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On m-Sectorial Extensions of Sectorial Operators

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We study maximal sectorial extensions of an arbitrary closed densely defined sectorial operator. In particular, abstract boundary conditions for these extensions are obtained. The results are applied for the parametrization of all m-sectorial extensions of a nonnegative symmetric operator in a planar model of two-point interactions.

Key words: sectorial operator, accretive operator, Friedrichs extension, Kreĭn–von Neumann extension, boundary pair, boundary triplet.

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Introduction

Let $\mathfrak{H}$ be a complex Hilbert space with an inner product $(\cdot,\cdot)$. We use the symbols $\operatorname{dom}(T)$, $\operatorname{ran}(T)$, $\ker(T)$ for the domain, the range, and the null-subspace of a linear operator T, respectively. The resolvent set of T is denoted by $\rho(T)$. The linear space of bounded operators acting between Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ is denoted by $\mathbf{L}(\mathfrak{H}_1,\mathfrak{H}_2)$ and the Banach algebra $\mathbf{L}(\mathfrak{H},\mathfrak{H})$ by $\mathbf{L}(\mathfrak{H})$.

A linear operator T in a complex Hilbert space $\mathfrak{H}$ is called *accretive* if its numerical range $W(T) \stackrel{\text{def}}{=} \{(Tu,u), u\in\operatorname{dom}(T), \|u\|=1\}$ is contained in the closed right half-plane, i.e., $\operatorname{Re}(Tu,u)\geqslant 0$ for all $u\in\operatorname{dom}(T)$.

An accretive operator T is called *maximal accretive* or *m-accretive* if T is closed and has no accretive extensions in $\mathfrak{H}$ [23, 28, 32, 33].

Let $\alpha\in[0,\pi/2)$. Denote by $\Theta(\alpha)$ the sector in the complex plane

$$\Theta(\alpha) \stackrel{\text{def}}{=} \{z\in\mathbb{C} : |\arg z|\leqslant\alpha\}.$$

A linear operator S is called *sectorial* with vertex at the origin and the semi-angle α [23] if $W(S) \subseteq \Theta(\alpha)$. In particular, if $\alpha = 0$, then $(Sx, x) \geq 0$ for all $x \in \mathrm{dom}(S)$, i.e., S is symmetric and nonnegative operator.

A linear operator S is called *maximal sectorial* (*m-sectorial* for short) if it is sectorial in the sense given above and m-accretive. If T is m-α-sectorial operator and if $\gamma \in (\alpha, \pi/2)$, then

$$\lambda \in \mathbb{C} \setminus \Theta(\gamma) \Longrightarrow ||(T - \lambda I)^{-1}|| \leq \frac{1}{|\lambda| \sin(\gamma - \alpha)}, \tag{0.1}$$

and the one-parameter semigroup $U(t) = \exp(-tT)$, $t \geq 0$, admits a holomorphic contractive continuation into the interior of the sector $\Theta(\pi/2 - \alpha)$ [23].

In our recent paper [11], for the general case of an *arbitrary* closed densely defined sectorial operator S, we proposed a new approach for solving the problem of parametrization of all m-accretive extensions. In particular, if $\{\mathcal{H}, \Gamma\}$ is a boundary pair of S, then there is a bijective correspondence between all m-accretive extensions $\widetilde{S}$ of S and all pairs $\langle \mathbf{Z}, X\rangle$, where $\mathbf{Z}$ is an m-accretive linear relation in $\mathcal{H}$ and $X : \mathrm{dom}(\mathbf{Z}) \to \overline{\mathrm{ran}(S_{\mathrm{F}})}$ is a linear operator such that $\|Xe\|^2 \leqslant \mathrm{Re}(\mathbf{Z}(e), e)_{\mathcal{H}}\ \forall e \in \mathrm{dom}(\mathbf{Z})$ (see Section 2 and Theorem 2.5, where the domains, actions, regular points, resolvents, and eigenvalues of m-accretive extensions are described). Our method is applicable, in particular, to a sectorial operator S having a unique m-sectorial extension (the Friedrichs extension S_{F}). In the present paper, assuming that S_{F} is not a unique m-sectorial extension of S, we apply our method for describing all its m-sectorial extensions, i.e., we find necessary and sufficient conditions on a pair $\langle \mathbf{Z}, X\rangle$ such that the corresponding $\widetilde{S}$ is an m-sectorial extension of S (see Theorem 3.3 and Theorem 3.9). Let S_{N} be the Kreĭn–von Neumann extension of S [5,6] and let $\mathrm{D}[S_{\mathrm{N}}]$ be the domain of a closed sesquilinear form associated with S_{N}. For a sectorial operator S satisfying the condition $\mathrm{dom}(S^*) \subseteq \mathrm{D}[S_{\mathrm{N}}]$ a survey of results on the m-accretive and m-sectorial extensions of S is given in [9]. The latter condition is valid iff $\mathrm{dom}(S_{\mathrm{F}}^*) + \mathrm{dom}(S_{\mathrm{N}}^*) = \mathrm{dom}(S^*)$; in particular, it holds for a coercive sectorial operator S.

Let A be a densely defined closed symmetric operator in $\mathfrak{H}$. The extensions $\widetilde{A}$ of A possessing the property $A \subset \widetilde{A} \subset A^*$ are called the *quasi-selfadjoint* (*proper, intermediate*) extensions of A. The problem of the existence and the description of all quasi-selfadjoint m-accretive extensions of a nonnegative symmetric operator via linear-fractional transformation was solved in [12] and via abstract boundary conditions in [4, 16, 17, 24, 30, 36, 37]. In Section 4, we use the approach proposed in [11] for studying these extensions.

An interesting example of a densely defined closed nonnegative symmetric operator is the following differential operator in the Hilbert space $L_2(\mathbb{R}^2)$:

$$\mathrm{dom}(A) = \left\{ f(x) \in W_2^2(\mathbb{R}^2) : f(y_1) = \cdots = f(y_m) = 0 \right\}, \quad Af = -\Delta f,$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $W_2^2(\mathbb{R}^2)$ is the Sobolev space, and $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the Laplacian. The operator A is the base in a planar model of point interactions with a finite number of centers [2]. The model describes the motion of a particle in a potential which is given by Schrödinger Hamiltonian supported on a discrete finite set of points ("sources"). These models are widely used in solid state, atomic and nuclear physics.

For the case of a single center ($m = 1$), the free Hamiltonian $-\Delta$, $\mathrm{dom}(-\Delta) = W_2^2(\mathbb{R}^2)$ is a unique nonnegative selfadjoint extension of A, which is also the Friedrichs extension A_{F} of A [1,18]. Therefore A has neither quasi-selfadjoint non-selfadjoint m-accretive extensions nor nonselfadjoint m-sectorial extensions [9]. In our paper [11] we described all m-accretive (non-quasi-selfadjoint) extensions of A. If $m \geq 2$, the free Hamiltonian is still the Friedrichs extension of A, but there are other nonnegative selfadjoint extensions [1] and, therefore, there exist non-selfadjoint m-accretive quasi-selfadjoint extensions and m-sectorial extensions. In this case, the Friedrichs and Kreĭn–von Neumann extensions are not disjoint ($\mathrm{dom}(A_{\mathrm{F}}) \cap \mathrm{dom}(A_{\mathrm{N}}) \supset \mathrm{dom}(A)$) [19], i.e., the condition $\mathrm{dom}(A^*) \subseteq \mathrm{D}[A_{\mathrm{N}}]$ does not hold [29]. In the last section, we apply the abstract results of this paper to the parametrization of all m-sectorial and all nonnegative selfadjoint extensions of A in the case of two-center point interactions ($m = 2$).

1. Preliminaries

1.1. Sectorial forms and operators. The basic definitions and results on sesquilinear forms can be found in [23]. If τ is a closed densely defined sectorial form in the Hilbert space $\mathfrak{H}$, then by the First Representation Theorem [23, 25], there exists a unique m-sectorial operator T in $\mathfrak{H}$ associated with τ in the following sense: $(Tu, v) = \tau[u, v]$, for all $u \in \mathrm{dom}(T)$ and for all $v \in \mathrm{dom}(\tau)$. The adjoint operator T^* is associated with the adjoint form $\tau^*[u, v] := \overline{\tau[v, u]}$. Denote by T_{R} the nonnegative selfadjoint operator associated with the real part $\tau_{\mathrm{R}}[u, v] := (\tau[u, v] + \tau^*[u, v])/2$ of the form τ. The operator T_{R} is called the *real part* of T. According to the Second Representation Theorem [23], the equality $\mathrm{dom}(\tau) = \mathrm{dom}(T_{\mathrm{R}}^{\frac{1}{2}})$ holds. Moreover, $\tau[u, v] = ((I + iG)T_R^{\frac{1}{2}}u, T_R^{\frac{1}{2}}v),\ u, v \in \mathrm{dom}(\tau)$, where G is a bounded selfadjoint operator in the subspace $\overline{\mathrm{ran}(T_{\mathrm{R}})}$ and $||G|| \leq \tan\alpha$ iff τ is α-sectorial.

In the sequel we will use the following notations for an m-sectorial operator T:

$$\mathrm{D}[T] \stackrel{\mathrm{def}}{=} \mathrm{dom}(T_{\mathrm{R}}^{\frac{1}{2}}), \quad \mathrm{R}[T] \stackrel{\mathrm{def}}{=} \mathrm{ran}(T_{\mathrm{R}}^{\frac{1}{2}}), \quad \widehat{T} = T{\upharpoonright}\overline{\mathrm{ran}(T)}, \quad \widehat{T}_{\mathrm{R}} = T_{\mathrm{R}}{\upharpoonright}\overline{\mathrm{ran}(T)}.$$

Note that for an m-accretive operator T the equality $\ker(T) = \ker(T^*)$ holds; if T is m-sectorial, then $\ker(T) = \ker(T^*) = \ker T_{\mathrm{R}}$, and this yields that $\ker(\widehat{T}) = \ker(\widehat{T}^*) = \ker(\widehat{T}_{\mathrm{R}}) = \{0\}$.

Proposition 1.1 ([6]). *If $T = T_{\rm R}^{\frac{1}{2}}(I+iG)T_{\rm R}^{\frac{1}{2}}$ is an m-α-sectorial operator in $\mathfrak{H}$ and $\gamma \in (\alpha, \pi/2)$, then*

$$\mathrm{R}[T] = \left\{ f \in \mathfrak{H} : \varlimsup_{\substack{\lambda\to 0,\\ \lambda\in\mathbb{C}\setminus\Theta(\alpha)}} \left|\left((T-\lambda I)^{-1}f, f\right)\right| < \infty \right\}, \tag{1.1}$$

$$\begin{aligned}\lim_{\substack{\lambda\to 0,\\ \lambda\in\mathbb{C}\setminus\Theta(\gamma)}} \left((T-\lambda I)^{-1}f, g\right) &= \widehat{T}^{-1}[f,g]\\ &= \left((I+iG)^{-1}\widehat{T}_R^{-\frac{1}{2}}f, \widehat{T}_R^{-\frac{1}{2}}g\right), \quad f,g\in \mathrm{R}[T],\end{aligned} \tag{1.2}$$

$$\lim_{\substack{\lambda\to 0,\\ \lambda\in\mathbb{C}\setminus\Theta(\gamma)}} T_{\rm R}^{\frac{1}{2}}(T-\lambda I)^{-1}T_{\rm R}^{\frac{1}{2}}g = (I+iG)^{-1}g, \quad g \in \mathrm{D}[T]\ominus\ker(T). \tag{1.3}$$

1.2. The Friedrichs and Kreĭn–von Neumann m-sectorial extensions. Let S be an α-sectorial operator. It is well known [23] that the form (Su,v), $u,v\in\mathrm{dom}(S)$, is closable. We let $S[u,v]$ denote the closure of it. The domain of $S[u,v]$ is denoted by $\mathrm{D}[S]$. With the closed form $S[u,v]$ is associated the maximal α-sectorial operator $S_{\rm F}$, which is called the *Friedrichs extension* of S [23]. So, $\mathrm{D}[S]=\mathrm{D}[S_{\rm F}]$ and $S_{\rm F}[u,v]=S[u,v]$ for all $u,v\in\mathrm{D}[S]$. Let $S_{\rm FR}$ be the real part of $S_{\rm F}$. Clearly, $\mathrm{D}[S]=\mathrm{dom}(S_{\rm FR}^{\frac{1}{2}})$.

In the case of a nonnegative symmetric operator S ($\alpha=0$), M.G. Kreĭn [25] discovered that the set of all its nonnegative selfadjoint extensions has a minimal element (in the sense of associated closed quadratic forms). This minimal element $S_{\rm N}$ was defined in [25] by means of a linear-fractional transformation. If $\alpha\neq 0$, then the corresponding m-sectorial analog of the extremal extension also exists and it can be defined in a similar way (see [5, 6, 9]). We preserve the same notation $S_{\rm N}$ and the name *Kreĭn–von Neumann extension* for the general case of not necessarily symmetric sectorial operator S. We notice that interesting applications of Kreĭn–von Neumann extension of nonnegative symmetric operator in the elasticity theory can be found in [13, 21].

