

On the Form of Dispersive Shock Waves of the Korteweg–de Vries Equation

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We show that the long-time behavior of solutions to the Korteweg–de Vries shock problem can be described as a slowly modulated one-gap solution in the dispersive shock region. The modulus of the elliptic function (i.e., the spectrum of the underlying Schrödinger operator) depends only on the size of the step of the initial data and on the direction, $\frac{x}{t} = \text{const}$, along which we determine the asymptotic behavior of the solution. In turn, the phase shift (i.e., the Dirichlet spectrum) in this elliptic function depends also on the scattering data, and is computed explicitly via the Jacobi inversion problem.

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1. Introduction

The Korteweg–de Vries (KdV) shock problem is concerned with the long-time behavior of the solution of the KdV equation

$$q_t(x, t) = 6q(x, t)q_x(x, t) - q_{xxx}(x, t), \quad (x, t) \in \mathbb{R} \times \mathbb{R},$$

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with the steplike initial profile

$$\begin{cases} q(x, 0) \rightarrow 0, & \text{as } x \rightarrow +\infty, \\ q(x, 0) \rightarrow -c^2, & \text{as } x \rightarrow -\infty. \end{cases} \quad (1.1)$$

This behavior is well understood on a physical level of rigor for the pure-step initial data (i.e., when $q(x, 0) = 0$ as $x > 0$, $q(x, 0) = -c^2$ as $x < 0$), and was studied using a quasi-classical Whitham approach in [13, 15, 16] with further treatments using the matched-asymptotic method in [1, 21]. Extensions to steplike finite-gap backgrounds were given in [3–6] and [23]. This led to the following three main regions with different asymptotical behavior of the solution:

1. Region $x < -6c^2t$, where the solution is asymptotically close to the background $-c^2$ up to a decaying dispersive tail.
2. Middle region $-6c^2t < x < 4c^2t$, also known as dispersive shock or elliptic region, where the solution can asymptotically be described by a modulated elliptic wave.
3. Soliton region $4c^2t < x$, where the solution is asymptotically given by a sum of solitons.

We refer to our paper [10] for further details and more on the history of this problem. In this note we want to revisit the middle region which is the most interesting and challenging one from a mathematical point of view. In particular, in the case of pure-step initial data, Gurevich and Pitaevskii ([15, 16], see also [21]) derived the following large-time asymptotical formula in the elliptic region:

$$q(x, t) \sim q_{\text{GP}}(x, t) = -2c^2 \text{dn}^2(2tc(6\xi - c^2(1 + \mathbf{m}^2(\xi)), \mathbf{m}(\xi)) + c^2(1 - \mathbf{m}^2(\xi)), \quad (1.2)$$

where $\xi = \frac{x}{12t}$, and the modulus $\mathbf{m}(\xi)$ is determined implicitly by

$$\frac{c^2}{6} \left(1 + \mathbf{m}^2(\xi) - \frac{2\mathbf{m}^2(\xi)(1 - \mathbf{m}^2(\xi))K(\mathbf{m}(\xi))}{[E(\mathbf{m}(\xi)) - (1 - \mathbf{m}^2(\xi))K(\mathbf{m}(\xi))]} \right) = \xi. \quad (1.3)$$

Here $\text{dn}(s, \mathbf{m})$ is the Jacobi elliptic function and $K(\mathbf{m})$, $E(\mathbf{m})$ are the standard complete elliptic integrals. The function $q_{\text{GP}}(x, t)$ is a stationary running wave of the KdV equation if the parameter ξ is a constant.

On the other hand, in [10], under the assumption

$$\int_0^{+\infty} e^{C_0 x} (|q(x)| + |q(-x) + c^2|) dx < \infty, \quad C_0 > c > 0, \quad (1.4)$$

we derived the formula (for precise conditions, see our paper)

$$q(x, t) \sim q_{\text{RH}}(x, t) = \frac{\Gamma(\xi)B'(\xi)}{6\pi} \frac{d^2}{dv^2} \log \theta_3 \left(\frac{tB(\xi) + \Delta(\xi)}{2\pi} + v \right) \Big|_{v=0} - \frac{z'(\xi)}{6}, \tag{1.5}$$

where all quantities are associated with the elliptic Riemann surface $y^2 = \lambda(\lambda + c^2)(\lambda + a^2(\xi))$ depending on the parameter $a(\xi) > 0$ for $\xi \in (-c^2/2, c^2/3)$ implicitly determined by

$$\int_{a(\xi)}^0 \left(\xi + \frac{c^2 - a^2(\xi)}{2} - s^2 \right) \sqrt{\frac{s^2 - a^2(\xi)}{c^2 - s^2}} ds = 0. \tag{1.6}$$

As is shown in [18, 19], the point $a(\xi)$ increases monotonically from 0 to $c > 0$ as ξ changes from $-c^2/2$ to $c^2/3$. The quantities B , Γ , z and Δ are defined in (2.9)–(2.12) below.

