Holomorphic Riemannian Maps

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We introduce holomorphic Riemannian maps between almost Hermitian manifolds as a generalization of holomorphic submanifolds and holomorphic submersions, give examples and obtain a geometric characterization of harmonic holomorphic Riemannian maps from almost Hermitian manifolds to Kähler manifolds.

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1.   Introduction

In 1992, Fischer introduced Riemannian maps between Riemannian manifolds in [6] as a generalization of the notions of isometric immersions and Riemannian submersions, for Riemannian submersions, [8] and [11], see also [5] and [16]. Let $F : (M_1, g_1) \longrightarrow (M_2, g_2)$ be a smooth map between Riemannian manifolds such that $0 < rankF < min\{m, n\}$, where $dimM_1 = m$ and $dimM_2 = n$. Then we denote the kernel space of F_* by $kerF_*$ and consider the orthogonal complementary space $\mathcal{H} = (kerF_*)^\perp$ to $kerF_*$. Then the tangent bundle of M_1 has the following decomposition:

$$TM_1 = kerF_* \oplus \mathcal{H}.$$

We denote the range of F_* by $rangeF_*$ and consider the orthogonal complementary space $(rangeF_*)^\perp$ to $rangeF_*$ in the tangent bundle TM_2 of M_2. Since $rankF < min\{m, n\}$, we always have $(rangeF_*)^\perp$. Thus the tangent bundle TM_2 of M_2 has the following decomposition:

$$TM_2 = (rangeF_*) \oplus (rangeF_*)^\perp.$$

Now, a smooth map $F : (M_1^m, g_1) \longrightarrow (M_2^n, g_2)$ is called Riemannian map at $p_1 \in M$ if the horizontal restriction $F_{*p_1}^h : (kerF_{*p_1})^\perp \longrightarrow (rangeF_{*p_1})$ is a linear

isometry between the inner product spaces $((kerF_{*p_1})^\perp, g_1(p_1) \mid_{(kerF_{*p_1})^\perp})$ and $(rangeF_{*p_1}, g_2(p_2) \mid_{(rangeF_{*p_1})})$, $p_2 = F(p_1)$. Therefore Fischer stated in [6] that a Riemannian map is a map which is as isometric as it can be. In other words, F_* satisfies the equation

$$g_2(F_*X, F_*Y) = g_1(X, Y) \tag{1.1}$$

for X, Y vector fields tangent to $\mathcal{H}$. It follows that isometric immersions and Riemannian submersions are particular Riemannian maps with $kerF_* = \{0\}$ and $(rangeF_*)^\perp = \{0\}$. It is known that a Riemannian map is a subimmersion [6].

In this paper, as a generalization of holomorphic submersions and holomorphic submanifolds, we introduce holomorphic Riemannian maps and study the harmonicity of these maps.

2. The Gauss Equation for Riemannian Maps

Let $(\bar{M}, g)$ be an almost Hermitian manifold. This means [16] that $\bar{M}$ admits a tensor field J of type (1, 1) on $\bar{M}$ such that $\forall X, Y \in \Gamma(T\bar{M})$, and we have

$$J^2 = -I, \quad g(X, Y) = g(JX, JY). \tag{2.1}$$

An almost Hermitian manifold $\bar{M}$ is called Kähler manifold if

$$(\bar{\nabla}_X J)Y = 0, \forall X, Y \in \Gamma(T\bar{M}), \tag{2.2}$$

where $\bar{\nabla}$ is the Levi–Civita connection on $\bar{M}$.

Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $\varphi : M \longrightarrow N$ is a smooth mapping between them. Then the differential φ_* of φ can be viewed as a section of the bundle $Hom(TM, \varphi^{-1}TN) \longrightarrow M$, where $\varphi^{-1}TN$ is the pullback bundle which has fibres $(\varphi^{-1}TN)_p = T_{\varphi(p)}N, p \in M$. $Hom(TM, \varphi^{-1}TN)$ has a connection ∇ induced from the Levi–Civita connection ∇^M and the pullback connection ∇^φ. Then the second fundamental form of φ is given by

$$(\nabla\varphi_*)(X, Y) = \nabla_X^\varphi \varphi_*(Y) - \varphi_*(\nabla_X^M Y) \tag{2.3}$$