Let

$$\mathfrak{N}_\lambda \stackrel{\rm def}{=} \mathfrak{H}\ominus\mathrm{ran}(S-\bar{\lambda}I)$$

be the defect subspace of a linear operator S. If S is closed and densely defined, then $\mathfrak{N}_\lambda=\ker(S^*-\lambda I)$.

For the operators $S_{\rm F}$, $S_{\rm N}$ and for an arbitrary m-sectorial extension $\widetilde{S}$ of S, the following relations are valid [5, 6, 9]:

$$\mathrm{D}[S]\cap\mathfrak{N}_\lambda=\{0\},\quad \mathrm{D}[S_{\rm N}]\cap\mathfrak{N}_\lambda=\mathrm{R}[S_{\rm F}]\cap\mathfrak{N}_\lambda, \tag{1.4}$$

$$\mathrm{D}[S_{\rm N}]=\mathrm{D}[S]\dot{+}\left(\mathfrak{N}_\lambda\cap\mathrm{D}[S_{\rm N}]\right),\quad \lambda\in\rho(S_{\rm F}^*). \tag{1.5}$$

Fix $z\in\rho(S^*_{\rm F})$ and define the linear manifold $\mathfrak{L}$:

$$\mathfrak{L}\stackrel{\rm def}{=}{\rm D}[S]\dot{+}\mathfrak{N}_z,\quad z\in\rho(S^*_{\rm F}). \tag{1.6}$$

It is easy to see [11] that $\mathfrak{L}$ does not depend on the choice of $z\in\rho(S^*_{\rm F})$. We will denote by $\mathcal{P}_{z,{\rm F}}$ and $\mathcal{P}_z$ the skew projectors in $\mathfrak{L}$ onto ${\rm D}[S]$ and $\mathfrak{N}_z$, corresponding to the decomposition (1.6). When $z=i$, we denote these projectors by $\mathcal{P}_{\rm F}$ and $\mathcal{P}_i$, respectively. Since ${\rm dom}(S^*)={\rm dom}(S^*_{\rm F})\dot{+}\mathfrak{N}_\lambda$ for $\lambda\in\rho(S^*_{\rm F})$, we have ${\rm dom}(S^*)\subset\mathfrak{L}$. From (1.5), it follows that ${\rm D}[S_{\rm N}]\subseteq\mathfrak{L}$. Hence, the inclusion ${\rm dom}(S^*)\subseteq{\rm D}[S_{\rm N}]$ holds iff ${\rm D}[S_{\rm N}]=\mathfrak{L}$ (see [5,6]).

We will use the notations $S_{\rm FR}$ and $S_{\rm NR}$ for "real parts" of the Friedrichs and Kreĭn–von Neumann extensions, respectively. The next relation was established in [6]:

$$\begin{gathered}S_{\rm N}[u,v]=\Big((I+iG_{\rm F})\left(S_{\rm FR}^{\frac12}\mathcal{P}_{z,{\rm F}}u+z(I-iG_{\rm F})^{-1}\widehat{S}_{\rm FR}^{-\frac12}\mathcal{P}_z u\right),\\ \left(S_{\rm FR}^{\frac12}\mathcal{P}_{z,{\rm F}}v+z(I-iG_{\rm F})^{-1}\widehat{S}_{\rm FR}^{-\frac12}\mathcal{P}_z v\right)\Big),\quad u,v\in{\rm D}[S_{\rm N}].\end{gathered} \tag{1.7}$$

The operator S has a unique m-sectorial extension iff, for some $\lambda\in\rho(S^*_{\rm F})$ (then for all $\lambda\in\rho(S^*_{\rm F})$):

$$\sup_{x\in{\rm dom}(S)}\frac{|(f_\lambda,x)|^2}{{\rm Re}(Sx,x)}=\infty,\quad\forall f_\lambda\in\mathfrak{N}_\lambda\setminus\{0\}.$$

Moreover, (see [5,6,9]),

$$S_{\rm N}\neq S_{\rm F}\iff{\rm D}[S_{\rm N}]\cap\mathfrak{N}_\lambda\neq\{0\}\iff{\rm R}[S_{\rm F}]\cap\mathfrak{N}_\lambda\neq\{0\},\quad\lambda\in\rho(S^*_{\rm F}). \tag{1.8}$$

Taking into account (1.1), (1.2), and (1.8), for $\mu\in\mathbb{C}\setminus\Theta(\alpha)$ we have

$$\varphi_\mu\in\mathfrak{N}_\mu\cap{\rm D}[S_{\rm N}]\iff\varliminf_{\substack{\lambda\to0,\\ \lambda\in\mathbb{C}\setminus\Theta(\alpha)}}\left|\left((S^*_{\rm F}-\lambda I)^{-1}\varphi_\mu,\varphi_\mu\right)\right|<\infty. \tag{1.9}$$

1.3. Boundary triplets and abstract boundary conditions for quasi-selfadjoint extensions of nonnegative symmetric operator. Let A be a closed densely defined symmetric operator in $\mathfrak{H}$. Recall the definition of a boundary triplet (boundary value space) [20] for A^*.

Definition 1.2. A triplet $\{\mathcal{H},\Gamma_1,\Gamma_0\}$ is called a boundary triplet of A^* if $\mathcal{H}$ is a Hilbert space and Γ_0,Γ_1 are bounded linear operators from the Hilbert space $H_+={\rm dom}(S^*)$ with the graph norm into $\mathcal{H}$ such that the map $\vec{\Gamma}=\langle\Gamma_0,\Gamma_1\rangle$ is a surjection from H_+ onto $\mathcal{H}^2=\mathcal{H}\oplus\mathcal{H}$, and the Green identity holds:

$$(A^*f,g)-(f,A^*g)=(\Gamma_1 f,\Gamma_0 g)_{\mathcal{H}}-(\Gamma_0 f,\Gamma_1 g)_{\mathcal{H}},\quad\forall f,\,g\in H_+. \tag{1.10}$$

In the sequel we use *linear relations* to describe the extensions in terms of abstract boundary conditions. One can find basic concepts and properties related to these objects in, for instance, [3,9,16,20,34]. The formulae

$$\operatorname{dom}(\widetilde{A}) = \left\{ u \in \operatorname{dom}(A^*) : \vec{\Gamma} u \in \widetilde{\mathbf{T}} \right\}, \quad \widetilde{A} = A^* \upharpoonright \operatorname{dom}(\widetilde{A}) \tag{1.11}$$

give a one-to-one correspondence between all quasi-selfadjoint extensions $\widetilde{A}$ of A ($A \subset \widetilde{A} \subset A^*$) and all linear relations $\widetilde{\mathbf{T}}$ in $\mathcal{H}$. Moreover, $\widetilde{A}^*$ corresponds to $\widetilde{\mathbf{T}}^*$. Therefore, an extension $\widetilde{A}$ is selfadjoint iff the relation $\widetilde{\mathbf{T}}$ is selfadjoint in $\mathcal{H}$.

As it was shown in [15,16], the operators A_0, A_1, defined as follows: $A_k = A^* \upharpoonright \operatorname{Ker} \Gamma_k$, $k = 0, 1$, are mutually transversal selfadjoint extensions of A, i.e., $\operatorname{dom}(A^*) = \operatorname{dom}(A_0) \dotplus \operatorname{dom}(A_1)$.

The operator-valued function $\Gamma_0(\lambda) := (\Gamma_0 \upharpoonright \mathfrak{N}_\lambda)^{-1}$ [15] is the γ-field corresponding to A_0 [26,27], that is, $\operatorname{ran}(\Gamma_0(\lambda)) = \mathfrak{N}_\lambda$ and

$$\Gamma_0(\lambda) = \Gamma_0(z) + (\lambda - z)(A_0 - zI)^{-1} \Gamma_0(z).$$

Note that as a consequence of (1.10), one can obtain the equality

$$\Gamma_0(\overline{\lambda}) = \left(\Gamma_1 (A_0 - \lambda I)^{-1} \right)^*. \tag{1.12}$$

V. Derkach and M. Malamud [15,16] defined the Weyl function $M_0(\lambda)$ by the equality

$$M_0(\lambda) = \Gamma_1 \Gamma_0(\lambda). \tag{1.13}$$

The function M_0 is the Kreĭn–Langer Q-function [26,27], i.e., belongs to the Nevanlinna class operator-valued functions (M_0 is holomorphic in the upper and lower half-planes, $M_0(\lambda)^* = M_0(\bar{\lambda})$ and $(M_0(\lambda) - M_0(\lambda)^*)/(\lambda - \bar{\lambda}) \geq 0$ for all λ, $\operatorname{Im} \lambda \neq 0$), and the identity

$$M_0(\lambda) - M_0(z) = (\lambda - z) \Gamma_0(\bar{z})^* \Gamma_0(\lambda) \tag{1.14}$$

holds. In terms of the boundary triplet, the connection between a quasi-selfadjoint extension $\widetilde{A}_{\widetilde{\mathbf{T}}}$ defined by relations (1.11), and its resolvent for all $\lambda \in \rho(A_0) \cap \rho(\widetilde{A}_{\widetilde{\mathbf{T}}})$ is given by the Kreĭn resolvent formula

$$\left(\widetilde{A}_{\widetilde{\mathbf{T}}} - \lambda I \right)^{-1} = \left(A_0 - \lambda I \right)^{-1} + \Gamma_0(\lambda) \left(\widetilde{\mathbf{T}} - M_0(\lambda) \right)^{-1} \Gamma_0(\overline{\lambda})^*. \tag{1.15}$$

Theorem 1.3 ([14–16,29]). *Let A be a closed nonnegative symmetric operator and let $\{\mathcal{H}, \Gamma_1, \Gamma_0\}$ be a boundary triplet of A^* such that $A_0 = A_{\mathrm{F}} (= A^* \upharpoonright \operatorname{Ker} \Gamma_0)$. Then A has a non-unique nonnegative selfadjoint extension iff*

$$\mathcal{D}_0 = \left\{ h \in \mathcal{H} : \lim_{x \uparrow 0} \left(M_0(x) h, h \right)_{\mathcal{H}} < \infty \right\} \neq \{0\},$$

and the quadratic form $\tau[h] = \lim_{x\uparrow 0}(M_0(x)h,h)_{\mathcal{H}}$, $\mathcal{D}[\tau] = \mathcal{D}_0$ *is bounded from below. Denote by* $M_0(0)$ *the selfadjoint linear relation in* $\mathcal{H}$ *associated with* τ. *Then the Kreĭn–von Neumann extension* $A_{\rm N}$ *can be defined by the boundary condition*

$$\mathrm{dom}(A_{\rm N}) = \{u \in \mathrm{dom}(A^*) : \langle \Gamma_0 u, \Gamma_1 u\rangle \in M_0(0)\}\,.$$

The relation $M_0(0)$ *is also the strong resolvent limit of* $M_0(x)$ *when* $x \to -0$. *Moreover,* A_0 *and* $A_{\rm N}$ *are disjoint iff* $\overline{\mathcal{D}_0} = \mathcal{H}$, *and transversal iff* $\mathcal{D}_0 = \mathcal{H}$.

Theorem 1.4 ([17]). *There is a one-to-one correspondence given by* (1.11) *between the* m*-accretive extensions* $\widetilde{A}_{\widetilde{\mathbf{T}}}$ *and the* m*-accretive linear relations* $\widetilde{\mathbf{T}}$ *satisfying the condition*

$$\mathrm{dom}(\widetilde{\mathbf{T}}) \subseteq \mathcal{D}_0, \quad \mathrm{Re}(\widetilde{\mathbf{T}}x, x) \geq \tau[x], \qquad x \in \mathrm{dom}(\widetilde{\mathbf{T}}). \tag{1.16}$$

The extension $\widetilde{A}_{\widetilde{\mathbf{T}}}$ *is* m*-*α*-sectorial iff the form* $(\widetilde{\mathbf{T}}x, y) - \tau[x,y]$ *is* α*-sectorial.*

2. Abstract Boundary Conditions for m-Accretive Extensions of Sectorial Operators

Next, we recall some definitions and results from [11]. Let the linear manifold $\mathfrak{L}$ be defined by (1.6) and let the sesquilinear form η be given by

$$\eta[u,v] \stackrel{\rm def}{=} S_{\rm FR}[\mathcal{P}_{-1,{\rm F}}u, \mathcal{P}_{-1,{\rm F}}v] + (\mathcal{P}_{-1}u, \mathcal{P}_{-1}v), \quad u,v \in \mathfrak{L}.$$

Then η is nonnegative and closed in $\mathfrak{H}$. Therefore $\mathfrak{L}$ is a Hilbert space w.r.t. the inner product

$$(u,v)_\eta = \eta(u,v) + (u,v)_{\mathfrak{H}}, \quad u,v \in \mathfrak{L}.$$

Definition 2.1 ([11]). A pair $\{\mathcal{H}, \Gamma\}$ is called a *boundary pair* of S, if $\mathcal{H}$ is a Hilbert space and $\Gamma \in \mathbf{L}(\mathfrak{L}, \mathcal{H})$ is such that $\ker(\Gamma) = \mathrm{D}[S]$, $\mathrm{ran}(\Gamma) = \mathcal{H}$.

