f(\lambda - \mu - m\omega)) \, \widehat{\sigma}^{ph}_{n_\alpha m_{\alpha'}}(\lambda) d\lambda, \tag{5.11}$$

$n \in \mathbb{N}_0$, $\alpha \in \{l, r\}$, where $\mu := \mu_l = \mu_r$ and $\alpha \neq \alpha'$.

(iii) *If the case (E) of Proposition 5.3 is realized and the system $\mathcal{S} = \{H, H_0\}$ is time reversible as well as mirror symmetric, then $J^{ph}_{\rho_0, Q^{el}_\alpha} = 0$.*

P r o o f. (i) The statement follows from Proposition 5.5(i) and Lemma 5.4.

(ii) By setting $\mu_\alpha = \mu_{\alpha'}$, formula (5.7) reduces to (5.11).

(iii) If $\mathcal{S} = \{H, H_0\}$ is time reversible and mirror symmetric, we get from Lemma 2.13 (ii) that $\widehat{\sigma}^{ph}_{n_\alpha m_{\alpha'}}(\lambda) = \widehat{\sigma}^{ph}_{n_{\alpha'} m_\alpha}(\lambda)$, $\lambda \in \mathbb{R}$, $n, m \in \mathbb{N}_0$, $\alpha, \alpha' \in \{l, r\}$, $\alpha \neq \alpha'$. Using that, we get from (5.11) that

$$J^{ph}_{\rho_0, Q^{el}_\alpha} = - \sum_{m,n \in \mathbb{N}_0} \frac{\mathfrak{e}}{2\pi} \int_{\mathbb{R}} (\rho^{ph}(n) f(\lambda - \mu - n\omega) - \rho^{ph}(m) f(\lambda - \mu - m\omega)) \, \widehat{\sigma}^{ph}_{n_{\alpha'} m_\alpha}(\lambda) d\lambda.$$

Interchanging m and n, we get

$$J^{ph}_{\rho_0, Q^{el}_\alpha} = - \sum_{m,n \in \mathbb{N}_0} \frac{\mathfrak{e}}{2\pi} \int_{\mathbb{R}} (\rho^{ph}(m) f(\lambda - \mu - m\omega) - \rho^{ph}(n) f(\lambda - \mu - n\omega)) \, \widehat{\sigma}^{ph}_{m_{\alpha'} n_\alpha}(\lambda) d\lambda.$$

Using that $\mathcal{S}$ is time reversible symmetric, we get from Lemma 2.13 (i) that

$$J^{ph}_{\rho_0, Q^{el}_\alpha} = - \sum_{m,n \in \mathbb{N}_0} \frac{\mathfrak{e}}{2\pi} \int_{\mathbb{R}} (\rho^{ph}(m) f(\lambda - \mu - m\omega) - \rho^{ph}(n) f(\lambda - \mu - n\omega)) \, \widehat{\sigma}^{ph}_{n_\alpha m_{\alpha'}}(\lambda) d\lambda,$$

which shows that $J^{ph}_{\rho_0, Q^{el}_\alpha} = -J^{ph}_{\rho_0, Q^{el}_\alpha}$. Hence, $J^{ph}_{\rho_0, Q^{el}_\alpha} = 0$. ∎

We note that by Proposition 5.3 the contact induced current is zero, i.e., $J^{c}_{\rho_0, Q^{el}_\alpha} = 0$. Hence, if $\mathcal{S}$ is time reversible and mirror symmetric, then the total current is zero, i.e., $J^{\mathcal{S}}_{\rho_0, Q^{el}_\alpha} = 0$.

R e m a r k 5.7. Let the case (E) of Proposition 5.3 be realized, that is, $\mu_l = \mu_r$. Moreover, we assume for simplicity that $0 =: v_r \leq v := v_l$.

(i) If $\beta = \infty$, then $\rho^{ph}(n) = \delta_{0n}$, $n \in \mathbb{N}_0$. From (5.6), we immediately get that $J^{ph}_{\rho^{el},Q^{el}_\alpha} = 0$. That means, if the temperature is zero, then the photon-induced electron current is zero.