Formula (1.5) is very reminiscent of the usual Its–Matveev formula for the one-gap solution of the KdV equation. Hence this raises two natural questions, namely, whether (1.5) agrees with (1.2) and whether they reduce to the Its–Matveev formula for the constant ξ , that is, whether (1.5) can be viewed as a slowly modulated one-gap solution of the KdV equation. The purpose of the present note is to give a positive answer to both questions.

2. Comparison between $q_{\text{RH}}(x, t)$ and the Its–Matveev Formula

We begin by recalling the well-known Its–Matveev formula for the finite-gap solution of the KdV equation in the case of one gap in the spectrum (see, for example, [14, 20, 22, 17]). To facilitate further comparison of this formula with $q_{\text{RH}}(x, t)$, we suppose that the spectrum of this one-gap solution is the set $\sigma = [-c^2, -a^2(\xi)] \cup [0, \infty)$. Let $\mathbb{M} = \mathbb{M}(\xi)$ be the elliptic Riemann surface associated with the function

$$\mathcal{R}(\lambda) := \mathcal{R}(\lambda, \xi) = \sqrt{\lambda(\lambda + c^2)(\lambda + a^2(\xi))},$$

where the cuts are taken along the spectrum σ , and $\mathcal{R}(\lambda)$ takes positive values on the upper side of the cut along the interval $[0, \infty)$. A point on \mathbb{M} is denoted by $p = (\lambda, \pm)$, $\lambda \in \mathbb{C}$, and the projection onto $\mathbb{C} \cup \{\infty\}$ is denoted by $\pi(p) = \lambda$. The sheet exchange map is given by $p^* = (\lambda, \mp)$ for $p = (\lambda, \pm)$. The sets

$$\Pi_U = \{(\lambda, +) \mid \lambda \in \mathbb{C} \setminus \sigma\} \subset \mathbb{M}, \quad \Pi_L = \{p^* \mid p \in \Pi_U\},$$

are called the upper, the lower sheets, respectively. Introduce the canonical bases of \mathfrak{a} and \mathfrak{b} cycles. The cycle \mathfrak{b} surrounds the interval $[-c^2, -a^2]$ clockwise on the

upper sheet, and the cycle \mathbf{a} supplements \mathbf{b} passing along the gap from $-a^2$ to 0 in the positive direction on the lower sheet and back from 0 to $-a^2$ on the upper sheet. Next, let $d\omega$ be a holomorphic differential on \mathbb{M} normalized by $\int_{\mathbf{a}} d\omega = 2\pi i$. Evidently,

$$d\omega = \tilde{\Gamma} \frac{d\lambda}{\mathcal{R}(\lambda)}, \quad \tilde{\Gamma} = 2\pi i \left(\int_{\mathbf{a}} \frac{d\lambda}{\mathcal{R}(\lambda)} \right)^{-1} > 0, \quad (2.1)$$

and

$$\tilde{\tau} = \int_{\mathbf{b}} d\omega < 0. \quad (2.2)$$

Define now the theta function of \mathbb{M} by

$$\theta(z) = \theta(z \mid \tilde{\tau}) = \sum_{m \in \mathbb{Z}} \exp\left\{ \frac{1}{2} \tilde{\tau} m^2 + zm \right\}.$$

Recall that this function is even and takes real values for $z \in i\mathbb{R} \cup \mathbb{R}$.