Let

$$\gamma(\lambda) = (\Gamma{\restriction}\mathfrak{N}_\lambda)^{-1}, \quad \lambda \in \rho(S_{\rm F}^*).$$

The following relations are valid:

$$\begin{gathered}\gamma(\lambda) = \gamma(z) + (\lambda - z)(S_{\rm F}^* - \lambda I)^{-1}\gamma(z), \qquad (2.1)\\ \mathcal{P}_{\rm F}\gamma(\lambda)e = (\lambda - i)(S_{\rm F}^* - \lambda I)^{-1}\gamma(i)e, \quad \mathcal{P}_i\gamma(\lambda)e = \gamma(i)e, \quad e \in \mathcal{H}.\end{gathered}$$

In particular, it follows that the norms $\|\cdot\|_{\mathfrak{H}}$ and $\|\cdot\|_\eta$ are equivalent on $\mathfrak{N}_\lambda$. Hence, $\gamma(\lambda) \in \mathbf{L}(\mathcal{H}, \mathfrak{H})$ for all $\lambda \in \rho(S_{\rm F}^*)$, and it is a holomorphic operator-valued function. The operator-valued function $\gamma(\lambda)$ is called a γ*-field* of S associated with the boundary pair $\{\mathcal{H}, \Gamma\}$. Clearly, $\gamma(\lambda)$ maps $\mathcal{H}$ onto $\mathfrak{N}_\lambda$. Hence $S^*\gamma(\lambda) = \lambda\gamma(\lambda)$ and $\ker(\gamma^*(\lambda)) = \mathrm{ran}(S - \bar{\lambda}I)$.

Define on $\mathfrak{L}$ one more sesquilinear form $l[u,v]$:

$$l[u,v] = S_{\rm F}[\mathcal{P}_{\rm F}u, \mathcal{P}_{\rm F}v] - i(\mathcal{P}_i u, \mathcal{P}_{\rm F}v) - i(\mathcal{P}_{\rm F}u, \mathcal{P}_i v) - i(\mathcal{P}_i u, \mathcal{P}_i v). \tag{2.2}$$

Due to the equality ${\rm Re}\, l[u] = \left\|S_{\rm FR}^{\frac{1}{2}}\mathcal{P}_{\rm F}u\right\|^2$, $u \in \mathfrak{L}$, the form $l[u,v]$ is accretive. Moreover, $\inf_{\varphi\in{\rm D}[S]}\{{\rm Re}\, l[u-\varphi]\} = 0$ for all $u \in \mathfrak{L}$ and $l[\varphi, v] = (\varphi, S^*v)$ for all $\varphi \in {\rm D}[S]$, $v \in {\rm dom}(S^*)$.

Relations (1.7) and (2.2) imply the following representation of the form $S_{\rm N}[\cdot,\cdot]$:

$$\begin{aligned} S_{\rm N}[u,v] = l[u,v] + \left[i\left(\gamma(i)\Gamma u, \gamma(i)\Gamma v\right) + \left((I - iG_{\rm F})^{-1}\widehat{S}_{\rm FR}^{-\frac{1}{2}}\gamma(i)\Gamma u, \widehat{S}_{\rm FR}^{-\frac{1}{2}}\gamma(i)\Gamma v\right)\right] \\ + 2i\left((I - iG_{\rm F})^{-1}\widehat{S}_{\rm FR}^{-\frac{1}{2}}\gamma(i)\Gamma u, S_{\rm FR}^{\frac{1}{2}}\mathcal{P}_{\rm F}v\right), \quad u, v \in {\rm D}[S_{\rm N}]. \end{aligned} \tag{2.3}$$

Definition 2.2 ([11])**.** The triplet $\{\mathcal{H}, G, \Gamma\}$ is called a *boundary triplet for* S^* if $\{\mathcal{H}, \Gamma\}$ is a boundary pair for S and $G\colon {\rm dom}(S^*) \to \mathcal{H}$ is a linear operator such that the relation

$$l^*[u,v] = (S^*u, v) - (Gu, \Gamma v)_{\mathcal{H}}, \quad \forall u \in {\rm dom}(S^*), \quad \forall v \in \mathfrak{L}, \tag{2.4}$$

is valid, where the form l is given by (2.2).

It is shown in [11] that there exists a unique operator $G\colon {\rm dom}(S^*) \to \mathcal{H}$ such that (2.4) holds and, moreover,

$$Gu = \gamma(i)^*(S^* - iI)u.$$

Next, we define the operator-valued functions $\mathcal{Q}(\lambda) \in \mathbf{L}(\mathcal{H})$, $\mathcal{G}(\lambda) \in \mathbf{L}(\mathfrak{H}, \mathcal{H})$, $\Phi(\lambda) \in \mathbf{L}(\mathfrak{H}, \mathfrak{H})$, $q(\lambda) \in \mathbf{L}(\mathcal{H}, \mathfrak{H})$, $\lambda \in \rho(S_{\rm F})$ associated with the boundary triplet for S^*, see [11]:

$$\begin{aligned} \mathcal{Q}(\lambda) &\stackrel{\rm def}{=} G\gamma(\lambda) = \gamma(i)^*(S_{\rm F}^* - iI)\gamma(\lambda) = (\lambda - i)\gamma(i)^*\gamma(\lambda), \qquad (2.5) \\ q(\lambda) &\stackrel{\rm def}{=} \left(G(S_{\rm F}^* - \bar{\lambda}I)^{-1}\right)^*, \\ \mathcal{G}(\lambda) &\stackrel{\rm def}{=} \left(S_{\rm FR}^{\frac{1}{2}}\mathcal{P}_{\rm F}\gamma(\bar{\lambda})\right)^*, \\ \Phi(\lambda) &\stackrel{\rm def}{=} \left(S_{\rm FR}^{\frac{1}{2}}(S_{\rm F}^* - \bar{\lambda}I)^{-1}\right)^*. \end{aligned}$$

Observe that the function $\mathcal{Q}(\lambda)$ is an analog of the Weyl function (1.13) corresponding to a boundary triplet of the adjoint to a symmetric operator, while $q(\lambda)$ is an analog of the function from (1.12).

Let L be a linear operator in $\mathfrak{L}$ defined as follows:

$$\begin{aligned} {\rm dom}(L) &= {\rm dom}(S_{\rm F}) \dotplus \mathfrak{N}_i, \\ L(u_{\rm F} + u_i) &= S_{\rm F}u_{\rm F} - iu_i, \quad u_{\rm F} \in {\rm dom}(S_{\rm F}),\ u_i \in \mathfrak{N}_i. \end{aligned} \tag{2.6}$$

Definition 2.3 ([11]). Let S be a closed densely defined sectorial operator and let $\{\mathcal{H}, \Gamma\}$ be a boundary pair for S. A triplet $\{\mathcal{H}, G_*, \Gamma\}$ is called a *boundary triplet for* L if $G_* : \mathrm{dom}(L) \to \mathcal{H}$ is a linear operator such that

$$l[u, v] = (Lu, v) - (G_* u, \Gamma v)_{\mathcal{H}}, \quad \forall u \in \mathrm{dom}(L),\ \forall v \in \mathfrak{L}.$$

The operator G_* is uniquely defined [11] and, moreover, for each $\lambda \in \rho(S_{\mathrm{F}})$,

$$\begin{aligned} G_* f &= \gamma(\bar{\lambda})^* (S_{\mathrm{F}} - \lambda I) f, & f &\in \mathrm{dom}(S_{\mathrm{F}}), \\ G_* q(\lambda) e &= \mathcal{Q}(\bar{\lambda})^* e, & e &\in \mathcal{H}. \end{aligned} \tag{2.7}$$

Thus, given a boundary pair $\{\mathcal{H}, \Gamma\}$ for an operator S, the boundary triplets corresponding to it are $\{\mathcal{H}, G, \Gamma\}$ for S^* and $\{\mathcal{H}, G_*, \Gamma\}$ for L, and we have the abstract Green formula

$$(Lu, v) - (u, S^* v) = (G_* u, \Gamma v)_{\mathcal{H}} - (\Gamma u, G v)_{\mathcal{H}}, \quad \forall u \in \mathrm{dom}(L),\ \forall v \in \mathrm{dom}(S^*).$$

Theorem 2.4 ([11]). *Let S be a densely defined closed sectorial operator. Let $\{\mathcal{H}, \Gamma\}$ be a boundary pair for S and $\{\mathcal{H}, G, \Gamma\}$ be a corresponding boundary triplet for S^*. If $\widetilde{S}$ is an m-accretive extension of S, then there exist the linear operators*

$$Z : \mathrm{dom}(\widetilde{S}) \to \mathcal{H} \text{ and } X : \mathrm{dom}(X) = \Gamma \mathrm{dom}(\widetilde{S}) \to \overline{\mathrm{ran}(S_{\mathrm{F}})}$$

such that:

1) $\mathrm{dom}(S) \subseteq \ker(Z)$;
2) $(\widetilde{S}u, v) = l[u, v] + (Zu, \Gamma v)_{\mathcal{H}} + 2(X\Gamma u, S_{\mathrm{FR}}^{\frac{1}{2}} \mathcal{P}_{\mathrm{F}} v)$, $\forall u \in \mathrm{dom}(\widetilde{S})$, $v \in \mathfrak{L}$;
3) $\mathbf{Z} = \{\langle \Gamma u, Zu \rangle, u \in \mathrm{dom}(\widetilde{S})\}$ *is an m-accretive linear relation in* $\mathcal{H}$;
4) $\|Xe\|^2 \leqslant \mathrm{Re}(\mathbf{Z}(e), e)_{\mathcal{H}}$ *for all* $e \in \mathrm{dom}(\mathbf{Z}) = \Gamma \mathrm{dom}(\widetilde{S})$.

Theorem 2.5 ([11]). *There is a bijective correspondence between all m-accretive extensions $\widetilde{S}$ of S and all pairs $\langle \mathbf{Z}, X \rangle$, where $\mathbf{Z}$ is an m-accretive linear relation in $\mathcal{H}$ and $X : \mathrm{dom}(\mathbf{Z}) \to \overline{\mathrm{ran}(S_{\mathrm{F}})}$ is a linear operator such that*

$$\|Xe\|^2 \leqslant \mathrm{Re}(\mathbf{Z}(e), e)_{\mathcal{H}} \quad \forall e \in \mathrm{dom}(\mathbf{Z}). \tag{2.8}$$

This correspondence is given by the boundary conditions for the domain and the action of $\widetilde{S}$ as follows: for all $\mathrm{Re}\,\lambda < 0$,

$$\begin{aligned} \mathrm{dom}(\widetilde{S}) = \{u \in \mathfrak{L} :\ & 1)\ u - (q(\lambda) - 2\Phi(\lambda) X)\Gamma u \in \mathrm{dom}(S_{\mathrm{F}}); \\ & 2)\ G_*(u + 2\Phi(\lambda) X \Gamma u) \in (\mathbf{Z} + 2\mathcal{G}(\lambda) X)\Gamma u\}, \\ \widetilde{S} u = S_{\mathrm{F}}\big(u - (q(\lambda) &- 2\Phi(\lambda)X)\Gamma u\big) + \lambda\big(q(\lambda) - 2\Phi(\lambda)X\big)\Gamma u. \end{aligned} \tag{2.9}$$

Set

$$\mathbf{W}(\lambda) := \mathbf{Z} - \mathcal{Q}(\bar{\lambda})^* + 2\mathcal{G}(\lambda), \quad \lambda \in \rho(S_{\mathrm{F}}). \tag{2.10}$$

Then

1) *a number* $\lambda \in \rho(S_{\mathrm{F}})$ *is a regular point of* $\widetilde{S}$ *iff* $\mathbf{W}(\lambda)^{-1} \in \mathbf{L}(\mathcal{H})$, *and*

$$(\widetilde{S} - \lambda I)^{-1} = (S_{\mathrm{F}} - \lambda I)^{-1} + (q(\lambda) - 2\Phi(\lambda)X)\mathbf{W}(\lambda)^{-1}\gamma(\bar{\lambda})^*, \tag{2.11}$$

$$\mathrm{dom}(\widetilde{S}) = \Big(I + (q(\lambda) - 2\Phi(\lambda)X)\mathbf{W}(\lambda)^{-1}\gamma(\bar{\lambda})^*(S_{\mathrm{F}} - \lambda I)\Big)\,\mathrm{dom}(S_{\mathrm{F}}), \tag{2.12}$$

$$\widetilde{S}u = (S_{\mathrm{F}} - \lambda I)f + \lambda u \tag{2.13}$$

for

$$u = \Big(I + (q(\lambda) - 2\Phi(\lambda)X)\mathbf{W}(\lambda)^{-1}\gamma(\bar{\lambda})^*(S_{\mathrm{F}} - \lambda I)\Big)\, f, \ f \in \mathrm{dom}(S_{\mathrm{F}}); \tag{2.14}$$

2) *a number* $\lambda \in \rho(S_{\mathrm{F}})$ *is an eigenvalue of* $\widetilde{S}$ *iff* $\ker(\mathbf{W}(\lambda)) \neq \{0\}$, *and*

$$\ker(\widetilde{S} - \lambda I) = (q(\lambda) - 2\Phi(\lambda)X)\ker(\mathbf{W}(\lambda)).$$

Remark 2.6. Relations (2.9) remain valid for all $\lambda \in \rho(\widetilde{S}) \cap \rho(S_{\mathrm{F}})$. The resolvent formula (2.11) is an analog of (1.15).

