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Solutions of Nonlinear Schrödinger Equation with Two Potential Wells in Linear/Nonlinear Media

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In the framework of nonlinear Schrödinger equation, we analytically studied the nonlinear localized states in the system with two potential holes in the cases of linear and nonlinear media in the holes as well as their linear and nonlinear environment. All the possible solutions for the system are found and studied. The frequency dependences of the field amplitudes for all types of possible stationary localized states are obtained.

Key words: nonlinear Schrödinger equation, nonlinear localized states, potential well.

Mathematics Subject Classification 2010: 35Q55; 37K10.

1. Introduction

In the paper, we consider the actual problem of analytical research of the character of localization of nonlinear stationary waves propagating in an anharmonic medium along thin plane-parallel layers with different physical properties. As is well known [1, 2], the nonlinearity of the medium can give rise to new physical effects such as dependence of the transparency of a medium on the power of the transmitted wave, spatial localization of nonlinear waves in periodic arrays of optical waveguides, etc. From the point of view of technological applications, of special interest are layered and modulated structures of various types.

In nonlinear optics, where layered and modulated structures can be applied in optical communication systems, optical fibers, photonic crystals, optical delay lines [1], the studying of localized states in the system with two linear/nonlinear defect layers, for instance, in optical switches [4, 5], and in periodic modulated structures [1, 3] are very actual. Also of importance is the study of the adsorption of polymer chains in the systems with interfaces modeled by δ-functions or potential wells (in some cases, by nonlinear or/and asymmetric ones) [6–8]. The study of the structures with potential wells of this type is also important in the theory of Bose–Einstein condensation [9].

In the present paper, in the framework of nonlinear Schrödinger equation we study analytically the character of localization of nonlinear stationary waves in a model system, which is a medium with linear or nonlinear properties at presence of the potential in the form of two rectangular wells also with linear and nonlinear properties. All possible solutions of the nonlinear Schrödinger equation for this system are found and studied under the conditions of continuity of the wave function and its first derivative at the boundaries of the potential wells and the environment.

The exact solutions are found and the character of localization of nonlinear stationary waves is studied for all possible combinations: (1) continuous linear medium in the system; (2) nonlinear medium in the potential wells and linear medium in the surrounding regions; (3) medium with linear properties in the potential wells and nonlinear medium in the surrounding regions; (4) continuous nonlinear medium in the system. The frequency dependences of the field amplitudes for all types of possible stationary localized states are obtained.

2. Formulation of the Problem

We describe a model system consisting of two equal potentials U_0 in the form of symmetrical rectangular wells with the width d and placed at the distance $2a$ from each other. Let us consistently consider all the possible cases when the medium inside the wells ($a < |x| < a + d$), as well as the environment, has linear and nonlinear properties.

To solve this problem, we divide the system of potentials under consideration along the axis of coordinates into five regions as shown in Fig. 1. We will seek solutions separately for each region (the potential in each region is assumed to be constant) and then make a "cross-linking" of these regions, taking into account the equality of wave functions and their derivatives at the boundaries. Thus, the problem reduces to finding the solutions $\Psi_i (i = 1, 2, \ldots, 5)$ of the one-dimensional stationary nonlinear Schrödinger equation in the form

$$\lambda\,\Psi = -\frac{d^2\Psi}{dx^2} + 2\Psi^3 - U_0\,\Psi, \tag{1}$$

satisfying the following boundary conditions on the boundaries of the regions of a partition:

$$\begin{cases} \Psi_1|_{-a-d-0} = \Psi_4|_{-a-d+0}, & \Psi'_1|_{-a-d-0} = \Psi'_4|_{-a-d+0}, \\ \Psi_4|_{-a-0} = \Psi_3|_{-a+0}, & \Psi'_4|_{-a-0} = \Psi'_3|_{-a+0}, \\ \Psi_3|_{a-0} = \Psi_5|_{a+0}, & \Psi'_3|_{a-0} = \Psi'_5|_{a+0}, \\ \Psi_5|_{a+d-0} = \Psi_2|_{a+d+0}, & \Psi'_5|_{a+d-0} = \Psi'_2|_{a+d+0}. \end{cases} \tag{2}$$

The eigenfunction Ψ is corresponded to the eigenvalue λ.