for $X, Y \in \Gamma(TM)$. It is known that the second fundamental form is symmetric. A smooth map $\varphi : (M, g_M) \longrightarrow (N, g_N)$ is said to be harmonic if $trace(\nabla\varphi_*) = 0$. On the other hand, the tension field of φ is the section $\tau(\varphi)$ of $\Gamma(\varphi^{-1}TN)$ defined by

$$\tau(\varphi) = div\varphi_* = \sum_{i=1}^{m}(\nabla\varphi_*)(e_i, e_i), \tag{2.4}$$

where $\{e_1, \ldots, e_m\}$ is the orthonormal frame on M. Then it follows that φ is harmonic if and only if $\tau(\varphi) = 0$.

For a Riemannian map, we have the following.

Lemma 2.1. [13] *Let F be a Riemannian map from a Riemannian manifold (M_1, g_1) to a Riemannian manifold (M_2, g_2). Then*

$$g_2((\nabla F_*)(X, Y), F_*(Z)) = 0, \forall X, Y, Z \in \Gamma((ker F_*)^\perp). \tag{2.5}$$

From now on, for simplicity, we will denote by ∇^2 both the Levi–Civita connection of (M_2, g_2) and its pullback along F. Then according to [10], for any vector field X on M_1 and any section V of $(range F_*)^\perp$, where $(range F_*)^\perp$ is the subbundle of $F^{-1}(TM_2)$ with the fiber $(F_*(T_pM))^\perp$-orthogonal complement of $F_*(T_pM)$ for g_2 over p, we have $\nabla_X^{F\perp} V$ which is the orthogonal projection of $\nabla_X^2 V$ on $(F_*(TM))^\perp$. In [10], the author also showed that $\nabla^{F\perp}$ is a linear connection on $(F_*(TM))^\perp$ such that $\nabla^{F\perp} g_2 = 0$. We now define $\mathcal{S}_V$ as

$$\nabla_{F_*X}^2 V = -\mathcal{S}_V F_* X + \nabla_X^{F\perp} V, \tag{2.6}$$

where $\mathcal{S}_V F_* X$ is the tangential component (a vector field along F) of $\nabla_{F_*X}^2 V$. It is easy to see that $\mathcal{S}_V F_* X$ is bilinear in V and $F_* X$ and $\mathcal{S}_V F_* X$ at p depend only on V_p and $F_{*p} X_p$. By direct computations, we obtain

$$g_2(\mathcal{S}_V F_* X, F_* Y) = g_2(V, (\nabla F_*)(X, Y)) \tag{2.7}$$

for $X, Y \in \Gamma((ker F_*)^\perp)$ and $V \in \Gamma((range F_*)^\perp)$. Since (∇F_*) is symmetric, it follows that $\mathcal{S}_V$ is a symmetric linear transformation of $range F_*$.

By using (2.6) and (2.3), we obtain the following equation which will be called the Gauss equation for a Riemannian map between Riemannian manifolds.

Lemma 2.2. *Let $F : (M_1, g_1) \longrightarrow (M_2, g_2)$ be a Riemannian map from a Riemannian manifold M_1 to a Riemannian manifold M_2. Then we have*

$$\begin{aligned}
g_2(R^2(F_*X, F_*Y)F_*Z, F_*T) &= g_1(R^1(X, Y)Z, T) + g_2((\nabla F_*)(X, Z), (\nabla F_*)(Y, T)) \\
&- g_2((\nabla F_*)(Y, Z), (\nabla F_*)(X, T))
\end{aligned} \tag{2.8}$$

for $X, Y, Z, T \in \Gamma((ker F_*)^\perp)$, where R^1 and R^2 denote the curvature tensors of ∇^1 and ∇^2 which are metric connections on M_1 and M_2, respectively.

3. Holomorphic Riemannian Maps

In this section, we define holomorphic Riemannian maps and obtain a geometric characterization of harmonic holomorphic Riemannian maps from a Kähler manifold to an almost Hermitian manifold.

Definition 1. *Let F be a Riemannian map from an almost Hermitian manifold (M_1, g_1, J_1) to an almost Hermitian manifold (M_2, g_2, J_2). Then we say that F is a holomorphic Riemannian map at $p \in M_1$ if*

$$J_2 F_* = F_* J_1. \tag{3.1}$$

If F is a holomorphic Riemannian map at every point $p \in M_1$, then we say that F is a holomorphic Riemannian map between M_1 and M_2.