Let S be a densely defined closed sectorial operator. Following [7, 8], we can define the linear operator S_z for all $z \in \mathbb{C}$, $\mathrm{Re}\, z \le 0$:

$$\mathrm{dom}(S_z) = \mathrm{dom}(S) \dot{+} \mathfrak{N}_z \text{ and } S_z h = S\varphi - z\varphi_z, \quad h = \varphi + \varphi_z \in \mathrm{dom}(S_z). \tag{2.15}$$

Proposition 2.7 ([7, 8])**.** *The operator* S_z *is an* m*-accretive extension of* S.

Proof. Proposition was proved in [7, 8] for $\mathrm{Re}\, z < 0$. Let us prove the statement for $z = ix$, $x \in \mathbb{R}$. Let $g = \varphi + \varphi_{ix}$, $\varphi \in \mathrm{dom}(S)$, $\varphi_{ix} \in \mathfrak{N}_{ix}$. Then

$$(S_{ix}g, g) = (S\varphi - ix\varphi_{ix}, \varphi + \varphi_{ix}) = (S\varphi, \varphi) - ix\|\varphi_{ix}\|^2 - 2i\,\mathrm{Im}(ix(\varphi_{ix}, \varphi)).$$

Hence $\mathrm{Re}(Sg, g) = \mathrm{Re}(S\varphi, \varphi) \ge 0$ for all $g \in \mathrm{dom}(S_{ix})$. Furthermore, it is easy to verify that

$$\begin{aligned}
&\mathrm{dom}(S^*_{ix}) = (S^*_{\mathrm{F}} - ixI)^{-1}(S + ixI)\mathrm{dom}(S) \dot{+} \mathfrak{N}_{ix},\\
&S^*_{ix}\left((S^*_{\mathrm{F}} - ixI)^{-1}(S + ixI)f + \varphi_{ix}\right) = S^*_{\mathrm{F}}(S^*_{\mathrm{F}} - ixI)^{-1}(S + ixI)f + ix\varphi_{ix},\\
&f \in \mathrm{dom}(S), \quad \varphi_{ix} \in \mathfrak{N}_{ix},\\
&\mathrm{Re}(S^*_{ix}h, h) = \mathrm{Re}\left(S^*_{\mathrm{F}}(S^*_{\mathrm{F}} - ixI)^{-1}(S + ixI)f, (S^*_{\mathrm{F}} - ixI)^{-1}(S + ixI)f\right) \ge 0
\end{aligned}$$

for $h = (S^*_{\mathrm{F}} - ixI)^{-1}(S + ixI)f + \varphi_{ix}$, $f \in \mathrm{dom}(S)$, $\varphi_{ix} \in \mathfrak{N}_{ix}$. This means that S^*_{ix} is accretive. Thus, S_{ix} and S^*_{ix} are accretive. It follows that S_{ix} is m-accretive. □

Note that in general from (2.15) it follows for $\operatorname{Re} z \le 0$ that

$$\operatorname{dom}(S_z^*) = \{g \in \operatorname{dom}(S^*) : (S^* + \bar{z}I)g \in \operatorname{ran}(S - \bar{z}I)\}, \quad S_z^* = S^* {\upharpoonright} \operatorname{dom}(S_z^*).$$

In addition, for the boundary operators in the boundary triplets in Definitions 2.2 and 2.3, the following equalities hold:

$$\ker(G) = \operatorname{dom}(S_i^*), \quad \ker(G_*) = \operatorname{dom}(S_i).$$

Next we give expressions for the operators $\mathbf{Z}_z$ and X_z in the pair $\langle \mathbf{Z}_z, X_z \rangle$ corresponding to S_z, $\operatorname{Re} z \le 0$, in accordance with Theorem 2.4.

Proposition 2.8. $\mathbf{Z}_z$ *is the graph of the operator* $Z_z = -\mathcal{Q}(z)$, $\operatorname{dom}(Z_z) = \mathcal{H}$, *and* $X_z = -\mathcal{G}(\bar{z})^*$. *In addition, for* $u \in \operatorname{dom}(S_z)$, $v \in \mathfrak{L}$,

$$(S_z u, v) = l[u, v] - \Big(\mathcal{Q}(z)\Gamma u, \Gamma v\Big)_{\mathcal{H}} - 2\Big(\mathcal{G}(\bar{z})^* \Gamma u, S_{\mathrm{FR}}^{\frac{1}{2}} \mathcal{P}_{\mathrm{F}} v\Big). \tag{2.16}$$

Proof. Define for $u \in \operatorname{dom}(S_z)$,

$$\begin{aligned} Z_z u &:= \gamma(i)^*(S_z + iI)u, \\ M_z u &:= \frac{1}{2}\left(\widehat{S}_{\mathrm{FR}}^{-\frac{1}{2}}(S_z u + i\mathcal{P}_i u) - (I + iG_{\mathrm{F}}) S_{\mathrm{FR}}^{\frac{1}{2}} \mathcal{P}_{\mathrm{F}} u\right). \end{aligned} \tag{2.17}$$

Observe that from here one obtains the inclusions $\operatorname{dom}(S) \subseteq \ker(Z)$ and $\operatorname{dom}(S) \subset \ker(M_z)$. In addition, due to definition of $\mathcal{L}$ (1.6), Definition 2.1 of a boundary pair, and (2.15), we have

$$\Gamma \operatorname{dom}(S_z) = \mathcal{H}.$$

According to the proof of Theorem 2.4 (see [11]), the relations

$$\mathbf{Z}_z = \{\langle \Gamma u, Z_z u \rangle, \ u \in \operatorname{dom}(S_z)\} \quad \text{and} \quad X_z \Gamma u = M_z u$$

hold. Then, taking into account that $u = \gamma(z)\Gamma u$ and relations (2.4), (2.5), (2.15), we have

$$Z_z u = \gamma(i)^*(S_z + iI)\gamma(z)\Gamma u = \gamma(i)^*(-z\gamma(z)\Gamma u + i\gamma(z)\Gamma u) = -\mathcal{Q}(z)\Gamma u.$$

Let $\Gamma u = e$, then $u = \varphi + \gamma(z)e$, $\varphi \in \operatorname{dom}(S)$, and

$$\begin{aligned} X_z \Gamma u &= M_z u = M_z \gamma(z) e \\ &= \frac{1}{2}\left(\widehat{S}_{\mathrm{FR}}^{-\frac{1}{2}}(S_z \gamma(z)e + iP_i \gamma(z)e) - (I + iG_{\mathrm{F}}) S_{\mathrm{FR}}^{\frac{1}{2}} \mathcal{P}_{\mathrm{F}} \gamma(z) e\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\widehat{S}_{\rm FR}^{-\frac{1}{2}}(-z\gamma(z)e + i\gamma(i)e) - (I+iG_{\rm F})S_{\rm FR}^{\frac{1}{2}}\mathcal{P}_{\rm F}\gamma(z)e\right)\\
&= \frac{1}{2}\left(\widehat{S}_{\rm FR}^{-\frac{1}{2}}(-S^*\gamma(z)e + S^*\gamma(i)e) - (I+iG_{\rm F})S_{\rm FR}^{\frac{1}{2}}\mathcal{P}_{\rm F}\gamma(z)e\right)\\
&= \frac{1}{2}\left(-\widehat{S}_{\rm FR}^{-\frac{1}{2}}S_{\rm F}^*\mathcal{P}_{\rm F}\gamma(z)e - (I+iG_{\rm F})S_{\rm FR}^{\frac{1}{2}}\mathcal{P}_{\rm F}\gamma(z)e\right)\\
&= \frac{1}{2}\left(-(I-iG_{\rm F})S_{\rm FR}^{\frac{1}{2}}\mathcal{P}_{\rm F}\gamma(z)e - (I+iG_{\rm F})\right)S_{\rm FR}^{\frac{1}{2}}\mathcal{P}_{\rm F}\gamma(z)e\\
&= -S_{\rm FR}^{\frac{1}{2}}\mathcal{P}_{\rm F}\gamma(z)e = -\mathcal{G}(\bar{z})^*\Gamma u.
\end{aligned}
$$

Equality (2.16) follows from Theorem 2.4. □

3. m-Sectorial Extensions

By Theorem 2.5, there is a bijective correspondence between all m-accretive extensions $\widetilde{S}$ of S and all pairs $\langle \mathbf{Z}, X\rangle$ satisfying condition (2.8). *In the sequel we will assume that a densely defined closed sectorial operator S admits more than one m-sectorial extension, i.e., one of the equivalent conditions in* (1.8) *is satisfied*. Our main goal is to establish additional conditions for a pair $\langle \mathbf{Z}, X\rangle$ guaranteeing that the corresponding m-accretive extension $\widetilde{S}$ is sectorial.

Next, we will need the following auxiliary result:

Lemma 3.1. *The following assertions hold.*

1. *If T is m-accretive and $\beta \in (0, \pi/2)$, then*

$$
\lim_{\substack{z\to 0,\\ \pi/2+\beta\le|\arg z|\le\pi}} z(T-zI)^{-1}h = \begin{cases} -h, & h\in \ker(T)\\ 0, & h\in \overline{\operatorname{ran}(T)}\end{cases}. \tag{3.1}
$$

2. *If T is m-α-sectorial and $\beta \in (\alpha, \pi/2)$, then*

$$
\lim_{\substack{z\to 0,\\ z\in\mathbb{C}\setminus\Theta(\beta)}} z(T-zI)^{-1}h = \begin{cases} -h, & h\in \ker(T)\\ 0, & h\in \overline{\operatorname{ran}(T)}\end{cases}. \tag{3.2}
$$

Proof. 1. Let $h \in \ker(T)$. Then $(T-zI)^{-1}h = -\frac{h}{z}$ for all $z \in \rho(T)\setminus\{0\}$. Therefore,

$$
\lim_{\substack{z\to 0,\\ \pi/2+\beta\le|\arg z|\le\pi}} z(T-zI)^{-1}h = -h.
$$

Now let $h \in \operatorname{ran}(T)$. Then $h = T\varphi$, $\varphi \in \operatorname{dom}(T)$ and

$$
z(T-zI)^{-1}h = z(T-zI)^{-1}T\varphi
$$

$$= z(T - zI)^{-1}(T - zI + zI)\varphi = z\varphi - z^2(T - zI)^{-1}\varphi.$$

Taking into account that

$$\|(T - zI)^{-1}\| \leqslant \frac{1}{|\operatorname{Re} z|}, \quad \operatorname{Re} z < 0,$$

and $|\operatorname{Re} z| \geq |z| \sin\beta$ for $\pi/2 + \beta \leq |\arg z| \leq \pi$, we get for all $\varphi \in \operatorname{dom}(T)$ that

$$\lim_{\substack{z\to 0,\\ \pi/2+\beta\leq|\arg z|\leq\pi}} z(T - zI)^{-1}T\varphi = 0.$$

Further, since $\operatorname{ran}(T)$ is dense in $\overline{\operatorname{ran}(T)}$ and

$$\|z(T - zI)^{-1}\| \leqslant \frac{1}{\sin\beta}, \quad \pi/2 + \beta \leq |\arg z| \leq \pi,$$

we have

$$\lim_{\substack{z\to 0,\\ \pi/2+\beta\leq|\arg z|\leq\pi}} z(T - zI)^{-1}h = 0$$

for all $h \in \overline{\operatorname{ran}(T)}$. Thus (3.1) is valid.