(ii) The photon-induced electron current might be zero even if $\beta < \infty$. Indeed, let $\mathcal{S} = \{H, H_0\}$ be time reversible symmetric and let the case (E) be realized. If $\omega \geq v + 4$, then $\mathfrak{h}^{el}(\lambda) := \mathfrak{h}^{el}_n(\lambda) = \mathfrak{h}^{el}(\lambda - n\omega)$, $n \in \mathbb{N}_0$. Hence one always has $n = m$ in formula (5.11), which immediately yields $J^{ph}_{\rho_0,Q^{el}_\alpha} = 0$.

(iii) The photo-induced electron current might be different from zero. In fact, let $\mathcal{S} = \{H, H_0\}$ be time reversible symmetric and let $v = 2$ and $\omega = 4$, then one gets that to calculate the $J^{ph}_{\rho_0,Q^{el}_l}$ one has to take into account $m = n + 1$ in formula (5.11). Therefore we find

$$J^{ph}_{\rho_0,Q^{el}_l} = -\sum_{n\in\mathbb{N}_0} \frac{\mathfrak{e}}{2\pi}$$
$$\times \int_{\mathbb{R}} d\lambda \Big(\rho^{ph}(n) f(\lambda - \mu - n\omega) - \rho^{ph}(n+1) f(\lambda - \mu - (n+1)\omega) \Big)\, \widehat{\sigma}^{ph}_{n_l\,(n+1)_r}(\lambda).$$

If ρ^{ph} is given by (5.3) and $f(\lambda) = f_{FD}(\lambda)$, cf. (3.7), then one easily verifies that

$$\frac{\partial}{\partial x} \rho^{ph}(x) f_{FD}(\lambda - \mu - x\omega) < 0, \quad x, \mu, \lambda \in \mathbb{R}.$$

Hence $\rho^{ph}(n) f_{FD}(\lambda - \mu - n\omega)$ is decreasing in $n \in \mathbb{N}_0$ for $\lambda, \mu \in \mathbb{R}$, which yields $\Big(\rho^{ph}(n) f(\lambda - \mu - n\omega) - \rho^{ph}(n+1) f(\lambda - \mu - (n+1)\omega) \Big) \geq 0$. Therefore, $J^{ph}_{\rho_0,Q^{el}_l} \leq 0$, which means that the photon-induced current leaves the left-hand side and enters the right-hand side. In fact, $J^{ph}_{\rho_0,Q^{el}_l} = 0$ implies that $\widehat{\sigma}^{ph}_{n_l\,(n+1)_r}(\lambda) = 0$ for $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}$, which means that there is no scattering from the left-hand side to the right-hand one and vice versa, which can be excluded generically.

5.2. Photon current

The photon current is related to the charge

$$Q := Q^{ph} = -I_{\mathfrak{h}^{el}} \otimes \mathfrak{n},$$

where $\mathfrak{n} = \mathrm{d}\Gamma(1) = b^* b$ is the photon number operator on $\mathfrak{h}^{ph} = \mathfrak{F}_+(\mathbb{C})$, which is self-adjoint and commutes with h^{ph}. It follows that Q^{ph} is also self-adjoint and commutes with H_0. It is not bounded, but since $\mathrm{dom}(\mathfrak{n}) = \mathrm{dom}(h^{ph})$, it is immediately obvious that $Q^{ph}(H_0 + \theta)^{-1}$ is bounded, whence $\mathfrak{n}$ is a tempered

charge. Its charge matrix with respect to the spectral representation $\Pi(H_0^{ac})$ of Lemma 2.11 is given by

$$Q_{ac}^{ph}(\lambda) = -\bigoplus_{n\in\mathbb{N}_0} nP_n(\lambda).$$

We recall that $P_n(\lambda)$ is the orthogonal projection form $\mathfrak{h}(\lambda)$ onto $\mathfrak{h}_n(\lambda) = \mathfrak{h}^{el}(\lambda - n\omega)$, $\lambda \in \mathbb{R}$. We are going to calculate the photon current or, as it is also called, the photon production rate.