It is known that the vertical and horizontal distributions of an almost Hermitian submersion are invariant with respect to the complex structure of the total manifold. Next, we will show that this is true for a holomorphic Riemannian map.

Lemma 3.1. *Let F be a holomorphic Riemannian map between the almost Hermitian manifolds (M_1, g_1, J_1) and (M_2, g_2, J_2). Then the distributions $ker F_*$ and $(ker F_*)^\perp$ are invariant with respect to J_1.*

P r o o f. For $X \in \Gamma(ker F_*)$, from (3.1) we have $F_*(J_1 X) = J_2 F_*(X) = 0$ which implies that $J_1 X \in \Gamma(ker F_*)$. In a similar way, one shows that $(ker F_*)^\perp$ is invariant. ■

Thus, it is easy to see that $(range F_*)^\perp$ is invariant under the action of J_2. We now give the examples of holomorphic Riemannian maps.

E x a m p l e 1. Every holomorphic submersion between almost Hermitian manifolds is a holomorphic Riemannian map with $(range F_*)^\perp = \{0\}$, for holomorphic (almost Hermitian) submersions, see [5, 15].

E x a m p l e 2. Every Kähler submanifold of a Kähler manifold is a holomorphic Riemannian map with $ker F_* = \{0\}$. For Kähler submanifolds, see [16].

In the following, R^{2m} denotes the Euclidean $2m$-space with the standard metric. An almost complex structure J on R^{2m} is said to be compatible if (R^{2m}, J) is complex analytically isometric to the complex number space C^m with the standard flat Kähler metric. We denote by J the compatible almost complex structure on R^{2m} defined by

$$J(a^1, \ldots, a^{2m}) = (-a^2, a^1, \ldots, -a^{2m}, a^{2m-1}).$$

E x a m p l e 3. Consider the Riemannian map given by

$$F : \quad \begin{array}{ccc} R^4 & \longrightarrow & R^4 \\ (x_1, x_2, x_3, x_4) & & (\frac{x_1+x_3}{\sqrt{2}}, \frac{x_2+x_4}{\sqrt{2}}, 0, 0). \end{array}$$

Then F is a holomorphic Riemannian map.

R e m a r k 1. We note that the notion of the invariant Riemannian map was introduced in [13] as a generalization of invariant immersion of almost Hermitian manifolds and holomorphic Riemannian submersions. One can see that every holomorphic Riemannian map is an invariant Riemannian map, but the converse is not true. In other words, an invariant Riemannian map may not be a holomorphic Riemannian map.

Since F is a subimmersion, it follows that the rank of F is constant on M_1, then the rank theorem for functions implies that $ker F_*$ is an integrable subbundle of TM_1, [1, p. 205]. We now study the harmonicity of holomorphic Riemannian maps. We first note that if M_1 and M_2 are Kähler manifolds and $F : M_1 \longrightarrow M_2$ is a holomorphic map, then F is harmonic [2]. But there is no guarantee that M_1 or M_2 is an almost Hermitian manifold.

Theorem 3.1. *Let F be a holomorphic Riemannian map from a Kähler manifold (M_1, g, J_1) to an almost Hermitian manifold (M_2, g_2, J_2). Then F is harmonic if and only if the distribution $F_*((ker F_*)^\perp)$ is minimal.*

P r o o f. Since $TM_1 = ker F_* \oplus (ker F_*)^\perp$, we can write $\tau = \tau^1 + \tau^2$, where τ^1 and τ^2 are the parts of τ in $ker F_*$ and $(ker F_*)^\perp$, respectively. First we compute $\tau^1 = \sum_{i=1}^{n_1} (\nabla F_*)(e_i, e_i)$, where $\{e_1, \ldots, e_{n_1}\}$ is a basis of $ker F_*$. From (2.3), we have

$$\tau^1 = - \sum_{i=1}^{n_1} F_*(\nabla_{e_i}^1 e_i). \tag{3.2}$$

We note that since $(ker F_*)$ is an invariant space with respect to J_1, then $\{J_1 e_i\}_{i=1}^{n_1}$ is also a basis of $ker F_*$. Thus we can write

$$\tau^1 = \sum_{i=1}^{n_1} (\nabla F_*)(J_1 e_i, J_1 e_i) = - \sum_{i=1}^{n_1} F_*(\nabla_{J_1 e_i}^1 J_1 e_i).$$