2. Relation (3.2) follows from (0.1). □

Proposition 3.2. *Let S be a densely defined closed α-sectorial operator, $\gamma(z)$ be its γ-field corresponding to the boundary pair $\{\mathcal{H}, \Gamma\}$ of S. Suppose $S_{\rm F} \neq S_{\rm N}$. Then for all $e \in \mathcal{H}$ such that $\gamma(\lambda)e \in \mathrm{D}[S_{\rm N}]$,*

$$\lim_{\substack{z\to 0,\\ z\in\mathbb{C}\setminus\Theta(\beta)}} z\gamma(z)e = 0, \quad \textit{where } \beta \in (\alpha, \pi/2).$$

Proof. Let $\gamma(\lambda)e \in \mathrm{D}[S_{\rm N}]$. Since $\mathrm{D}[S_{\rm N}] \cap \mathfrak{N}_\lambda = \mathrm{R}[S_{\rm F}] \cap \mathfrak{N}_\lambda$, then $\gamma(\lambda)e \in \mathrm{R}[S_{\rm F}]$. Since $\overline{\mathrm{R}[S_{\rm F}]} = \overline{\operatorname{ran}(S_{\rm F})} = \overline{\operatorname{ran}(S_{\rm F}^*)}$, from Lemma 3.1 and (2.1), we have

$$\lim_{\substack{z\to 0,\\ z\in\mathbb{C}\setminus\Theta(\beta)}} z\gamma(z)e = \lim_{\substack{z\to 0,\\ z\in\mathbb{C}\setminus\Theta(\beta)}} \big(z\gamma(\lambda)e + (z - \lambda)z(S_{\rm F}^* - zI)^{-1}\gamma(\lambda)e\big) = 0. \quad \square$$

Theorem 3.3. *Let S be a densely defined closed sectorial operator, $\gamma(z)$ be its γ-field corresponding to the boundary pair $\{\mathcal{H}, \Gamma\}$ of S. Define a set in $\mathcal{H}$:*

$$\mathcal{D}_0 := \left\{ e \in \mathcal{H} : \lim_{\substack{z\to 0,\\ z\in\mathbb{C}\setminus\Theta(\alpha)}} |(\mathcal{Q}(z)e, e)_{\mathcal{H}}| < \infty \right\}. \tag{3.3}$$

Then

$$\gamma(\mu)\mathcal{D}_0 = \mathfrak{N}_\mu \cap \mathrm{D}[S_{\rm N}]$$

for all $\mu \in \mathbb{C} \setminus \Theta(\alpha)$ *and*

$$\mathcal{D}_0 = \Gamma \mathrm{D}[S_{\mathrm{N}}].$$

Moreover, for all $e, g \in \mathcal{D}_0$ *the following limits exist:*

$$\begin{aligned}\Omega_0[e,g] = -\lim_{\substack{z\to 0\\ z\in\mathbb{C}\setminus\Theta(\beta)}} (\mathcal{Q}(z)e,g) &= i\,(\gamma(i)e,\gamma(i)g)\\ &\quad + \left((I-iG_{\mathrm{F}})^{-1}\widehat{S}_{\mathrm{FR}}^{-\frac{1}{2}}\gamma(i)e, \widehat{S}_{\mathrm{FR}}^{-\frac{1}{2}}\gamma(i)g\right)\\ &= i\,(\gamma(i)e,\gamma(i)g) + S_{\mathrm{F}}^{*-1}\left[\gamma(i)e,\gamma(i)g\right], \qquad (3.4)\end{aligned}$$

$$X_0 e := -\lim_{\substack{z\to 0\\ z\in\mathbb{C}\setminus\Theta(\beta)}} \mathcal{G}(\bar{z})^* e = i(I-iG_{\mathrm{F}})^{-1}\widehat{S}_{\mathrm{FR}}^{-\frac{1}{2}}\gamma(i)e, \quad \beta\in(\alpha,\pi/2). \qquad (3.5)$$

Proof. Let $e \in \mathcal{H}$. Using (2.1) and (2.5), we have

$$(\mathcal{Q}(z)e,e)_{\mathcal{H}} = (z-i)(\gamma(i)e + (z-i)((S_{\mathrm{F}}^* - zI)^{-1}\gamma(i)e,\gamma(i)e)$$

for $z \in \mathbb{C} \setminus \Theta(\alpha)$. Hence,

$$((S_{\mathrm{F}}^* - zI)^{-1}\gamma(i)e,\gamma(i)e) = -\frac{1}{z-i}(\gamma(i)e,\gamma(i)e) + \frac{1}{(z-i)^2}(\mathcal{Q}(z)e,e)_{\mathcal{H}}.$$

The latter equality and (1.1) yield

$$\begin{aligned}\lim_{\substack{z\to 0,\\ z\in\mathbb{C}\setminus\Theta(\alpha)}} \left|(\mathcal{Q}(z)e,e)_{\mathcal{H}}\right| < \infty \iff &\lim_{\substack{z\to 0,\\ z\in\mathbb{C}\setminus\Theta(\alpha)}} \left|((S_{\mathrm{F}}^* - zI)^{-1}\gamma(i)e,\gamma(i)e)\right| < \infty\\ \iff &\gamma(i)e \in \mathrm{R}[S_{\mathrm{F}}] \cap \mathfrak{N}_i.\end{aligned}$$

Let $\mathcal{D}_0$ be defined by (3.3). Then, using (1.4), (1.9) and Proposition 3.2, we obtain

$$e \in \mathcal{D}_0 \iff \gamma(i)e \in \mathfrak{N}_i \cap \mathrm{D}[S_{\mathrm{N}}].$$

Hence $\gamma(\mu)\mathcal{D}_0 = \mathfrak{N}_\mu \cap \mathrm{D}[S_{\mathrm{N}}]$ for all $\mu \in \mathbb{C} \setminus \Theta(\alpha)$. Observe that $\mathcal{D}_0$ is a linear manifold. Equality (1.5) yields that $\Gamma \mathrm{D}[S_{\mathrm{N}}] = \mathcal{D}_0$.

From the equality $\gamma(z) = \gamma(i) + (z-i)(S_{\mathrm{F}}^* - zI)^{-1}\gamma(i)$, taking account of the inclusion $\gamma(i)\mathcal{D}_0 \subseteq \overline{\mathrm{ran}(S_{\mathrm{F}}^*)}$ and applying Proposition 3.2, we obtain

$$\lim_{\substack{z\to 0,\\ z\in\mathbb{C}\setminus\Theta(\beta)}} z\gamma(z)e = 0, \quad \text{for } e \in \mathcal{D}_0, \quad \beta \in (\alpha,\pi/2).$$

We will now prove the remaining equalities of the theorem.

Let $e, g \in \mathcal{H}$ and $z \in \mathbb{C} \setminus \Theta(\beta)$, $\beta \in (\alpha,\pi/2)$. Using (1.2), we have

$$\Omega_0[e,g] = -\lim_{z\to 0}(\mathcal{Q}(z)e,g)$$

$$= -\lim_{z\to 0}\big((z-i)(\gamma(i)e + (z-i)((S_{\rm F}^* - zI)^{-1}\gamma(i)e, \gamma(i)g)\big)$$
$$= i\,(\gamma(i)e, \gamma(i)g) + \left((I - iG_{\rm F})^{-1}\widehat{S}_{\rm FR}^{-\frac{1}{2}}\gamma(i)e, \widehat{S}_{\rm FR}^{-\frac{1}{2}}\gamma(i)g\right).$$

Analogously, using (1.3), for X_0 we have

$$X_0 e = -\lim_{z\to 0}\mathcal{G}(\bar z)^* e = -\lim_{z\to 0} S_{\rm FR}^{\frac{1}{2}}\mathcal{P}_{\rm F}\gamma(\bar z)e$$
$$= -\lim_{z\to 0}(\bar z - i)S_{\rm FR}^{\frac{1}{2}}(S_{\rm F}^* - \bar z I)^{-1}S_{\rm FR}^{\frac{1}{2}}\widehat{S}_{\rm FR}^{-\frac{1}{2}}\gamma(i)e = i(I - iG_{\rm F})^{-1}\widehat{S}_{\rm FR}^{-\frac{1}{2}}\gamma(i)e. \quad \square$$

Clearly, the form $\Omega_0[e,g]$ can also be rewritten as follows:

$$\Omega_0[e,g] = i\,(\gamma(i)e, \gamma(i)g) - i\left(X_0 e, \widehat{S}_{\rm FR}^{-\frac{1}{2}}\gamma(i)g\right), \quad e, g \in \mathcal{D}_0.$$

Using the expressions for Ω_0 and X_0, by straightforward calculations, one can deduce that

$$\operatorname{Re}\Omega_0[e] = \left\|(I + iG_{\rm F})^{-1}\widehat{S}_{\rm FR}^{-\frac{1}{2}}\gamma(i)e\right\|^2 = \|X_0 e\|^2, \quad e \in \mathcal{D}_0. \tag{3.6}$$

It follows that the sesquilinear form $\Omega_0[e,g]$ is accretive and, moreover, the form $\operatorname{Re}\Omega_0$ is closed in the Hilbert space $\mathcal{H}$. Observe that the form

$$\mathfrak{t}_0[e,g] := \Omega_0[e,g] - i(\gamma(i)e, \gamma(i)g) = S_{\rm F}^{*-1}[\gamma(i)e, \gamma(i)g], \quad e, g \in \mathcal{D}_0,$$

is closed and sectorial in $\mathcal{H}$. Let the linear relation $\mathfrak{T}_0$ be associated with $\mathfrak{t}_0$ by the First Representation Theorem (see [34] for nondensely defined closed sectorial forms). Then define $\mathbf{Z}_0 = \mathfrak{T}_0 + iP_{\overline{\mathcal{D}}_0}\gamma^*(i)\gamma(i)$, where $P_{\overline{\mathcal{D}}_0}$ is the orthogonal projection in $\mathcal{H}$ onto the subspace $\overline{\mathcal{D}}_0$. The linear relation $\mathbf{Z}_0$ is m-accretive and is associated with the form Ω_0 in the sense

$$(\mathbf{Z}_0 e, g)_{\mathcal{H}} = \Omega_0[e,g] \quad \text{for all} \quad e \in \operatorname{dom}(\mathbf{Z}_0) \quad \text{and all} \quad g \in \mathcal{D}_0.$$

Theorem 3.4. *Let $\{\mathcal{H}, \Gamma\}$ be a boundary pair of S. Then the pair $\langle \mathbf{Z}_0, X_0\rangle$ corresponds to the Kreĭn–von Neumann extension $S_{\rm N}$ of the operator S in accordance with Theorem 2.4.*

Proof. It follows from (2.3) and Theorem 3.3 that

$$S_{\rm N}[u,v] = l[u,v] + \Omega_0[\Gamma u, \Gamma v] + 2\left(X_0\Gamma u, S_{\rm FR}^{\frac{1}{2}}\mathcal{P}_{\rm F} v\right), \quad u, v \in {\rm D}[S_{\rm N}]. \tag{3.7}$$

Let the pair $\langle Z_{\rm N}, X_{\rm N}\rangle$ corresponds to $S_{\rm N}$ in accordance with Theorem 2.4, $\mathrm{dom}(Z_{\rm N}) = \mathrm{dom}(S_{\rm N})$, $\mathrm{dom}(X_{\rm N}) = \Gamma\mathrm{dom}(S_{\rm N})$, that is,

$$\begin{aligned}(S_{\rm N}u, v) = l[u, v] + (Z_{\rm N}u, \Gamma v)_{\mathcal{H}}\\ + 2\left(X_{\rm N}\Gamma u, S_{\rm FR}^{\frac{1}{2}}\mathcal{P}_{\rm F}v\right), \quad u \in \mathrm{dom}(S_{\rm N}),\ v \in \mathfrak{L}.\end{aligned} \tag{3.8}$$

Then (3.7) and (3.8) imply for $v \in \mathrm{D}[S]$ that $\left(X_0\Gamma u, S_{\rm FR}^{\frac{1}{2}}v\right) = \left(X_{\rm N}\Gamma u, S_{\rm FR}^{\frac{1}{2}}v\right)$. Hence $X_{\rm N} = X_0{\upharpoonright}\Gamma\mathrm{dom}(S_{\rm N})$. Further,

$$\Omega_0[\Gamma u, \Gamma v] = (Z_{\rm N}u, \Gamma v)_{\mathcal{H}}, \quad u \in \mathrm{dom}(S_{\rm N}),\ v \in \mathrm{D}[S_{\rm N}].$$

Therefore, the m-accretive linear relation $\mathbf{Z}_{\rm N} = \{\{\Gamma u, Z_{\rm N}u\},\ u \in \mathrm{dom}(S_{\rm N})\}$ is associated with the form Ω_0. This proves the equality $\mathbf{Z}_{\rm N} = \mathbf{Z}_0$. □

Remark 3.5. If the set $\mathcal{D}_0$ in Theorem 3.3 is trivial, then the operator S admits a unique m-sectorial extension, namely the Friedrichs extension $S_{\rm F}$.

Let

$$S_{\rm N}[u, v] = \left((I + iG_{\rm N})S_{\rm NR}^{\frac{1}{2}}u, S_{\rm NR}^{\frac{1}{2}}v\right), \quad u, v \in \mathrm{D}[S_{\rm N}].$$

Since $S_{\rm N}[u, v] = S_{\rm F}[u, v]$, for all $u, v \in \mathrm{D}[S]$, there exists an isometry $U_{\rm F}$ mapping $\overline{\mathrm{ran}(S_{\rm F})}$ onto $\overline{\mathrm{ran}(S_{\rm N})}$ such that (see [6]):

$$\begin{aligned} S_{\rm NR}^{\frac{1}{2}}u &= U_{\rm F}S_{\rm FR}^{\frac{1}{2}}u, \quad u \in \mathrm{D}[S],\\ G_{\rm N}U_{\rm F} &= U_{\rm F}G_{\rm F},\\ S_{\rm NR}^{\frac{1}{2}}\varphi_\mu &= \mu U_{\rm F}(I - iG_{\rm F})^{-1}\widehat{S}_{\rm FR}^{-\frac{1}{2}}\varphi_\mu, \quad \varphi_\mu \in \mathfrak{N}_\mu \cap \mathrm{D}[S_{\rm N}].\end{aligned}$$

It follows that

$$S_{\rm NR}^{\frac{1}{2}}u = U_{\rm F}S_{\rm FR}^{\frac{1}{2}}\mathcal{P}_{\rm F}u + U_{\rm F}X_0\Gamma u, \tag{3.9}$$

A description of all closed sesquilinear forms associated with m-sectorial extensions of the operator S in terms of the boundary pairs has been obtained by the first author. For this purpose, the following notion of the boundary pair is defined.