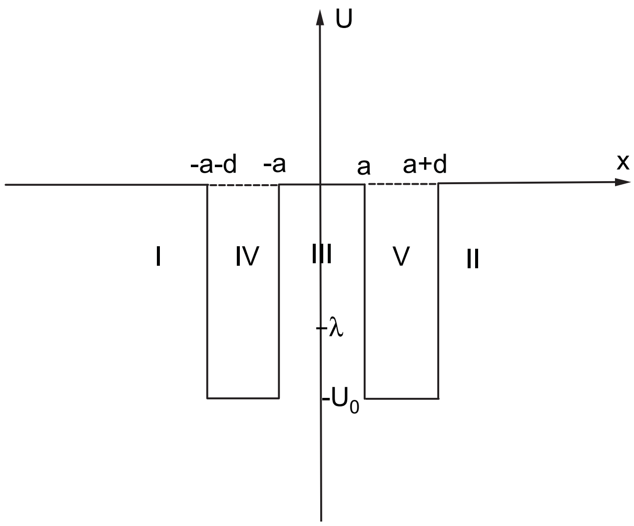

Fig. 1. Investigated system with potential wells.

Notice that in the regions (I–III) the values of λ can be only non-positive, $\lambda \leq 0$, and in the regions (IV–V) the values of λ are in the range of $-U_0 \leq \lambda \leq 0$. Let us consider each of the cases separately.

3. Linear Medium in the Potential Wells and Linear Environment

For a given system, Eq. (1) in the respective regions is reduced to the equations

$$\lambda\, \Psi_{1,2,3} = -\frac{d^2\Psi_{1,2,3}}{dx^2}, \tag{3}$$

$$\lambda\, \Psi_{4,5} = -\frac{d^2\Psi_{4,5}}{dx^2} - U_0\, \Psi_{4,5}. \tag{4}$$

The general solution of Eqs. (3) has the form

$$\Psi_{1,2,3} = A_{1,2,3}\, e^{-\sqrt{|\lambda|}\,x} + B_{1,2,3}\, e^{\sqrt{|\lambda|}\,x}. \tag{5}$$

Due to the finiteness of the wave function, the solutions Ψ_1 and Ψ_2 must vanish at $x \to \infty$, thus the coefficients of these solutions A_1 and B_2 should be set to be equal to zero. Accounting for symmetry of the wave functions $\Psi_2(x) = \Psi_1(-x)$ and $\Psi_3(x) = \Psi_3(-x)$ allows us to simplify the form of the solutions of (3):

$$\Psi_1 = A\, e^{\sqrt{|\lambda|}\,x}, \Psi_2 = A\, e^{-\sqrt{|\lambda|}\,x}, \tag{6}$$

$$\Psi_3 = A_3 \left(e^{-\sqrt{|\lambda|}\,x} + e^{\sqrt{|\lambda|}\,x} \right) = 2A_3 \mathrm{ch} \left(\sqrt{|\lambda|}\, x \right), \tag{7}$$

where $A \equiv A_2 = B_1$ and $A_3 = B_3$.

The solution of Eq. (4) is conveniently written in the form

$$\Psi_{4,5} = C_{4,5}\, \cos \varepsilon x + D_{4,5}\, \sin \varepsilon x, \tag{8}$$

where $\varepsilon^2 = U_0 - |\lambda|$.

By satisfying the boundary conditions (2) and taking into account the symmetry of the functions $\Psi_4(-x) = \Psi_5(x)$, the coefficients C and D in (8) can be expressed in terms of the coefficients A and A_3 which are connected by the relation

$$A = 2A_3 \left[\mathrm{ch} \left(\sqrt{|\lambda|}\, a \right)\, \cos \varepsilon d + \mu\, \mathrm{sh} \left(\sqrt{|\lambda|}\, a \right)\, \sin \varepsilon d \right], \tag{9}$$

where $\mu = \frac{\sqrt{|\lambda|}}{\varepsilon}$.