5.2.1. Contact-induced photon current. The following proposition is in accordance with physical intuition.

Proposition 5.8. *Let $\mathcal{S} = \{H, H_0\}$ be the JCL-model. Then $J^c_{\rho_0,Q^{ph}} = 0$.*

P r o o f. We note that $q^{el}_{ac}(\lambda) = I_{\mathfrak{h}^{el}(\lambda)}$, $\lambda \in \mathbb{R}$. Inserting this into (3.9), we get $J^{s_c}_{\rho^{el},q^{el}} = 0$. Applying Proposition 4.2, we prove $J^c_{\rho_0,Q^{ph}} = 0$. ∎

The result reflects the fact that the lead contact does not contribute to the photon current which is plausible.

5.2.2. Photon current. From Proposition 5.8 we get that only the photon-induced photon current $J^{ph}_{\rho_0,Q^{ph}}$ contributes to the photon current $J^{\mathcal{S}}_{\rho_0,Q^{ph}}$. Since $J^{\mathcal{S}}_{\rho_0,Q^{ph}} = J^{ph}_{\rho_0,Q^{ph}}$, we call $J^{ph}_{\rho_0,Q^{ph}}$ simply the *photon current*.

Using the notation $\widehat{T}^{ph}_{nm}(\lambda) := P_n(\lambda)\,\widehat{T}_{ph}(\lambda) \restriction \mathfrak{h}^{el}(\lambda - m\omega)$, $\lambda \in \mathbb{R}$, $m, n \in \mathbb{N}_0$, we set

$$\widetilde{T}^{ph}_{nm}(\lambda) = \widehat{T}^{ph}_{nm}(\lambda) s_c(\lambda - m\omega), \quad \lambda \in \mathbb{R}, \quad m, n \in \mathbb{N}_0 \tag{5.12}$$

and

$$\widetilde{T}^{ph}_{n_\varkappa m_\alpha}(\lambda) := P_{n_\varkappa}(\lambda)\widetilde{T}^{ph}_{nm}(\lambda) \restriction \mathfrak{h}^{el}_\alpha(\lambda - m\omega), \quad \lambda \in \mathbb{R}, \tag{5.13}$$

$m, n \in \mathbb{N}_0$, $\alpha, \varkappa \in \{l, r\}$, as well as $\widetilde{\sigma}^{ph}_{n_\varkappa\, m_\alpha}(\lambda) := \mathrm{tr}(\widetilde{T}^{ph}_{n_\varkappa m_\alpha}(\lambda)^* \widetilde{T}^{ph}_{n_\varkappa m_\alpha}(\lambda))$, $\lambda \in \mathbb{R}$.

Proposition 5.9. *Let $\mathcal{S} = \{H, H_0\}$ be the JCL-model.*

(i) *Then*

$$J^{ph}_{\rho_0,Q^{ph}} = \sum_{\substack{m,n\in\mathbb{N}_0\\ \alpha,\varkappa\in\{l,r\}}} (n-m)\rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}} f(\lambda - \mu_\alpha - m\omega)\widetilde{\sigma}^{ph}_{n_\varkappa m_\alpha}(\lambda)d\lambda. \tag{5.14}$$

(ii) *If ρ^{el} commutes with s_c, then*

$$J^{ph}_{\rho_0,Q^{ph}} = \sum_{\substack{m,n\in\mathbb{N}_0\\ \alpha,\varkappa\in\{l,r\}}} (n-m)\rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}} f(\lambda - \mu_\alpha - m\omega)\,\widehat{\sigma}^{ph}_{n_\varkappa m_\alpha}(\lambda)d\lambda. \tag{5.15}$$