Next, following [20], introduce two meromorphic differentials of the second kind $d\Omega_1$ and $d\Omega_3$ with the vanishing \mathbf{a} periods and with the only pole at infinity of the form

$$d\Omega_1 = \frac{i}{2} \frac{\lambda - h}{\mathcal{R}(\lambda)} d\lambda, \quad d\Omega_3 = -\frac{3i}{2} \frac{(\lambda - \nu_1)(\lambda - \nu_2)}{\mathcal{R}(\lambda)} d\lambda, \quad (2.3)$$

where

$$h = \int_{\mathbf{a}} \frac{\lambda d\lambda}{\mathcal{R}(\lambda)} \left(\int_{\mathbf{a}} \frac{d\lambda}{\mathcal{R}(\lambda)} \right)^{-1} \in (-a^2, 0), \quad (2.4)$$

and the points $\nu_j \in \mathbb{R}$ are chosen such that $\int_{\mathbf{a}} d\Omega_3 = 0$, and

$$2\nu_1 + 2\nu_2 + c^2 + a^2 = 0. \quad (2.5)$$

The last equality guarantees the absence of the term of order $\lambda^{-1/2}d\lambda$ in representation for $d\Omega_3$. Note that at least one of the points ν_i lies in the gap $(-a^2, 0)$. Denote

$$iV = \int_{\mathbf{b}} d\Omega_1, \quad \text{and} \quad iW = \int_{\mathbf{b}} d\Omega_3. \quad (2.6)$$

The values V and W are called the wave number and the frequency. Evidently, $V, W \in \mathbb{R}$.

Next recall the Abel map $A(p) = \int_{\infty}^p d\omega$. Let p_0 be a point on \mathbb{M} with the projection in the gap, $\pi(p_0) \in [-a^2, 0]$. For this point, the value $A(p_0) + \mathcal{K}$ is pure imaginary (cf. [8]), where $\mathcal{K} = -\frac{\tilde{\tau}}{2} + \pi i$ is the Riemann constant. Then

the Its–Matveev formula for the one-gap solution with initial Dirichlet divisor p_0 reads (cf. [20, 22]):

$$q_{\text{IM}}(x, t) = -2 \frac{d^2}{dx^2} \log \theta (iVx - 4iWt - A(p_0) - \mathcal{K}) - a^2 - c^2 - 2h, \quad (2.7)$$

where h is defined by (2.4).

Now we will study in more details formula (1.5), where the quantities are defined as follows: $\tau \in i\mathbb{R}_+$ is the period given by

$$\tau := \tau(\xi) = - \int_{-c^2}^{-a^2(\xi)} \frac{d\lambda}{\sqrt{\lambda(\lambda + c^2)(\lambda + a^2(\xi))}} \left(\int_{-a^2(\xi)}^0 \frac{d\lambda}{\sqrt{\lambda(\lambda + c^2)(\lambda + a^2(\xi))}} \right)^{-1}, \quad (2.8)$$

and

$$\theta_3(v) = \theta_3(v \mid \tau) = \sum_{m \in \mathbb{Z}} \exp\{(m^2\tau + 2mv)\pi i\}$$

is the associated theta function. Furthermore,

$$B(\xi) = 24 \int_{a(\xi)}^c \left(\xi + \frac{c^2 - a^2(\xi)}{2} - s^2 \right) \sqrt{\frac{s^2 - a^2(\xi)}{c^2 - s^2}} ds; \quad (2.9)$$

$$\Gamma(\xi) = -\frac{1}{2} \left(\int_{-a(\xi)}^{a(\xi)} ((c^2 - s^2)(a^2(\xi) - s^2))^{-1/2} ds \right)^{-1}; \quad (2.10)$$

$$z(\xi) = \frac{12\xi(c^2 - a(\xi)^2) + 3c^4 + 9a(\xi)^4 - 6a(\xi)^2c^2}{2}; \quad (2.11)$$

$$\Delta(\xi) = 2 \int_{a(\xi)}^c \frac{\log |(\overline{T}(is)T_1(is))|}{\sqrt{(c^2 - s^2)(s^2 - a(\xi)^2)}} ds \left(\int_{-a(\xi)}^{a(\xi)} \frac{ds}{\sqrt{(c^2 - s^2)(a^2(\xi) - s^2)}} \right)^{-1}; \quad (2.12)$$

T and T_1 are the left and the right transmission coefficients of the initial data (1.1). In formulas (2.9), (2.10), and (2.12) the positive value of square root is taken.