Finally, using the normalization condition for a linear medium, $N = \int\limits_{-\infty}^{+\infty} |\Psi|^2\, dx = 1$, we find the solutions of the Schrödinger equation in each of the five regions:

$$\Psi_{1,2}(x) = \frac{\mathrm{ch}^2\left(\sqrt{|\lambda|}\, a\right) + \mu^2\, \mathrm{sh}^2\left(\sqrt{|\lambda|}\, a\right) - \mu^2\, \mathrm{sh}^2\left(\sqrt{|\lambda|}\, a\right)}{\sqrt{\Omega}\sqrt{(1+\mu^2)}\sqrt{\mathrm{ch}^2\left(\sqrt{|\lambda|}\, a\right) + \mu^2\, \mathrm{sh}^2\left(\sqrt{|\lambda|}\, a\right)}}\, e^{\sqrt{|\lambda|}\,(a+d\pm x)}, \tag{10}$$

$$\Psi_3 = \frac{\mathrm{ch}\left(\sqrt{|\lambda|}x\right)}{\sqrt{\Omega}}, \tag{11}$$

$$\Psi_{4,5} = \frac{\mathrm{ch}\sqrt{|\lambda|}\, a\, \cos \varepsilon a - \mu \mathrm{sh}\sqrt{|\lambda|}\, a\, \sin \varepsilon a}{\sqrt{\Omega}}\, \cos \varepsilon x$$
$$\mp \frac{\mathrm{ch}\sqrt{|\lambda|}\, a\, \sin \varepsilon a + \mu\, \mathrm{sh}\sqrt{|\lambda|}\, a\, \cos \varepsilon a}{\sqrt{\Omega}}\, \sin \varepsilon x, \tag{12}$$

where it is assumed that

$$\Omega = a + d\left(\mathrm{ch}^2\sqrt{|\lambda|}\,a + \mu^2\,\mathrm{sh}^2\sqrt{|\lambda|}\,a\right) + \frac{1}{\sqrt{\lambda}}\left(\mathrm{sh}\left(2\sqrt{|\lambda|}\,a\right) + \mathrm{ch}^2\sqrt{|\lambda|}\,a\right)$$
$$+ 2\,\frac{\mu}{\varepsilon}\,\frac{\mathrm{sh}\sqrt{|\lambda|}\,a\left(\mathrm{ch}\sqrt{|\lambda|}\,a - \mathrm{sh}\sqrt{|\lambda|}\,a\right)}{\mathrm{ch}^2\sqrt{|\lambda|}\,a + \mu^2\,\mathrm{sh}^2\sqrt{|\lambda|}\,a}. \qquad (13)$$

4. Nonlinear Medium in Potential Wells and Linear Environment

It is obvious that the solutions in the regions with a linear medium do not differ from the corresponding solutions (6), (7). The equations that describe the media in potential wells become nonlinear,

$$\lambda\,\Psi_{4,5} = -\frac{d^2\Psi_{4,5}}{dx^2} + 2\Psi_{4,5}^3 - U_0\,\Psi_{4,5}, \qquad (14)$$

and can be reduced to the equation for the Duffing oscillator with soft nonlinearity. Similarly to the Duffing equation, in the main anharmonic approximation they describe small-amplitude oscillations in the symmetric potential and have periodic solutions in the energy interval $0 < E < E_0 = \frac{\varepsilon^4}{8}$ [10].

By multiplying (14) for Ψ by $2\Psi'$ and then integrating, we reduce it to the form

$$(\Psi')^2 = \Psi^4 - \varepsilon^2\,\Psi^2.$$

Separating the variables in the resulting equation and using an integral of motion (energy), we can get its solution in an implicit form

$$\int\limits_0^{\Psi} \frac{d\Psi}{\sqrt{2E + \Psi^4 - \varepsilon^2\,\Psi^2}} = x - x_0. \qquad (15)$$

Expressing the energy E by the dimensionless parameter $q = \sqrt{\frac{\varepsilon^4}{8E}} - \sqrt{\frac{\varepsilon^4}{8E} - 1}$ and changing the variable according to $\Psi = \frac{\varepsilon\,q}{\sqrt{1+q^2}}$, an implicit solution (15) can be transformed to the elliptic integral of the first kind:

$$F\left(\arcsin\frac{\Psi}{\Psi*}, q\right) = \int\limits_0^{\frac{\Psi}{\Psi*}} \frac{dz}{\sqrt{(1-z^2)\,(1-q^2z^2)}} = \frac{\varepsilon\,(x - x_0)}{\sqrt{1+q^2}}, \qquad (16)$$

where $\Psi* = \frac{\varepsilon\,q}{\sqrt{1+q^2}}$ corresponds to the amplitude of the oscillations of the Duffing pendulum, and q is the elliptic integral modulus.