Definition 3.6 ([5, 9])**.** The pair $\{\mathcal{H}', \Gamma'\}$ is called a boundary pair of the operator S if $\mathcal{H}'$ is a Hilbert space, and $\Gamma' : \mathrm{D}[S_{\rm N}] \to \mathcal{H}'$ is a linear operator such that $\ker(\Gamma') = \mathrm{D}[S]$, $\mathrm{ran}(\Gamma') = \mathcal{H}'$.

Remark 3.7. Since $\mathrm{D}[S]$ is a subspace in $\mathrm{D}[S_{\mathrm{N}}]$, the boundary pairs $\{\mathcal{H}', \Gamma'\}$ for the operator S exist. As has been mentioned in Subsection 1.2, in general, $\mathrm{D}[S_{\mathrm{N}}] \subseteq \mathfrak{L}$ and the equality $\mathrm{D}[S_{\mathrm{N}}] = \mathfrak{L}$ holds true iff $\mathrm{dom}(S^*) \subset \mathrm{D}[S_{\mathrm{N}}]$. Thus, a boundary pair in the sense of Definition 3.6 is not, in general, a boundary pair in the sense of Definition 2.1.

Theorem 3.8 ([5, 9])**.** *Let $\{\mathcal{H}', \Gamma'\}$ be a boundary pair of S in the sense of Definition 3.6. Then the formula*

$$\widetilde{S}[u,v] = S_{\mathrm{N}}[u,v] + \omega'\left[\Gamma' u, \Gamma' v\right] + 2\left(X'\Gamma' u, S_{\mathrm{NR}}^{\frac{1}{2}} v\right),$$
$$u, v \in \mathrm{D}[\widetilde{S}] = \Gamma'^{-1}\mathrm{D}[\omega'] \qquad (3.10)$$

establishes a bijective correspondence between all closed forms associated with the m-sectorial extensions $\widetilde{S}$ of S and all pairs $\langle \omega', X' \rangle$, where

1) *ω' is closed and sectorial sesquilinear in the Hilbert space $\mathcal{H}'$;*
2) *$X' : \mathrm{dom}(\omega') \to \overline{\mathrm{ran}(S)}$ is a linear operator such that for some $\delta \in [0,1)$:*

$$\|X' e\|^2 \le \delta^2 \,\mathrm{Re}\,\omega'[e], \quad \forall e \in \mathrm{dom}(\omega').$$

Let $\{\mathcal{H}, \Gamma\}$ be a boundary pair for S in the sense of Definition 2.1. Set

$$\begin{aligned} &\mathcal{H}' = \mathcal{D}_0 \ (= \mathrm{dom}(\Omega_0)),\\ &(e,g)_{\mathcal{H}'} = (e,g)_{\mathcal{H}} + \mathrm{Re}\,\Omega_0[e,g] = (e,g)_{\mathcal{H}} + (X_0 e, X_0 g),\\ &\Gamma' = \Gamma{\restriction}\mathrm{D}[S_{\mathrm{N}}] = \Gamma{\restriction}\left(\mathrm{D}[S] \dotplus \gamma(i)\mathcal{D}_0\right). \end{aligned} \qquad (3.11)$$

Then $\mathcal{H}'$ is a Hilbert space with respect to the inner product $(\cdot,\cdot)_{\mathcal{H}'}$, and $\{\mathcal{H}', \Gamma'\}$ is a boundary pair of S in the sense of Definition 3.6. Note that

1) the operators X_0 and $\gamma(\lambda)$ are continuous from $\mathcal{H}'$ into $\mathfrak{H}$,
2) the sesquilinear form Ω_0 is continuous in $\mathcal{H}'$.

Further, using Theorem 2.4 and representation (3.7) for the form $S_{\mathrm{N}}[u,v]$, we are going to establish additional conditions on the pair $\langle \mathbf{Z}, X\rangle$ ensuring that the corresponding (by Theorem 2.5) extension is m-sectorial.

Theorem 3.9. *Let $\{\mathcal{H}, \Gamma\}$ be a boundary pair of S (Definition 2.1) and let the linear manifold $\mathcal{D}_0$, form Ω and the linear operator X_0 be given by* (3.3), (3.4)*, and* (3.5)*, respectively. Then the pair $\langle \mathbf{Z}, X\rangle$ determines an m-sectorial extension $\widetilde{S}$ of S (see Theorem 2.5 and Remark 2.6) iff the following conditions are fulfilled:*

1) $\mathrm{dom}(\mathbf{Z}) \subseteq \mathcal{D}_0$*;*

2) *the sesquilinear form*

$$\omega[e,g] = (\mathbf{Z}e, g)_{\mathcal{H}} - \Omega_0[e,g] - 2((X - X_0)e, X_0 g)$$
$$= (\mathbf{Z}e, g)_{\mathcal{H}} + \Omega_0^*[e,g] - 2(Xe, X_0 g),\ \ e, g \in \operatorname{dom}(\mathbf{Z}) = \Gamma\operatorname{dom}(\widetilde{S}) \quad (3.12)$$

is sectorial and admits a closure in the Hilbert space $\mathcal{H}'$*;*

3) $\|(X - X_0)e\|^2 \leqslant \delta^2 \operatorname{Re}\omega[e]$, $e \in \operatorname{dom}(\mathbf{Z})$ *for some* $\delta \in [0,1)$.

Moreover, the closed sesquilinear form associated with $\widetilde{S}$ *is given by*

$$\widetilde{S}[u,v] = l[u,v] + \mathbf{Z}[\Gamma u, \Gamma v] + 2\left(\overline{X}\Gamma u, S_{\mathrm{FR}}^{\frac{1}{2}}\mathcal{P}_{\mathrm{F}} v\right), \quad u, v \in \mathrm{D}[\widetilde{S}] = \Gamma^{-1}\operatorname{dom}(\overline{\omega}), \qquad (3.13)$$

where $\overline{X}$ *is a continuous extension of* X *on the domain* $\operatorname{dom}(\overline{\omega})$ *of the closure* $\overline{\omega}$ *of* ω*, and*

$$\mathbf{Z}[e,g] := \overline{\omega}[e,g] - \Omega_0^*[e,g] + 2\left(\overline{X}e, X_0 g\right), \quad e, g, \in \operatorname{dom}(\overline{\omega}). \qquad (3.14)$$

Proof. Let $\widetilde{S}$ be an m-sectorial extension of S determined by $\langle \mathbf{Z}, X\rangle$ in accordance with Theorem 2.4. Note that since $\widetilde{S}$ is an m-sectorial extension of S, we have $\operatorname{dom}(\widetilde{S}) \subset \mathrm{D}[\widetilde{S}] \subseteq \mathrm{D}[S_{\mathrm{N}}]$, and $\Gamma\operatorname{dom}(\widetilde{S})$ is the core of the linear manifold $\Gamma\mathrm{D}[\widetilde{S}]$. Then,

$$\left(\widetilde{S}u, v\right) = l[u,v] + (\mathbf{Z}\Gamma u, \Gamma v)_{\mathcal{H}} + 2\left(X\Gamma u, S_{\mathrm{FR}}^{\frac{1}{2}}\mathcal{P}_{\mathrm{F}} v\right), \quad u, v \in \operatorname{dom}(\widetilde{S}).$$

Using (3.7), one obtains

$$\left(\widetilde{S}u, v\right) = S_{\mathrm{N}}[u,v] + (\mathbf{Z}\Gamma u, \Gamma v)_{\mathcal{H}} - \Omega_0[\Gamma u, \Gamma v] + 2\left((X - X_0)\Gamma u, S_{\mathrm{FR}}^{\frac{1}{2}}\mathcal{P}_{\mathrm{F}} v\right).$$

From (3.9) $S_{\mathrm{FR}}^{\frac{1}{2}}\mathcal{P}_{\mathrm{F}} v = U_{\mathrm{F}}^* S_{\mathrm{NR}}^{\frac{1}{2}} v - X_0\Gamma v$. Hence,

$$\begin{aligned}\left(\widetilde{S}u, v\right) &= S_{\mathrm{N}}[u,v] + (\mathbf{Z}\Gamma u, \Gamma v)_{\mathcal{H}} - \Omega_0[\Gamma u, \Gamma v] \\ &\qquad - 2((X - X_0)\Gamma u, X_0\Gamma v) + 2\left(U_{\mathrm{F}}(X - X_0)\Gamma u, S_{\mathrm{NR}}^{\frac{1}{2}} v\right) \\ &= S_{\mathrm{N}}[u,v] + \omega[\Gamma u, \Gamma v] + 2\left(U_{\mathrm{F}}(X - X_0)\Gamma u, S_{\mathrm{NR}}^{\frac{1}{2}} v\right) \\ &= S_{\mathrm{N}}[u,v] + \omega[\Gamma' u, \Gamma' v] + 2\left(\widetilde{X}\Gamma' u, S_{\mathrm{NR}}^{\frac{1}{2}} v\right), \quad u, v \in \operatorname{dom}(\widetilde{S}),\end{aligned}$$

where ω is given by (3.12), and $\widetilde{X} = U_{\rm F}(X - X_0)$. From Theorem 3.8, it follows that ω is a sectorial form, $\mathrm{dom}(\omega) = \mathrm{dom}(\mathbf{Z}) \subseteq \mathcal{D}_0 = \mathcal{H}'$, and

$$\left\|\widetilde{X}e\right\|^2 = \|(X - X_0)e\|^2 \leq \delta^2 \,\mathrm{Re}\,\omega[e]$$

for all $e \in \mathrm{dom}(\mathbf{Z})$ where $\delta \in [0,1)$. Moreover, the form ω admits the closure $\overline{\omega}$ in the Hilbert space $\mathcal{H}'$, and $\widetilde{X}$ has a continuous extension to $\mathrm{dom}\,(\overline{\omega})$ as a linear operator from $\mathrm{dom}\,(\overline{\omega})$ with the inner product

$$(e,g)_{\overline{\omega}} = (e,g)_{\mathcal{H}'} + \mathrm{Re}\,\overline{\omega}[e,g].$$

Since X_0 is continuous from $\mathcal{H}'$ into $\mathfrak{H}$, the operator X admits a continuation $\overline{X}$ on $\mathrm{dom}\,(\overline{\omega})$. It follows that the form $\mathbf{Z}$ given by (3.14) is well defined and the closed form $\widetilde{S}[u,v]$ associated with $\widetilde{S}$ is of the form (3.13).

Conversely, let conditions 1)–3) of the theorem be fulfilled. Denote by $\overline{\omega}$ the closure in $\mathcal{H}'$ of the sesquilinear form ω given by (3.12), and by $\overline{X'}$ the continuation of the operator $\widetilde{X} = U_{\rm F}(X - X_0)$ on $\mathrm{dom}(\overline{\omega})$, which exists due condition 2). Then, by Theorem 3.8, the pair $\langle\overline{\omega}, \overline{X'}\rangle$ determines by (3.10) a closed sectorial form $\widetilde{S}[u,v]$ associated with some m-sectorial extension $\widetilde{S}$ of S. □

Remark 3.10. We can rewrite condition 3) of Theorem 3.9 in a slightly different form. Let us find the real part of the form $\omega[e,e]$. We have

$$\omega[e,e] = (\mathbf{Z}e,e)_{\mathcal{H}} - \Omega_0[e,e] - 2((X - X_0)e, X_0e).$$

Using (3.6), we obtain

$$\mathrm{Re}\,\omega[e,e] = \mathrm{Re}(\mathbf{Z}e,e)_{\mathcal{H}} + \|(X - X_0)e\|^2 - \|Xe\|^2.$$

Then the inequalities

$$\|(X - X_0)e\|^2 \leqslant \delta^2 \,\mathrm{Re}\,\omega[e] = \delta^2 \left(\mathrm{Re}(\mathbf{Z}e,e)_{\mathcal{H}} + \|(X - X_0)e\|^2 - \|Xe\|^2\right)$$

and $0 \leq \delta < 1$ imply

$$M\|(X - X_0)e\|^2 \leqslant \mathrm{Re}(\mathbf{Z}e,e)_{\mathcal{H}} - \|Xe\|^2, \quad \text{where } M = \frac{1-\delta^2}{\delta^2} > 0.$$

Thus, condition 3) can be rewritten as

$$\mathrm{Re}(\mathbf{Z}e,e)_{\mathcal{H}} - \|Xe\|^2 \geqslant M\|(X - X_0)e\|^2, \quad M > 0.$$