Denote $-a^2(\xi) = \gamma(\xi) := \gamma$,

$$\mu = -\xi - \frac{c^2 + \gamma}{2}, \quad (2.13)$$

and put $\lambda = -s^2$ in (1.6). Then (1.6) is equivalent to

$$\int_{\gamma}^0 G(\lambda) d\lambda = 0, \quad G(\lambda, \xi) := \frac{(\lambda - \mu)(\lambda - \gamma)}{\mathcal{R}(\lambda)}, \quad (2.14)$$

or

$$F(\gamma, \xi) := \int_{\gamma}^0 \left(\lambda + \xi + \frac{\gamma + c^2}{2} \right) \sqrt{\frac{\lambda - \gamma}{\lambda(\lambda + c^2)}} d\lambda = 0. \quad (2.15)$$

Equation (2.15) determines the function $\gamma(\xi)$ implicitly, and the function $\mu(\xi)$ by (2.13).

Lemma 2.1. *For $\xi \in (-c^2/2, c^2/3)$ the function $\gamma(\xi)$ decays monotonically from 0 to $-c^2$, and $\mu(\xi) \in (\gamma(\xi), 0)$. Moreover,*

$$\frac{d}{d\xi} \gamma(\xi) = 4 \frac{h(\xi) - \gamma(\xi)}{3\gamma(\xi) + 2\xi + c^2}, \quad (2.16)$$

where $h(\xi)$ is defined by formula (2.4) with $\mathcal{R}(\lambda) = \sqrt{\lambda(\lambda - \gamma(\xi))(\lambda + c^2)}$.

P r o o f. The existence and uniqueness of $\gamma(\xi)$ is proved in [18]. Formally, differentiating (2.15) with respect to ξ gives $\frac{d\gamma}{d\xi} = -\frac{\partial F}{\partial \xi} \left(\frac{\partial F}{\partial \gamma} \right)^{-1}$. In turn, this implies

$$\frac{d\gamma}{d\xi} = \frac{4\mathcal{S}}{3\gamma + 2\xi + c^2}, \quad \mathcal{S} = \int_{\gamma}^0 \frac{\lambda - \gamma}{\mathcal{R}(\lambda)} d\lambda \left(\int_{\gamma}^0 \frac{d\lambda}{\mathcal{R}(\lambda)} \right)^{-1},$$

which implies (2.16). Evidently, $\mathcal{S} > 0$ for $\gamma < 0$. Therefore, to prove monotonicity of $\gamma(\xi)$, it is sufficient to prove that

$$f(\xi) = 3\gamma(\xi) + 2\xi + c^2$$

is negative for $\xi \in (-c^2/2, c^2/3)$, and that $\gamma(\xi)$ is also negative there. We observe that $F(-c^2, c^2/3) = 0$, therefore $\gamma(c^2/3) = -c^2$. Since $f(c^2/3) = -4c^2/3 < 0$, then $\gamma'(c^2/3) < 0$. Thus $\gamma(\xi)$ grows continuously starting from $-c^2$ as ξ decreases starting from $c^2/3$. The function f is a continuous function of ξ , and monotonicity of γ can only stop if there is a change of sign of f . Let ξ_0 be a point where $f(\xi_0) = 0$. Then ξ_0 satisfies the system

$$\begin{cases} 3\gamma(\xi_0) + 2\xi_0 + c^2 = 0 \\ 2\mu(\xi_0) - 2\xi_0 - c^2 - \gamma(\xi_0) = 0. \end{cases}$$

Thus, $\mu(\xi_0) = \gamma(\xi_0)$. But for $\gamma < 0$, formula (2.14) holds iff $\mu \in (\gamma, 0)$. This means that $\mu(\xi_0) = \gamma(\xi_0) = 0$, that is, $\xi_0 = -c^2/2$. ■

Lemma 2.2. Let $\gamma(\xi)$ and $\mu(\xi)$ be as in Lemma 2.1 and $h(\xi)$ as in (2.4). Let $G(\lambda, \xi)$ be defined by (2.14) and $d\Omega_j$, $j = 1, 3$ by (2.3)–(2.5). Then, for any $\xi \in \left(-\frac{c^2}{2}, \frac{c^2}{3}\right)$, the following representation is valid:

$$G d\lambda = \frac{2i}{3}d\Omega_3 - 2i\xi d\Omega_1. \tag{2.17}$$

Moreover, the following formula holds:

$$\frac{\partial}{\partial \xi}G(\lambda, \xi) = \frac{\lambda - h(\xi)}{\mathcal{R}(\lambda, \xi)}. \tag{2.18}$$

P r o o f. By (2.14), the differential $G d\lambda$ has the vanishing \mathfrak{a} period. Moreover, using (2.5), (2.13), one checks that $G d\lambda - \frac{2i}{3}d\Omega_3 + 2i\xi d\Omega_1$ has no pole at ∞ and hence must vanish. This proves (2.17).