Since M_1 is a Kähler manifold and $ker F_*$ is integrable, using (3.1), we obtain

$$\tau^1 = - \sum_{i=1}^{n_1} J_2 F_*(\nabla_{e_i}^1 J_1 e_i).$$

Using again (3.1), we derive

$$\tau^1 = \sum_{i=1}^{n_1} F_*(\nabla_{e_i}^1 e_i). \tag{3.3}$$

Thus (3.2) and (3.3) imply that $\tau^1 = 0$. On the other hand, using Lemma 2.1 and (2.3), we obtain

$$\tau^2 = g_2(\sum_{s=1}^{m_2} \sum_{a=1}^{n_1} (\nabla_{e_a}^F F_*(e_a), \mu_s)\mu_s = H_{(rangeF_*)},$$

where $H_{(rangeF_*)}$ is the mean curvature vector field of $(rangeF_*)$. Then our assertion follows from the above equation and (3.3). ∎

Next, by using (2.1) and (2.2), we have the following.

Lemma 3.2. *Let F be a holomorphic Riemannian map from an almost Hermitian manifold (M_1, g_1, J_1) to a Kähler manifold (M_2, g_2, J_2). Then we have*

$$(\nabla F_*)(X, J_1 Y) = (\nabla F_*)(Y, J_1 X) = J_2 (\nabla F_*)(X, Y) \tag{3.4}$$

for $X, Y \in \Gamma((ker F_*)^{\perp})$.

Lemma 3.3. *Let F be a holomorphic Riemannian map from an almost Hermitian manifold (M_1, g_1, J_1) to a Kähler manifold (M_2, g_2, J_2). Then we have*

$$\begin{aligned}
g_1(R^1(X, J_1 X)J_1 X, X) &= g_2(R^2(F_* X, J_2 F_* X)J_2 F_* X, F_* X) \\
&\quad - 2 \parallel (\nabla F_*)(X, X) \parallel^2
\end{aligned} \tag{3.5}$$

for $X \in \Gamma((ker F_*)^{\perp})$.

P r o o f. Putting $Y = J_1 X$, $Z = J_1 X$ and $T = X$ in (2.8) and by using (3.1) and (3.4), we obtain (3.5). ∎

As a result of Lemma 3.3, we have the following result for the leaves of $(ker F_*)^{\perp}$.

Theorem 3.2. *Let F be a holomorphic Riemannian map from an almost Hermitian manifold (M_1, g_1, J_1) to a complex space form $(M_2(c), g_2, J_2)$ of constant holomorphic sectional curvature c such that $(ker F_*)^{\perp}$ is integrable. Then the integral manifold of $(ker F_*)^{\perp}$ is a complex space form $M'(c)$ if and only if $(\nabla F_*)(X, X) = 0$ for $X \in \Gamma((ker F_*)^{\perp})$.*

Concluding Remarks. It is known that the complex techniques in relativity have been very effective tools for understanding spacetime geometry [9]. Indeed, complex manifolds have two interesting classes of Kähler manifolds. One is Calabi–Yau manifolds which have their applications in superstring theory [3]. The other one is Teichmuler spaces applicable to relativity [14]. It is also important to note that CR-structures have been extensively used in spacetime geometry of relativity [12]. For complex methods in general relativity, see [4].

In [6], Fischer proposed an approach to build a quantum model and he pointed out that the success of such a program of building a quantum model of nature using Riemannian maps would provide an interesting relationship between Riemannian maps, harmonic maps, and Lagrangian field theory on the mathematical side, and Maxwell's equation, Schrödinger's equation and their proposed generalization on the physical side. It is also important to note that Riemannian maps satisfy the Eikonal equation which is a bridge between geometric optics and physical optics. For Riemannian maps and their applications in spacetime geometry, see [7]. As a unification of Riemannian maps and complex geometry, holomorphic Riemannian maps may have their applications in mathematical physics and physical optics.

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