Finally, by expressing the sine of the amplitude through elliptic sine $sn(u,q) \equiv \sin am(u,q)$ and taking into account the condition of periodicity $sn(u+K,q) = \frac{cn(u,q)}{dn(u,q)}$ [11], the solution (16) can be written in an explicit form:

$$\Psi = \gamma q \, \frac{cn(\gamma x, q)}{dn(\gamma x, q)}, \tag{17}$$

where $\gamma = \frac{\varepsilon}{\sqrt{1+q^2}}$.

Under the requirement that the solutions in linear (6), (7) and nonlinear (17) media be "cross-linked" on the borders of the regions, from the boundary conditions (2), we obtain

$$\Psi_{1,2} = q\gamma \, \frac{cn\left(\gamma(a+d), q\right)}{dn\left(\gamma(a+d), q\right)} e^{\sqrt{|\lambda|}(a+d\pm x)}, \tag{18}$$

$$\Psi_3 = \frac{\sqrt{|\lambda|\, q^2\, \gamma^2\, cn^2(\gamma a, q)\, dn^2(\gamma a, q) - q^2\, q'^4\, \gamma^4\, sn^2(\gamma a, q)}}{\sqrt{|\lambda|}\, dn^2(\gamma a, q)} \, \mathrm{ch}\left(\sqrt{|\lambda|}\, x\right), \tag{19}$$

$$\Psi_{4,5}(x) = \gamma \, q \, \frac{cn(\gamma x, q)}{dn(\gamma x, q)}, \tag{20}$$

where $\gamma \equiv \gamma_4 = \gamma_5$, $q \equiv q_4 = q_5$, and $q' = \sqrt{1-q^2}$. The parameter q as a function of the parameters of the system a, d and λ is determined by the equality

$$\sqrt{|\lambda|}\, cn\left(\gamma\,(a+d), q\right) = q'^2\, \gamma \, \frac{sn\left(\gamma(a+d), q\right)}{dn\left(\gamma(a+d), q\right)}. \tag{21}$$

Solutions (18)–(21) allow us to determine the full number of elementary excitations due to the nonlinearity of the medium in potential wells:

$$\begin{aligned} N &= 2\,\gamma^2\, d - 2\,\gamma\left[E\left(am(\gamma(a+d), q), q\right) - E\left(am(\gamma a, q), q\right)\right] \\ &+ \frac{q^2\,\gamma^2}{\sqrt{|\lambda|}}\, \frac{cn^2\left(\gamma(a+d), q\right)}{dn^2\left(\gamma(a+d), q\right)} + \left(2a + \mathrm{sh}2\sqrt{|\lambda|}\, a\right) \\ &\times \frac{|\lambda|\, q^2\, \gamma^2\, cn^2(\gamma a, q)\, dn^2(\gamma a, q) - q^2\, q'^4\, \gamma^4\, sn^2(\gamma a, q)}{|\lambda|^{3/2}\, dn^4(\gamma a, q)} \\ &+ 2\,\gamma\, q^2\left[\Lambda_1\left(\gamma(a+d), q\right) - \Lambda_1(\gamma a, q)\right]. \end{aligned} \tag{22}$$

Here $\Lambda_1(\gamma a, q) = \frac{sn(\gamma a, q)\, cn(\gamma a, q)}{dn(\gamma a, q)}$, $E(\varphi, q)$ is the elliptic integral of the second kind, and $am(\varphi, q) = \arcsin\left[sn(\varphi, q)\right]$ is the elliptic amplitude [11].

5. Linear Medium in the Potential Wells and Nonlinear Environment

In this case, the environment outside the potential wells is described by the nonlinear equations in the form

$$\lambda\, \Psi_{1,2,3} = -\frac{d^2 \Psi_{1,2,3}}{dx^2} + 2\, \Psi^3_{1,2,3}, \tag{23}$$

the solutions of which in the symmetric regions I and II are given by

$$\Psi_{1,2} = \mp \frac{\varepsilon_0}{\mathrm{sh}\,(\varepsilon_0 (x \mp x_0))}, \tag{24}$$

where $\varepsilon_0^2 = |\lambda|$ and $x_0 > -(a+d)$.