To get (2.18), we evaluate

$$\begin{aligned} \frac{\partial}{\partial \xi}G(\lambda, \xi) - \frac{\lambda - h(\xi)}{\mathcal{R}(\lambda, \xi)} &= \frac{\partial}{\partial \xi} \left(\frac{(\lambda - \mu)\sqrt{\lambda - \gamma}}{\sqrt{\lambda(\lambda + c^2)}} \right) - \frac{\lambda - h}{\mathcal{R}(\lambda)} \\ &= \frac{(-2\mu' - \gamma')\lambda + 2\mu'\gamma + \mu\gamma' - 2\lambda + 2h}{2\mathcal{R}(\lambda)}. \end{aligned}$$

Formula (2.13) implies

$$2\mu' + \gamma' + 2 = 0. \tag{2.19}$$

Thus, to justify (2.18), one has to prove the equality

$$2\mu'\gamma + \mu\gamma' + 2h = 0.$$

By virtue of (2.16), (2.19), and (2.13), we get

$$\begin{aligned} 2\mu'\gamma + \mu\gamma' + 2h &= (-\gamma' - 2)\gamma - \gamma' \left(\xi + \frac{c^2 + \gamma}{2} \right) + 2h \\ &= -\frac{1}{2}\gamma'(3\gamma + c^2 + 2\xi) - 2\gamma + 2h = 0, \end{aligned}$$

which proves (2.18). ■

Lemma 2.3. Let $B(\xi)$, $\Gamma(\xi)$, and $z(\xi)$ be defined by (2.9), (2.10), and (2.11), respectively. Let $V(\xi)$ and $W(\xi)$ be the wave number and the frequency, defined by (2.6), and let $h(\xi)$ be as in (2.4). Then the following identities hold:

$$tB(\xi) = -4W(\xi)t + V(\xi)x, \tag{2.20}$$

$$(a) \frac{d}{d\xi}B(\xi) = 12V(\xi), \quad (b) 4\pi\Gamma(\xi) = -V(\xi), \tag{2.21}$$

$$\frac{1}{6} \frac{d}{d\xi}z(\xi) = c^2 + a^2(\xi) + 2h(\xi). \tag{2.22}$$

P r o o f. To get (2.20), we make change of variables $\lambda = -s^2$ in (2.9), and take into account (2.17), (2.13), $\xi = \frac{x}{12t}$, and the definition of the \mathfrak{b} period on the Riemann surface $\mathbb{M}(\xi)$. Then

$$\begin{aligned} tB(\xi) &= -12t \int_{-a^2}^{-c^2} (\lambda - \mu) \sqrt{\frac{-a^2 - \lambda}{-\lambda(\lambda + c^2)}} d\lambda = 12t \int_{-c^2}^{\gamma} G(\lambda, \xi) d\lambda \\ &= 4it \int_{\mathfrak{b}} d\Omega_3 - ix \int_{\mathfrak{b}} d\Omega_1 = -4tW(\xi) + xV(\xi). \end{aligned}$$

This proves (2.20). Formula (2.21), (a) follows from (2.17) and (2.18):

$$\frac{d}{d\xi} B(\xi) = 12 \frac{d}{d\xi} \int_{-c^2}^{\gamma} G(\lambda, \xi) d\lambda = 12\gamma'(\xi)G(\gamma(\xi), \xi) - 12i \int_{\mathfrak{b}} d\Omega_1 = 12V(\xi).$$

Next, by definition of the \mathfrak{a} period, formula (2.10) reads:

$$\Gamma(\xi) = \left(2 \int_0^{-a^2} \frac{d\lambda}{\sqrt{-\lambda(\lambda + c^2)(\lambda + a^2)}} \right)^{-1} = \frac{i}{\int_{\mathfrak{a}} \frac{d\lambda}{\mathcal{R}(\lambda)}} = \frac{\tilde{\Gamma}}{2\pi}, \quad (2.23)$$

where $\tilde{\Gamma}$ is the normalization constant from (2.1). On the other side, formulas (2.1), (2.3), and the residue theorem [12] yield

$$\int_{\mathfrak{a}} d\omega \int_{\mathfrak{b}} d\Omega_1 = 2\pi i \int_{\mathfrak{b}} d\Omega_1 = 2\pi i \operatorname{Res}_{\infty} \left(d\Omega_1(p) \int_{\infty}^p d\omega \right), \quad p = (\lambda, +).$$