(iii) *If* ρ^{el} *commutes with* s_c *and* $\mathcal{S} = \{H, H_0\}$ *is time reversible symmetric, then*

$$J^{ph}_{\rho_0,Q^{ph}} = \sum_{\substack{m,n\in\mathbb{N}_0, n>m \\ \varkappa,\alpha\in\{l,r\}}} \frac{1}{2\pi}\int_{\mathbb{R}} d\lambda \times (n-m)\left(\rho^{ph}(m) f(\lambda-\mu_\alpha-m\omega) - \rho^{ph}(n) f(\lambda-\mu_\varkappa-n\omega)\right)\widehat{\sigma}^{ph}_{n_\varkappa m_\alpha}(\lambda), \tag{5.16}$$

where $\alpha' \in \{l,r\}$ *and* $\alpha' \neq \alpha$.

P r o o f. (i) From (4.4), we get

$$J^{ph}_{\rho_0,Q^{ph}} = -\sum_{m,n\in\mathbb{N}_0} n\rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}} d\lambda \,\mathrm{tr}\left(\widehat{\rho}^{el}_{ac}(\lambda-m\omega) \times \left(P_n(\lambda)\delta_{mn} - \widehat{S}^{ph}_{nm}(\lambda)^* q^{el}_{ac}(\lambda-n\omega)\,\widehat{S}^{ph}_{nm}(\lambda)\right)\right).$$

Hence,

$$J^{ph}_{\rho_0,Q^{ph}} = -\sum_{m\in\mathbb{N}_0} m\rho^{ph}(m) \times \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{tr}\left(\widehat{\rho}^{el}_{ac}(\lambda-m\omega)\,\left(P_m(\lambda) - \widehat{S}^{ph}_{mm}(\lambda)^* P_m(\lambda)\,\widehat{S}^{ph}_{mm}(\lambda)\right)\right) d\lambda + \sum_{\substack{m,n\in\mathbb{N}_0 \\ m\neq n}} n\rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{tr}\left(\widehat{\rho}^{el}_{ac}(\lambda-m\omega)\,\widehat{S}^{ph}_{nm}(\lambda)^* P_n(\lambda)\,\widehat{S}^{ph}_{nm}(\lambda)\right) d\lambda.$$

Using the relation $P_m(\lambda) = I_{\mathfrak{h}(\lambda)} - \sum_{n\in\mathbb{N}_0, m\neq n} P_n(\lambda)$, $\lambda\in\mathbb{R}$, we get

$$J^{ph}_{\rho_0,Q^{ph}} = -\sum_{\substack{m,n\in\mathbb{N}_0 \\ m\neq n}} m\rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{tr}\left(\widehat{\rho}^{el}_{ac}(\lambda-m\omega)\,\left(\widehat{S}^{ph}_{nm}(\lambda)^* P_n(\lambda)\,\widehat{S}^{ph}_{nm}(\lambda)\right)\right) d\lambda + \sum_{\substack{m,n\in\mathbb{N}_0 \\ m\neq n}} n\rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{tr}\left(\widehat{\rho}^{el}_{ac}(\lambda-m\omega)\,\widehat{S}^{ph}_{nm}(\lambda)^* P_n(\lambda)\,\widehat{S}^{ph}_{nm}(\lambda)\right) d\lambda.$$

Since $\widehat{T}_{ph}(\lambda) = \widehat{S}_{ph}(\lambda) - I_{\mathfrak{h}(\lambda)}$, $\lambda\in\mathbb{R}$, we find

$$J^{ph}_{\rho_0,Q^{ph}} = -\sum_{m,n\in\mathbb{N}_0} (m-n)\rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{tr}\left(\widehat{\rho}^{el}_{ac}(\lambda-m\omega)\,\widehat{T}^{ph}_{nm}(\lambda)^*\,\widehat{T}^{ph}_{nm}(\lambda)\right) d\lambda.$$

Using (4.5) and definition (5.12), one gets

$$J^{ph}_{\rho_0,Q^{ph}} = -\sum_{m,n\in\mathbb{N}_0}(m-n)\rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{tr}\left(\rho^{el}_{ac}(\lambda-m\omega)\widetilde{T}^{ph}_{nm}(\lambda)^*\widetilde{T}^{ph}_{nm}(\lambda)\right)d\lambda\,.$$