We will seek the solutions of Eq. (23) in region III in the form

$$\Psi = \frac{A}{cn(Bx)}.$$

Because of the correlation between the constants, $A = q' B$, where $B = \gamma_3 = \frac{\varepsilon_0}{\sqrt{2\,q^2 - 1}}$ and $\frac{1}{\sqrt{2}} < q = q_3 \leq 1$, we obtain

$$\Psi_3 = \frac{q'\,\gamma_3}{cn(\gamma_3\, x, q_3)}. \tag{25}$$

The linear medium in potential wells is described by equations (4). The solutions of these Eqs. (8), which satisfy the boundary conditions (2), for nonlinear environment take the form

$$\Psi_{4,5} = \frac{q'\,\gamma_3}{cn\,(\gamma_3\, a, q)}\, \left[\gamma_3\, \Lambda_2\, (\gamma_3\, a, q)\, \sin \varepsilon (a \pm x) + \cos \varepsilon (a \pm x)\right], \tag{26}$$

where $\Lambda_2\,(\gamma_3\, a, q) = \frac{sn\,(\gamma_3\, a, q)\, dn\,(\gamma_3\, a, q)}{cn\,(\gamma_3\, a, q)}$.

The total number of elementary excitations is given by the expression

$$\begin{aligned} N = 2 \Bigg\{ & \sqrt{\varepsilon_0^2 + \frac{q'^2_3\, \gamma_3^2\ \alpha^2}{cn^2\,(\gamma_3\, a, q_3)}} - \varepsilon_0 + \gamma_3 \left[q'^2\,(\gamma_3, q) + \Lambda_2\,(\gamma_3\, a, q) \right. \\ & \left. - E\,(am(\gamma_3\, a, q), q) \right] + \frac{q'^2\, \gamma_3^2}{2\, cn^2\,(\gamma_3\, a, q)} \Bigg[d\, \left(1 + \gamma_3^2\, \Lambda_2^2\,(\gamma_3\, a, q)\right) \\ & + \frac{1}{2\,\varepsilon} [4\, \gamma_3\, \Lambda_2\, \sin^2 \varepsilon\, d + (1 - \gamma_3^2\, \Lambda_2^2) \sin 2\, \varepsilon\, d] \Bigg] \Bigg\}, \end{aligned} \tag{27}$$

where the parameter $\alpha = \cos\varepsilon d - \gamma_3\,\Lambda_2(\gamma_3 a, q_3)\sin\varepsilon d$ depends on the parameters of both potential wells and is conditioned by the nonlinear medium surrounding the potential wells.

6. Nonlinear Medium in the Potential Wells and Nonlinear Environment

Let the studied model system be a continuous nonlinear medium both in the potential wells and around them. The Schrödinger equation in the corresponding regions is reduced to (23) and (14). Their solutions have already been found in Secs. 4 and 5:

$$\Psi_{1,2} = \mp\frac{\varepsilon_0}{\operatorname{sh}\left(\varepsilon_0(x\mp x_0)\right)}, \qquad \Psi_3 = \frac{q_3'\,\gamma_3}{cn(\gamma_3 x, q_3)}, \qquad \Psi_{4,5} = q\,\gamma\frac{cn\,(\gamma x, q)}{dn\,(\gamma x, q)}. \quad (28)$$

The total number of elementary excitations is determined by the formula

$$N = 2\Big\{-\varepsilon_0 + \gamma_3\left[\gamma^2 d + {q'}^2(\gamma_3, q) + \Lambda_2(\gamma_3 a, q) - E\left(am(\gamma_3 a, q), q\right)\right] + \gamma\left[E\left(am\left(\gamma(a+d), q\right), q\right) - E\left(am(\gamma a, q), q\right)\right] - \gamma q^2\left[\Lambda_1\left(\gamma(a+d), q\right) - \Lambda_1(\gamma a, q)\right] + \sqrt{\varepsilon_0^2 + q^2\gamma^2\frac{cn^2\left(\gamma(a+d), q\right)}{dn^2\left(g(a+d), q\right)}}\Big\}. \quad (29)$$

7. Conclusions

In the paper, we studied the possible localized states in the system containing two potential wells for all possible combinations of linear and nonlinear media in potential wells and regions outside them. The obtained results can be useful for the study of localized states in systems with defects/interfaces in various fields of physics, physical chemistry, biophysics, etc. In particular, they can be used for the description of adsorption of polymer chains at the interfaces, in fiber optics, where layered and modulated media are used in fiber communication systems, optical switches, delay lines, in the theory of Bose–Einstein condensation, etc.

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