Since in the local parameter $z = \lambda^{-1/2}$, $z \rightarrow 0$, we have

$$d\Omega_1 = \left(\frac{i}{2\sqrt{\lambda}} + O\left(\frac{1}{\lambda^{3/2}}\right) \right) d\lambda = -\frac{i}{z^2}(1 + o(1))dz,$$

and

$$\int_{\infty}^{\lambda} d\omega = -\frac{2\tilde{\Gamma}}{\sqrt{\lambda}} + O\left(\frac{1}{\lambda^{3/2}}\right) = -2\tilde{\Gamma}z(1 + o(1)),$$

then, by (2.23),

$$\operatorname{Res}_{\infty} \left(d\Omega_1(p) \int_{\infty}^p d\omega \right) = -2i\tilde{\Gamma} = -4\pi i\Gamma.$$

Together with (2.6), this proves (2.21), (b). To prove the remaining formula (2.22), we represent $z(\xi)$ via $\gamma(\xi)$ and apply (2.16). Then we get

$$\begin{aligned} \frac{d}{d\xi}z(\xi) &= \frac{1}{2} \frac{d}{d\xi} (12\xi(c^2 + \gamma(\xi)) + 3c^4 + 9\gamma(\xi)^2 + 6\gamma(\xi)c^2) \\ &= 6(c^2 + \gamma(\xi)) + (6\xi + 9\gamma(\xi) + 3c^2) \gamma'(\xi) \\ &= 6(c^2 + \gamma(\xi)) + 12(h(\xi) - \gamma(\xi)) = 6(c^2 + a^2(\xi)) + 12h(\xi). \quad \blacksquare \end{aligned}$$

Now we are ready to compare formula (1.5) with (2.7). Comparing the periods τ defined by (2.8) and $\tilde{\tau}$ defined by (2.1), (2.2), we observe that $2\pi i\tau = \tilde{\tau}$, and therefore

$$\theta(z \mid \tilde{\tau}) = \theta_3\left(\frac{z}{2\pi i} \mid \tau\right).$$

Put

$$\mathcal{B}(\xi) = \frac{\Gamma(\xi)}{6\pi} \left(\frac{d}{d\xi} \mathcal{B}(\xi) \right), \quad \mathcal{C}(\xi) = \frac{1}{6} \frac{d}{d\xi} z(\xi). \quad (2.24)$$

Then $q_{\text{RH}}(x, t)$ can be represented as

$$\begin{aligned} q_{\text{RH}}(x, t) &= \mathcal{B}(\xi) \frac{d^2}{dv^2} \log \theta_3 \left(\frac{it\mathcal{B}(\xi) + i\Delta(\xi) + 2i\pi v}{2\pi i} \right) \Big|_{v=0} + \mathcal{C}(\xi), \\ &= -4\pi^2 \mathcal{B}(\xi) \frac{d^2}{dv^2} \log \theta(it\mathcal{B}(\xi) + i\Delta(\xi) + v) \Big|_{v=0} + \mathcal{C}(\xi). \quad (2.25) \end{aligned}$$

On the other hand, by (2.22), we have

$$\begin{aligned} q_{\text{IM}}(x, t) - \mathcal{C}(\xi) \\ = -2(iV(\xi))^2 \frac{d^2}{dv^2} \log \theta(iV(\xi)x - 4iW(\xi)t - A(p_0, \xi) - \mathcal{K}(\xi) + v) \Big|_{v=0}. \end{aligned}$$