Since $\rho^{el}_{ac}=\rho^{el}_{l}\oplus\rho^{el}_{r}$, where ρ^{el}_{α} is given by (5.2), we find

$$J^{ph}_{\rho_0,Q^{ph}} = -\sum_{\substack{m,n\in\mathbb{N}_0\\ \alpha,\varkappa\in\{l,r\}}}(m-n)\rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}}f(\lambda-\mu_\alpha-m\omega)\mathrm{tr}\left(\widetilde{T}^{ph}_{n_\varkappa m_\alpha}(\lambda)^*\widetilde{T}^{ph}_{n_\varkappa m_\alpha}(\lambda)\right)d\lambda$$

where (5.13) is used. Using $\widetilde{\sigma}^{ph}_{n_\varkappa m_\alpha}(\lambda)=\mathrm{tr}(\widetilde{T}^{ph}_{n_\varkappa m_\alpha}(\lambda)^*\widetilde{T}^{ph}_{n_\varkappa m_\alpha}(\lambda))$, we prove (5.14).

(ii) If ρ^{el}_{ac} commutes with s_c, then $\widehat{\rho}^{el}_{ac}(\lambda)=\rho^{el}_{ac}(\lambda)$, $\lambda\in\mathbb{R}$, which yields that one can replace $\widetilde{\sigma}^{ph}_{n_\varkappa m_\alpha}(\lambda)$ by $\widehat{\sigma}^{ph}_{n_\varkappa m_\alpha}(\lambda)$, $\lambda\in\mathbb{R}$. Therefore (5.15) holds.

(iii) Obviously we have

$$J^{ph}_{\rho_0,Q^{ph}} = \sum_{\substack{m,n\in\mathbb{N}_0,n>m\\ \alpha,\varkappa\in\{l,r\}}}(n-m)\rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}}f(\lambda-\mu_\alpha-m\omega)\,\widehat{\sigma}^{ph}_{n_\varkappa m_\alpha}(\lambda)d\lambda + \sum_{\substack{m,n\in\mathbb{N}_0,n<m\\ \alpha,\varkappa\in\{l,r\}}}(n-m)\rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}}f(\lambda-\mu_\alpha-m\omega)\,\widehat{\sigma}^{ph}_{n_\varkappa m_\alpha}(\lambda)d\lambda\,. \quad (5.17)$$

Moreover, a straightforward computation shows that

$$\sum_{\substack{m,n\in\mathbb{N}_0,n<m\\ \alpha,\varkappa\in\{l,r\}}}(n-m)\rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}}f(\lambda-\mu_\alpha-m\omega)\,\widehat{\sigma}^{ph}_{n_\varkappa m_\alpha}(\lambda)d\lambda = \sum_{\substack{m,n\in\mathbb{N}_0,n>m\\ \alpha,\varkappa\in\{l,r\}}}(m-n)\rho^{ph}(n)\frac{1}{2\pi}\int_{\mathbb{R}}f(\lambda-\mu_\varkappa-n\omega)\,\widehat{\sigma}^{ph}_{m_\alpha n_\varkappa}(\lambda)d\lambda.$$

Since $\mathcal{S}=\{H,H_0\}$ is time reversible symmetric, we find

$$\sum_{\substack{m,n\in\mathbb{N}_0,n<m\\ \alpha,\varkappa\in\{l,r\}}}(n-m)\rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}}f(\lambda-\mu_\alpha-m\omega)\,\widehat{\sigma}^{ph}_{m_\alpha n_\varkappa}(\lambda)d\lambda = \sum_{\substack{m,n\in\mathbb{N}_0,n>m\\ \alpha,\varkappa\in\{l,r\}}}(m-n)\rho^{ph}(n)\frac{1}{2\pi}\int_{\mathbb{R}}f(\lambda-\mu_\varkappa-n\omega)\,\widehat{\sigma}^{ph}_{n_\varkappa m_\alpha}(\lambda)d\lambda. \quad (5.18)$$