4. Nonnegative Symmetric Operator and its Quasi-Selfadjoint m-Accretive Extensions

In this section, we will consider a densely defined closed nonnegative symmetric operator A, and parameterize all its quasi-selfadjoint m-accretive extensions in terms of abstract boundary conditions. We will use a boundary pair and the boundary triplets defined in Definitions 2.1, 2.2, and 2.3. In this case, if $\{\mathcal{H},\Gamma\}$ is a boundary pair for A in the sense of Definition 2.1, then the sesquilinear form Ω_0 and the linear operator X_0 defined on the linear manifold $\mathcal{D}_0=\Gamma \mathrm{D}[A_{\mathrm{N}}]$ (see Theorem 3.3) are of the form

$$\Omega_0[e,g]=i\left(\gamma(i)e,\gamma(i)g\right)+\left(\widehat{A}_{\mathrm{F}}^{-\frac{1}{2}}\gamma(i)e,\widehat{A}_{\mathrm{F}}^{-\frac{1}{2}}\gamma(i)g\right),$$
$$X_0e=i\widehat{A}_{\mathrm{F}}^{-\frac{1}{2}}\gamma(i)e,\quad e,g\in\mathcal{D}_0.$$

In addition, from (1.7), it follows that

$$A_{\mathrm{N}}[u,v]=\left(\left(A_{\mathrm{F}}^{\frac{1}{2}}\mathcal{P}_{z,\mathrm{F}}u+z\widehat{A}_{\mathrm{F}}^{-\frac{1}{2}}\mathcal{P}_zu\right),\left(A_{\mathrm{F}}^{\frac{1}{2}}\mathcal{P}_{z,\mathrm{F}}v+z\widehat{A}_{\mathrm{F}}^{-\frac{1}{2}}\mathcal{P}_zv\right)\right),\tag{4.1}$$

where $u,v\in\mathrm{D}[A_{\mathrm{N}}]=\mathrm{D}[A_{\mathrm{F}}]\dot{+}\left(\mathfrak{N}_z\cap\mathrm{ran}\left(A_{\mathrm{F}}^{\frac{1}{2}}\right)\right)=\mathrm{D}[A_{\mathrm{F}}]\dot{+}\gamma(z)\mathcal{D}_0$. It is established in [10] that the following assertions are equivalent for an m-accretive extension $\widetilde{A}$ of A:

1) A is a quasi-selfadjoint extension;
2) $\mathrm{dom}\big(\widetilde{A}\big)\subseteq\mathrm{D}[A_{\mathrm{N}}]$ and $\mathrm{Re}(\widetilde{A}f,f)\ge A_{\mathrm{N}}[f]$ for all $f\in\mathrm{dom}(\widetilde{A})$.

Observe that the operator L defined in (2.6) is of the form

$$\mathrm{dom}(L)=\mathrm{dom}(A^*),\quad Lu=A^*u-2iu_i,$$

where $u=u_{\mathrm{F}}+u_i$, $u_{\mathrm{F}}\in\mathrm{dom}(A_{\mathrm{F}})$, $u_i\in\mathfrak{N}_i$. If $\{\mathcal{H},\Gamma\}$ is a boundary pair for A (see Definition 2.1), then

$$Lu=A^*u-2i\gamma(i)\Gamma u,\quad u\in\mathrm{dom}(A^*).$$

Proposition 4.1. *Let A be a closed densely defined nonnegative symmetric operator in $\mathfrak{H}$ and let $\{\mathcal{H},\Gamma\}$ be its boundary pair (in the sense of Definition 2.1). Assume $\mathcal{D}_0\ne\{0\}$. Then a pair $\langle\mathbf{Z},X\rangle$ determines a quasi-selfadjoint m-accretive extension $\widetilde{A}$ of A in accordance with Theorem 2.5 iff the following conditions hold true:*

1) $\mathrm{dom}(\mathbf{Z})\subseteq\mathcal{D}_0$,
2) $X=X_0\upharpoonright\mathrm{dom}(\mathbf{Z})=i\widehat{A}_{\mathrm{F}}^{-\frac{1}{2}}\gamma(i)\upharpoonright\mathrm{dom}(\mathbf{Z})$.

Proof. Let $\widetilde{A}$ be a quasi-selfadjoint m-accretive extension of the operator A. Then $\mathrm{dom}(\widetilde{A}) \subseteq \mathrm{D}[A_{\mathrm{N}}]$. By Theorem 2.4, this implies the inclusion $\mathrm{dom}(\mathbf{Z}) \subseteq \Gamma\mathrm{D}[A_{\mathrm{N}}] = \mathcal{D}_0$. Taking into account the decomposition $\mathrm{dom}(A^*) = \mathrm{dom}(A_{\mathrm{F}}) \dot{+} \mathfrak{N}_i$, from (2.17) for $\mathrm{dom}(\widetilde{A}) \ni u = u_{\mathrm{F}} + u_i$, $u_{\mathrm{F}} \in \mathrm{dom}(A_{\mathrm{F}})$, $u_i \in \mathfrak{N}_i$, we have

$$\begin{aligned} X\Gamma u = Mu &= \frac{1}{2}\left(\widehat{A_{\mathrm{FR}}^{-\frac{1}{2}}}(\widetilde{A}u + i\mathcal{P}_i u) - (I + iG_{\mathrm{F}})A_{\mathrm{FR}}^{\frac{1}{2}}\mathcal{P}_{\mathrm{F}}u\right) \\ &= \frac{1}{2}\left(\widehat{A_{\mathrm{F}}^{-\frac{1}{2}}}(A^*u + iu_i) - A_{\mathrm{F}}^{\frac{1}{2}}u_{\mathrm{F}}\right) = \frac{1}{2}\left(\widehat{A_{\mathrm{F}}^{-\frac{1}{2}}}(A_{\mathrm{F}}u_{\mathrm{F}} + 2iu_i) - A_{\mathrm{F}}^{\frac{1}{2}}u_{\mathrm{F}}\right) \\ &= i\widehat{A_{\mathrm{F}}^{-\frac{1}{2}}}\gamma(i)\Gamma u = X_0\Gamma u. \end{aligned}$$

Now consider a pair $\langle \mathbf{Z}, X\rangle$, where $\mathbf{Z}$ is an m-accretive linear relation in $\mathcal{H}$ such that (**a**) $\mathrm{dom}(\mathbf{Z}) \subseteq \mathcal{D}_0$ and (**b**) $\mathrm{Re}(\mathbf{Z}e, e)_{\mathcal{H}} \geq ||X_0 e||^2$ for all $e \in \mathrm{dom}(\mathbf{Z})$. This pair determines an m-accretive extension $\widetilde{A}$. Let us prove that $\widetilde{A} \subseteq A^*$. Note that for all $u \in \mathfrak{L}$, $v \in \mathfrak{H}$,

$$(\Phi(\lambda)X_0\Gamma u, v) = i\left(\widehat{A_{\mathrm{F}}^{-\frac{1}{2}}}\gamma(i)\Gamma u, A_{\mathrm{F}}^{\frac{1}{2}}(A_{\mathrm{F}} - \bar{\lambda}I)^{-1}v\right) = i\left((A_{\mathrm{F}} - \lambda I)^{-1}\gamma(i)\Gamma u, v\right).$$

So,

$$\Phi(\lambda)X_0\Gamma u = i(A_{\mathrm{F}} - \lambda I)^{-1}\gamma(i)\Gamma u \subset \mathrm{dom}(A_{\mathrm{F}}). \tag{4.2}$$

Using (4.2), one gets

$$q(\lambda) - 2\Phi(\lambda)X_0 = \gamma(i) + (\lambda - i)(A_{\mathrm{F}} - \lambda I)^{-1}\gamma(i) = \gamma(\lambda). \tag{4.3}$$

From boundary conditions (2.9), for $u \in \mathfrak{L}$, we have

$$u \in \mathrm{dom}(\widetilde{A}) \Rightarrow u - (q(\lambda) - 2\Phi(\lambda)X_0)\Gamma u \in \mathrm{dom}(A_{\mathrm{F}}) \Rightarrow u - \gamma(\lambda)\Gamma u \in \mathrm{dom}(A_{\mathrm{F}})$$

and, therefore, $u \in \mathrm{dom}(A_{\mathrm{F}}) \dot{+} \mathfrak{N}_\lambda = \mathrm{dom}(A^*)$. Further, for $u = \mathcal{P}_{\lambda,F}u + \mathcal{P}_\lambda u$,

$$\begin{aligned} \widetilde{A}u &= A_{\mathrm{F}}\big(u - (q(\lambda) - 2\Phi(\lambda)X_0)\Gamma u\big) + \lambda\big(q(\lambda) - 2\Phi(\lambda)X_0\big)\Gamma u \\ &= A_{\mathrm{F}}\big(u - \gamma(\lambda)\Gamma u\big) + \lambda\gamma(\lambda)\Gamma u = A_{\mathrm{F}}\mathcal{P}_{\lambda,F}u + \lambda\mathcal{P}_\lambda u = A^*(\mathcal{P}_{\lambda,F}u + \mathcal{P}_\lambda u). \end{aligned}$$

So, $\widetilde{A} \subseteq A^*$. □

Theorem 4.2. *Let $\{\mathcal{H}, \Gamma\}$ and $\{\mathcal{H}, G_*, \Gamma\}$ be a boundary pair for A and the corresponding boundary triplet for L (see Definition 2.3). Assume $\mathcal{D}_0 \neq \{0\}$. Then there is a bijective correspondence between all m-accretive quasi-selfadjoint extensions $\widetilde{A}$ of A and all m-accretive linear relations $\mathbf{Z}$ in $\mathcal{H}$ such that* $\mathrm{dom}(\mathbf{Z}) \subseteq \mathcal{D}_0$, *and*

$$\mathrm{Re}(\mathbf{Z}e, e) \geqslant \left\|\widehat{A_{\mathrm{F}}^{-\frac{1}{2}}}\gamma(i)e\right\|^2, \quad \forall e \in \mathrm{dom}(\mathbf{Z}).$$

This correspondence is given by

$$\operatorname{dom}(\widetilde{A}) = \left\{u \in \operatorname{dom}(A^*) : G_* u \in \left(\mathbf{Z} - 2i\gamma(i)^*\gamma(i)\right)\Gamma u\right\}, \quad \widetilde{A}u = A^* u. \tag{4.4}$$

Moreover,

1) *a number $\lambda \in \rho(A_{\rm F})$ is a regular point of $\widetilde{A}$ iff*

$$\left(\mathbf{Z} - \frac{\lambda + i}{\lambda - i}\mathcal{Q}(\lambda)\right)^{-1} \in \mathbf{L}(\mathcal{H})$$

and

$$(\widetilde{A} - \lambda I)^{-1} = (A_{\rm F} - \lambda I)^{-1} + \gamma(\lambda)\left(\mathbf{Z} - \frac{\lambda + i}{\lambda - i}\mathcal{Q}(\lambda)\right)^{-1}\gamma(\bar{\lambda})^*; \tag{4.5}$$

2) *a number $\lambda \in \rho(A_{\rm F})$ is an eigenvalue of $\widetilde{A}$ iff*

$$\ker\left(\mathbf{Z} - \frac{\lambda + i}{\lambda - i}\mathcal{Q}(\lambda)\right) \neq \{0\}$$

and

$$\ker(\widetilde{A} - \lambda I) = \gamma(\lambda)\ker\left(\mathbf{Z} - \frac{\lambda + i}{\lambda - i}\mathcal{Q}(\lambda)\right).$$

Proof. We will use (2.9). Due to (4.3), the boundary condition 1) in (2.9) is fulfilled. Let us transform boundary condition 2). Due to (2.7), we have for $\lambda \in \rho(A_{\rm F})$,

$$G_*(f + q(\lambda)e) = \gamma(\bar{\lambda})^*(A_{\rm F} - \lambda I)f + \mathcal{Q}(\bar{\lambda})^* e, \quad f \in \operatorname{dom}(A_{\rm F}),\ f \in \operatorname{dom}(A_{\rm F}).$$

Hence

$$\begin{aligned} G_*(u + 2\Phi(\lambda)X\Gamma u) &= G_*(u + (q(\lambda) - \gamma(\lambda))\Gamma u) \\ &= G_*\left(u + 2i(A_{\rm F} - \lambda I)^{-1}\gamma(i)\Gamma u\right) \\ &= G_* u + 2i\gamma(\bar{\lambda})^*\gamma(i)\Gamma u \\ &= \gamma(\bar{\lambda})^*(A_{\rm F} - \lambda I)\mathcal{P}_{\lambda,F} u + \mathcal{Q}(\bar{\lambda})^*\Gamma u. \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathbf{W}(\lambda) &= \mathbf{Z} - \mathcal{Q}(\bar{\lambda})^* + 2\mathcal{G}(\lambda)X_0 \\ &= \mathbf{Z} - (\lambda + i)\gamma(\bar{\lambda})^*\gamma(i) + 2(\lambda + i)\gamma(i)^*\Phi(\lambda)X_0 \\ &= \mathbf{Z} - (\lambda + i)\left(\gamma(i)^* + (\lambda + i)\gamma(i)^*(A_{\rm F} - \lambda I)^{-1}\right)\gamma(i) \\ &\qquad - 2(\lambda + i)i\gamma(i)^*(A_{\rm F} - \lambda I)^{-1}\gamma(i) \end{aligned}$$

$$= \mathbf{Z} - (\lambda+i)\gamma(i)^* \left(I + (\lambda+i)(A_{\rm F}-\lambda I)^{-1} - 2i(A_{\rm F}-\lambda I)^{-1}\right)\gamma(i)$$
$$= \mathbf{Z} - (\lambda+i)\gamma^*(i)\gamma(\lambda) = \mathbf{Z} - \frac{\lambda+i}{\lambda-i}\mathcal{Q}(\lambda).$$