Substituting (2.20) and (2.21) into (2.24) and then into (2.25), we get $\mathcal{B}(\xi) = -V^2(\xi)$ and $it\mathcal{B}(\xi) = iV(\xi)x - 4iW(\xi)t$. We conclude that $q_{\text{RH}}(x, t) = q_{\text{IM}}(x, t)$ iff there exists a point p_0 , $\pi(p_0) \in [-a^2, 0]$ such that the following equality is fulfilled:

$$-A(p_0, \xi) - \mathcal{K}(\xi) = i\Delta(\xi) \pmod{2\pi i},$$

where $\Delta(\xi)$ is defined by (2.12). Since

$$A(-c^2) = \pi i \pmod{2\pi i}, \quad A(-a^2) - A(-c^2) = \frac{\tilde{\tau}}{2}, \quad \mathcal{K} = -\frac{\tilde{\tau}}{2} + \pi i,$$

and $\Delta(\xi)$ is a real value, then the point $p_0(\xi)$ can be found as the unique solution of the Jacobi inversion problem (cf. [12]):

$$\int_{-a^2(\xi)}^{p_0(\xi)} d\omega = -i\Delta(\xi). \quad (2.26)$$

In summary we have proved

Theorem 2.4. *For any fixed $\xi = \frac{x}{12t} \in (-c^2/2, c^2/3)$ the function $q_{RH}(x, t)$ is the usual one-gap solution of the KdV equation:*

$$q_{RH}(x, t) = -2 \frac{d^2}{dx^2} \log \theta (iV(\xi)x - 4iW(\xi)t - A(p_0, \xi) - \mathcal{K}(\xi)) - a(\xi)^2 - c^2 - 2h(\xi),$$

associated with the spectrum $[-c^2, -a^2(\xi)] \cup [0, \infty)$ and the Dirichlet divisor $p_0(\xi)$ defined via the Jacobi inversion problem (2.26).

Note that since $\theta(z + 2\pi i) = \theta(z)$, we see that $q_{RH}(x, t)$ is a periodic function with respect to x and t of the periods $\frac{2\pi}{V(\xi)}$ and $\frac{\pi}{2W(\xi)}$, respectively.

3. Copmarison between $q_{RH}(x, t)$ and the Gurevich–Pitaevskii Formula

Our next aim is to find a connection between the two functions $m(\xi)$ and $a(\xi)$ implicitly given by (1.3) and (1.6).

Lemma 3.1. *The following is true:*

$$m^2(\xi) = \frac{a^2(\xi)}{c^2}.$$

P r o o f. Represent (1.6) as

$$0 = \left(-\frac{c^2 + a^2(\xi)}{2} + \xi \right) \int_0^{a(\xi)} \sqrt{\frac{a^2(\xi) - s^2}{c^2 - s^2}} ds + \int_0^{a(\xi)} \sqrt{(a^2(\xi) - s^2)(c^2 - s^2)} ds = \left(\xi - \frac{c^2 + a^2(\xi)}{2} \right) I_1(\xi) + I_2(\xi). \quad (3.1)$$

Put $m := m(\xi) = \frac{a(\xi)}{c}$. Then (cf. [9], formulas 781.61 and 781.22)

$$I_1 = c(E(m) - (1 - m^2)K(m)), \quad I_2 = \frac{c^3}{3} \{ (m^2 - 1)K(m) + (m^2 + 1)E(m) \},$$

where $K(m)$ and $E(m)$ are the standard complete elliptic integrals for $0 < m < 1$. Substituting this into (3.1), we obtain

$$\begin{aligned} \xi &= c^2 \left\{ \frac{m^2 + 1}{2} - \frac{1}{3} \frac{(m^2 - 1)K(m) + (m^2 + 1)E(m)}{E(m) - (1 - m^2)K(m)} \right\} \\ &= \frac{c^2}{6} \frac{E(m)(m^2 + 1) - (1 - m^2)(m^2 + 1)K(m) + 2m^2(m^2 - 1)K(m)}{E(m) - (1 - m^2)K(m)} \\ &= \frac{c^2}{6} \left((m^2 + 1) + \frac{2m^2(m^2 - 1)K(m)}{E(m) - (1 - m^2)K(m)} \right). \end{aligned}$$

Thus, (1.3) and (1.6) define the same function: $\mathbf{m}(\xi) = m(\xi)$. ■

Lemma 3.1 implies that the period (2.8) corresponds in the standard way (cf. [2]) to the elliptic modulus m , and the following formula is valid:

$$\operatorname{dn}^2(s, m) = \frac{d^2}{du^2} \log \Theta(u | \tau) + \frac{E(m)}{K(m)},$$

where

$$\Theta(u | \tau) = \theta_3 \left(\frac{u}{2K(m)} + \frac{1}{2} | \tau \right) = \theta_3 \left(\frac{u}{2K(m)} + \frac{1}{2} \right),$$

and $\theta_3(s)$ is as in (1.5).