Inserting (5.18) into (5.17), we obtain (5.16). ∎

Corollary 5.10. *Let $\mathcal{S} = \{H, H_0\}$ be the JCL-model and let $f = f_{FD}$. If case* (E) *of Proposition 5.3 is realized and $\mathcal{S} = \{H, H_0\}$ is time reversible symmetric, then $J^{ph}_{\rho_0,Q^{ph}} \geq 0$.*

P r o o f. We set $\mu := \mu_l = \mu_r$. One has

$$\begin{aligned}&\rho^{ph}(m)f(\lambda-\mu-m\omega)-\rho^{ph}(n)f(\lambda-\mu-n\omega)\\ &\quad= \ e^{-m\beta\omega}(1-e^{-(n-m)\beta\omega})f_{FD}(\lambda-\mu-m\omega)f_{FD}(\lambda-\mu-n\omega)\geq 0\end{aligned}$$

for $n > m$. From (5.16), we get $J^{ph}_{\rho_0,Q^{ph}} \geq 0$. ∎

R e m a r k 5.11. Let us comment the results. If $J^{ph}_{\rho_0,Q^{ph}} \geq 0$, then the system $\mathcal{S}$ is called light emitting. Similarly, if if $J^{ph}_{\rho_0,Q^{ph}} \leq 0$, then we call it light absorbing. Of course, if $\mathcal{S}$ is light emitting and absorbing, then $J^{ph}_{\rho_0,Q^{ph}} = 0$.

(i) If $\beta = \infty$, then $\rho^{ph}(m) = \delta_{0m}$, $m \in \mathbb{N}_0$. Inserting this into (5.14), we get

$$J^{ph}_{\rho_0,Q^{ph}} = \sum_{\substack{n\in\mathbb{N}_0\\ \alpha,\varkappa\in\{l,r\}}} n\frac{1}{2\pi}\int\limits_{\mathbb{R}} f(\lambda-\mu_\alpha)\widetilde{\sigma}^{ph}_{n_\varkappa 0_\alpha}(\lambda)d\lambda \geq 0.$$

Hence $\mathcal{S}$ is light emitting.

(ii) Let us show that $\mathcal{S}$ might be light emitting even if $\beta < \infty$. We consider the case (E) of Proposition 5.3. If $\mathcal{S}$ is time reversible symmetric, then it follows from Corollary 5.10 that the system is light emitting.

If the system $\mathcal{S}$ is time reversible and mirror symmetric, then $J^{ph}_{\rho_0,Q^{el}_\alpha} = 0$, $\alpha \in \{l, r\}$, by Corollary 5.6 (iii). Since $J^{c}_{\rho_0,Q^{el}} = 0$ by Proposition 5.3, we get that $J^{\mathcal{S}}_{\rho_0,Q^{el}_\alpha} = 0$ but the photon current is larger than zero. Thus our JCL-model is light emitting by a zero total electron current $J^{\mathcal{S}}_{\rho_0,Q^{el}_\alpha}$.

Let $v_r = 0$, $v_l = 2$ and $\omega = 4$. Hence $\mathcal{S}$ is not mirror symmetric. Then we get from Remark 5.7 (iii) that $J^{ph}_{\rho_0,Q^{el}_l} = -J^{ph}_{\rho_0,Q^{el}_r} \leq 0$. Hence there is an electron current from the left to the right lead. Notice that by Proposition 5.3, $J^{c}_{\rho_0,Q^{el}_l} = 0$. Hence, $J^{\mathcal{S}}_{\rho_0,Q^{el}_l} \leq 0$.

(iii) To realize a light absorbing situation, we consider the case (S) of Proposition 5.3 and assume that $\mathcal{S}$ is time reversible symmetric. Notice that by Lemma 5.4, s_c commutes with ρ^{el}. We make the choice

$$v_r = 0, \quad v_l \geq 4, \quad \omega = v_l, \quad \mu_l = 0, \quad \mu_r = \omega = v_l.$$

It turns out that with respect to representation (5.16) one has only $m = n - 1$,

$\varkappa = r$ and $\alpha = l$. Hence,

$$J^{ph}_{\rho_0,Q^{ph}} = \sum_{n\in\mathbb{N}} \frac{1}{2\pi} \int_{\mathbb{R}} d\lambda \times \Big(\rho^{ph}(n-1)f(\lambda-(n-1)\omega) - \rho^{ph}(n)f(\lambda-(n+1)\omega)\Big)\, \widehat{\sigma}^{ph}_{n_l(n-1)_r}(\lambda).$$

Since $f(\lambda) = f_{FD}(\lambda)$, we find

$$\begin{aligned}\rho^{ph}(n-1)f(\lambda-(n-1)\omega) - \rho^{ph}(n)f(\lambda-(n+1)\omega) \\ = \rho^{ph}(n-1)f(\lambda-(n-1)\omega)f(\lambda-(n+1)\omega) \\ \times \Big(1 + e^{\beta(\lambda-(n+1)\omega)} - e^{-\beta\omega}(1+e^{\beta(\lambda-\omega(n-1))})\Big)\end{aligned}$$

or

$$\begin{aligned}\rho^{ph}(n-1)f(\lambda-(n-1)\omega) - \rho^{ph}(n)f(\lambda-(n+1)\omega) \\ = \rho^{ph}(n-1)f(\lambda-(n-1)\omega)f(\lambda-(n+1)\omega)(1-e^{-\beta\omega})(1-e^{\beta(\lambda-\omega n)}).\end{aligned}$$

Since $\lambda - n\omega \geq 0$, we find $\rho^{ph}(n-1)f(\lambda-(n-1)\omega) - \rho^{ph}(n)f(\lambda-(n+1)\omega) \leq 0$, which yields $J^{ph}_{\rho_0,Q^{ph}} \leq 0$.

To calculate $J^{ph}_{\rho_0,Q^{el}_l}$, we use formula (5.7). Setting $\alpha = l$, we get $\alpha' = r$, which yields

$$J^{ph}_{\rho_0,Q^{el}_l} = -\sum_{m,n\in\mathbb{N}_0} \frac{\mathfrak{e}}{2\pi} \int_{\mathbb{R}} d\lambda \times \Big(\rho^{ph}(n)f(\lambda-\mu_r-n\omega) - \rho^{ph}(m)f(\lambda-\mu_l-m\omega)\Big)\, \widehat{\sigma}^{ph}_{n_l m_r}(\lambda)\,.$$

One checks that $\widehat{\sigma}^{ph}_{0_l 0_r}(\lambda) = 0$ and $\widehat{\sigma}^{ph}_{n_l m_r}(\lambda) = 0$ for $m \neq n+1$, $n \in \mathbb{N}$. Hence,

$$J^{ph}_{\rho_0,Q^{el}_l} = -\sum_{n\in\mathbb{N}} \frac{\mathfrak{e}}{2\pi} \int_{\mathbb{R}} d\lambda \times \Big(\rho^{ph}(n)f(\lambda-\mu_r-n\omega) - \rho^{ph}(n-1)f(\lambda-\mu_l-(n+1)\omega)\Big)\, \widehat{\sigma}^{ph}_{n_l(n+1)_r}(\lambda)\,.$$

Since $\mu_r = \omega$ and $\mu_l = 0$, we find

$$J^{ph}_{\rho_0,Q^{el}_l} = -\sum_{n\in\mathbb{N}} \frac{\mathfrak{n}}{2\pi} \int_{\mathbb{R}} f(\lambda-(n+1)\omega)\rho^{ph}(n-1)(1-e^{-\beta\omega})\, \widehat{\sigma}^{ph}_{n_l(n+1)_r}(\lambda)\, d\lambda \leq 0.$$

Hence there is a current going from left to right induced by photons. We recall that $J^{c}_{\rho_0,Q^{el}_l} = 0$.

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