For the remainder, we will fix ξ and omit it from our notation. Fixing ξ , we also fix $m = a c^{-1}$ and, consequently, we will also omit m in the complete elliptic integrals: $K := K(m)$, $E := E(m)$. Therefore,

$$\begin{aligned} q_{GP}(x, t) &= -\frac{c^2}{2K^2} \frac{d^2}{dv^2} \log \theta_3 \left(\frac{tc(6\xi - c^2(1 + m^2))}{K} + \frac{1}{2} + v \right) \\ &\quad + c^2(1 - m^2) - \frac{2E}{K}. \end{aligned} \tag{3.2}$$

Thus, to convince ourselves that this formula coincides with (1.5) (possibly, up to a phase shift), it is sufficient to show:

Lemma 3.2. *The following equalities hold:*

$$-\frac{c^2}{2K^2} = \frac{\Gamma}{6\pi} \frac{dB}{d\xi}, \tag{3.3}$$

$$-\frac{6c\xi - c^3(1 + m^2)}{K} = \frac{B}{2\pi}; \tag{3.4}$$

$$c^2 - \frac{E}{K} = -h, \tag{3.5}$$

where B , Γ , and h are defined by (2.9), (2.10), and (2.4), respectively.

P r o o f. Formulas (2.21), (2.10), and 781.01 of [9] imply

$$\frac{\Gamma}{6\pi} \frac{dB}{d\xi} = -8\Gamma^2 = -\frac{1}{2} \left(\int_0^a \frac{ds}{\sqrt{(c^2 - s^2)(a^2 - s^2)}} \right)^{-2} = -\frac{c^2}{2K^2}. \quad (3.6)$$

Next, by (2.4) and formula 781.11 from [9], we have

$$-h = \int_0^a \frac{s^2 ds}{\sqrt{(c^2 - s^2)(a^2 - s^2)}} \left(\int_0^a \frac{ds}{\sqrt{(c^2 - s^2)(a^2 - s^2)}} \right)^{-1} = \frac{c(K - E)}{\frac{1}{c}K}.$$

This gives (3.5). To prove (3.4), we use (2.20) and (2.21). Namely, we have

$$\frac{tB}{2\pi} = -2\Gamma x - 4Wt = -2(x - 2(c^2 + a^2)t)\Gamma.$$

Here we used equation (4.4.13) of [22] which implies the equality $W = (c^2 + a^2)\Gamma$. By (3.6), we have $\Gamma = \frac{c}{4K}$. Thus,

$$\frac{B}{2\pi} = -\frac{tc}{K} (6\xi - (c^2 + a^2)),$$

which proves (3.4). Note that the opposite sign with respect to (3.2) of the first summand in θ_3 is not essential as θ_3 is even. ■

We proved that formulas (1.5) and (1.2) represent the same function up to the phase shift. Namely, instead of the summand $(2\pi)^{-1}\Delta$ in the argument of the theta function for $q_{RH}(x, t)$, in the same formula for $q_{GP}(x, t)$ we have the summand $\frac{1}{2}$. Recall now that the transmission coefficients for the pure step potential have the representation (cf. [7, 11])

$$T(k) = \frac{2ik}{w(k)}, \quad T_1(k) = \frac{2i\sqrt{k^2 + c^2}}{w(k)}, \quad w(k) = ik + i\sqrt{k^2 + c^2}.$$

Since in (2.12) $k = is$, $s \in [0, c]$, then

$$|w(k)|^2 = |-s + i\sqrt{c^2 - s^2}|^2 = c^2,$$

and

$$|T(is)T_1(is)| = 4sc^{-2}\sqrt{c^2 - s^2}, \quad s \in [0, c].$$

Therefore, the value of the phase shift for the pure step case is given by

$$\begin{aligned} \frac{\Delta_{ps}}{2\pi} &= \frac{\int_a^c \log \left(4sc^{-2}\sqrt{c^2 - s^2} \right) \left((c^2 - s^2)(s^2 - a^2) \right)^{-1/2} ds}{2\pi \int_0^a \left((c^2 - s^2)(s^2 - a^2) \right)^{-1/2} ds} \\ &= \frac{1}{2\pi K(m)} \int_m^1 \frac{\log \left(4s\sqrt{1 - s^2} \right)}{\sqrt{(1 - s^2)(s^2 - m^2)}} ds. \end{aligned}$$

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