Then

$$\mathbf{Z} + 2\mathcal{G}(\lambda)X_0 = \mathbf{Z} + \mathcal{Q}(\bar\lambda)^* - \frac{\lambda+i}{\lambda-i}\mathcal{Q}(\lambda).$$

So, for the boundary condition 2) from (2.9), one has

$$G_*u + 2i\gamma(\bar\lambda)^*\gamma(i)\Gamma u \in \left(\mathbf{Z} + \mathcal{Q}(\bar\lambda)^* - \frac{\lambda+i}{\lambda-i}\mathcal{Q}(\lambda)\right)\Gamma u$$
$$\Longleftrightarrow G_*u \in \left(\mathbf{Z} + \mathcal{Q}(\bar\lambda)^* - \frac{\lambda+i}{\lambda-i}\mathcal{Q}(\lambda) - 2i\gamma(\bar\lambda)^*\gamma(i)\right)\Gamma u$$
$$\Longleftrightarrow G_*u \in \left(\mathbf{Z} + (\lambda+i)\gamma(\bar\lambda)^*\gamma(i) - \frac{\lambda+i}{\lambda-i}\mathcal{Q}(\lambda) - 2i\gamma(\bar\lambda)^*\gamma(i)\right)\Gamma u$$
$$\Longleftrightarrow G_*u \in \left(\mathbf{Z} + \frac{\lambda-i}{\lambda+i}\mathcal{Q}(\bar\lambda)^* - \frac{\lambda+i}{\lambda-i}\mathcal{Q}(\lambda)\right)\Gamma u.$$

Further, using that $\mathcal{Q}(\lambda) = (\lambda-i)\gamma(i)^*\gamma(i)$, we get

$$\begin{aligned}
\big(\mathbf{Z} + (\lambda-i)\gamma(\bar\lambda)^*\gamma(i) &- (\lambda+i)\gamma(i)^*\gamma(\lambda)\big)\Gamma u\\
&= \big(\mathbf{Z} + (\lambda-i)\left(\gamma(i)^* + (\lambda+i)\gamma(i)^*(A_{\rm F}-\lambda I)^{-1}\right)\gamma(i)\\
&\quad -(\lambda+i)\gamma(i)^*\left(\gamma(i) + (\lambda-i)(A_{\rm F}-\lambda I)^{-1}\gamma(i)\right)\big)\Gamma u\\
&= \big(\mathbf{Z} + \gamma(i)^*\left((\lambda-i)I + (\lambda^2+1)(A_{\rm F}-\lambda I)^{-1}\right)\\
&\quad -\left((\lambda+i)I + (\lambda^2+1)(A_{\rm F}-\lambda I)^{-1}\right)\big)\gamma(i)\Gamma u\\
&= (\mathbf{Z} - 2i\gamma(i)^*\gamma(i))\Gamma u. \qquad \square
\end{aligned}$$

Remark 4.3. The boundary condition (4.4) can also be written for any $\lambda \in \rho(\widetilde{A}) \cap \rho(A_{\rm F})$ as

$$\operatorname{dom}(\widetilde{A}) = \Big\{u \in \operatorname{dom}(A^*) :$$
$$\gamma(\bar\lambda)^*(A_{\rm F}-\lambda I)(u - \gamma(\lambda)\Gamma u) \in \left(\mathbf{Z} - \frac{\lambda+i}{\lambda-i}\mathcal{Q}(\lambda)\right)\Gamma u\Big\},$$

and

$$\widetilde{A}u = A^*u = A_{\rm F}(u - \gamma(\lambda)\Gamma u) + \lambda\gamma(\lambda)\Gamma u.$$

From Theorems 3.9 and 4.2, we obtain

Corollary 4.4. *Let* $\mathbf{Z}$ *be an m-accretive linear relation corresponding to a quasi-selfadjoint m-accretive extension* $\widetilde{A}$ *of* A *by Theorem 4.2. Then the extension* $\widetilde{A}$ *is sectorial (nonnegative) iff*

1) $\mathrm{dom}(\mathbf{Z}) \subseteq \mathcal{H}'(= \mathrm{dom}(\Omega_0) = \mathcal{D}_0)$;
2) *the form* $\widetilde{\omega}[e,g] = (\mathbf{Z}e, g)_{\mathcal{H}} - \Omega_0[e,g]$ *is sectorial (nonnegative).*

Remark 4.5. The form $\widetilde{\omega}$ admits a closure in the Hilbert space $\mathcal{H}'$ defined by (3.11). Actually, since $\widetilde{\omega}[e,g] = (\mathbf{Z}e, g)_{\mathcal{H}} - \Omega_0[e,g]$ is sectorial, the form

$$\eta[e,f] = (\mathbf{Z}e, g)_{\mathcal{H}} - i(\gamma(i)e, \gamma(i)f),\ e, f \in \mathcal{H}'(= \mathcal{D}_0)$$

is sectorial as well. If $\lim\limits_{n\to\infty} e_n = 0$ in $\mathcal{H}'$ and $\lim\limits_{m,n\to\infty} \widetilde{\omega}[e_n - e_m] = 0$, then

$$\lim_{n\to\infty} e_n = 0 \text{ in } \mathcal{H},\ \lim_{n\to\infty} \mathrm{Re}\,\Omega[e_n] = \lim_{n\to\infty} \|X_0 e_n\|^2 = 0 \text{ and } \lim_{n\to\infty} \gamma(i)e_n = 0 \text{ in } \mathfrak{H}.$$

Since the linear relation $\mathbf{Z}$ is m-accretive and $\mathbf{Z} - i\gamma^*(i)\gamma(i)$ is sectorial, we get $\lim\limits_{n\to\infty} (\mathbf{Z}e_n, e_n)_{\mathcal{H}} = 0$ (see [23]).

Next we will find relationships between

- a boundary triplet $\{\mathcal{H}, \Gamma_1, \Gamma_0\}$ for A^* given by Definition 1.2 and the boundary triplets $\{\mathcal{H}, G, \Gamma\}$, $\{\mathcal{H}, G_*, \Gamma\}$ from Definitions 2.2 and 2.3;
- the parameterizations of the quasi-selfadjoint m-accretive extensions given by Theorems 1.3 and 4.2.

Let $\{\mathcal{H}, \Gamma_1, \Gamma_0\}$ be a boundary triplet for A^* (see Definition 1.2) such that $\ker(\Gamma_0) = \mathrm{dom}(A_{\mathrm{F}})$. Then

1) since $\mathrm{dom}(A_{\mathrm{F}})$ is the core of $\mathrm{D}[A]$ and $\ker(\Gamma_0) = \mathrm{dom}(A_{\mathrm{F}})$, we can define a boundary pair $\{\mathcal{H}, \overline{\Gamma}_0\}$, where $\overline{\Gamma}_0$ is a continuation of Γ_0 onto $\mathfrak{L} = \mathrm{D}[A] \dotplus \mathfrak{N}_i$ from $\mathrm{dom}(A^*) = \mathrm{dom}(A_{\mathrm{F}}) \dotplus \mathfrak{N}_i$;
2) it follows that
$$\gamma(\lambda) = \left(\overline{\Gamma}_0 {\upharpoonright} \mathfrak{N}_\lambda\right)^{-1} = \Gamma_0(\lambda);$$
3) because relation (1.14) can be rewritten as
$$M_0(\lambda) - M_0(z) = (\lambda - z)\gamma(\bar{z})^*\gamma(\lambda),$$
using (2.5), one gets
$$\mathcal{Q}(\lambda) = (\lambda - i)\gamma(i)^*\gamma(\lambda) = \frac{\lambda - i}{\lambda + i}(M_0(\lambda) - M_0(-i));$$
so,
$$M_0(\lambda) - M_0(-i) = \frac{\lambda + i}{\lambda - i}\mathcal{Q}(\lambda); \tag{4.6}$$

4) equation (4.6) yields that the linear manifolds $\mathcal{D}_0$ in Theorems 1.3 and 3.3 coincide and

$$\tau[h,g] = (M_0(-i)h,g)_{\mathcal{H}} + \Omega_0[h,g],\ h,g \in \mathcal{D}_0;$$

5) comparing resolvent formulae (1.15) and (4.5), we obtain that the linear relation $\mathbf{Z}$ from Theorem 4.2 and the linear relation $\widetilde{\mathbf{T}}$ from Theorem 1.3 (see (1.11), (1.16)) are connected by the equality

$$\mathbf{Z} = \widetilde{\mathbf{T}} - M(-i). \tag{4.7}$$

Proposition 4.6. *Let $\{\mathcal{H},\Gamma\}$ be a boundary pair for a nonnegative symmetric operator A. Let $\widetilde{A}$ be a quasi-selfadjoint m-accretive extension of A and let $\mathbf{Z}$ be the corresponding linear relation in $\mathcal{H}$ (see Theorem 4.2). Then*

$$\mathbf{Z}^* + 2i\gamma(i)^*\gamma(i)$$

corresponds to the adjoint extension $\widetilde{A}^*$.

Proof. The proof becomes easy if we recall that to the adjoint extension $\widetilde{A}^*$ there corresponds the adjoint linear relation $\widetilde{\mathbf{T}}^*$. Since $\widetilde{\mathbf{T}} = \mathbf{Z} + M(-i)$. Then we have $\widetilde{\mathbf{T}}^* = \mathbf{Z}^* + M(-i)^*$. Again, taking the equality $M(z)^* = M(\bar{z})$ into account, from (4.7) and (1.14) one gets that the adjoint extension $\widetilde{A}^*$ corresponds to

$$\widetilde{\mathbf{T}}^* - M(-i) = \mathbf{Z}^* + M(-i)^* - M(-i) = \mathbf{Z}^* + 2i\gamma(i)^*\gamma(i). \qquad \square$$

5. m-Sectorial Extensions of a Symmetric Operator in the Model of Two-Point Interactions on a Plane

Let $y_1, y_2 \in \mathbb{R}^2$. In the Hilbert space $L_2(\mathbb{R}^2)$, consider the operator A given by

$$\operatorname{dom}(A) = \left\{f(x) \in W_2^2(\mathbb{R}^2) : f(y_1) = f(y_2) = 0,\ k = 1,2\right\}, \quad Af = -\Delta f, \tag{5.1}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $W_2^2(\mathbb{R}^2)$ is the Sobolev space, and $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the Laplacian.

The operator A is a densely defined closed nonnegative symmetric operator with defect indices $(2,2)$ [2]. Such operators are basic in the models of point interactions [2]. In the case of one point, the corresponding operator

$$\operatorname{dom}(A_y) = \left\{f(x) \in W_2^2(\mathbb{R}^2) : f(y) = 0\right\}, \quad A_y f = -\Delta f,$$

admits a unique nonnegative selfadjoint extension [1, 18], the free Hamiltonian,

$$\operatorname{dom}(A_{\mathrm{F}}) = W_2^2(\mathbb{R}^2), \quad A_{\mathrm{F}} f = -\Delta f,$$

Thus, A_y has no m-sectorial and quasi-selfadjoint m-accretive extensions. All m-accretive extensions of A_y were described in [11]. For interactions with two and more centers the relation $A_{\rm F} \neq A_{\rm N}$ holds [1]. In this section, we apply Theorems 2.5 and 3.9 to parametrize all m-sectorial extensions of the operator A. It is convenient to use the Fourier transform and the momentum representation of A:

$$\begin{aligned}&\widehat{A}\widehat{f}(p) = |p|^2\widehat{f}(p),\\ &\operatorname{dom}(\widehat{A}) = \Big\{\widehat{f}(p) \in L_2(\mathbb{R}^2, dp): \quad 1)\, |p|^2\widehat{f}(p) \in L_2(\mathbb{R}^2, dp),\\ &\qquad\qquad 2) \int_{\mathbb{R}^2} \widehat{f}(p)e^{ipy_1}\,dp = \int_{\mathbb{R}^2} \widehat{f}(p)e^{ipy_2}\,dp = 0\Big\},\end{aligned}$$

where

$$\widehat{f}(p) = (\mathcal{F}f)(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} f(x)e^{-ix\cdot p}\,dx, \quad p = (p_1, p_2),$$

is the Fourier transform of $f(x) \in L_2(\mathbb{R}^2, dx)$. For a one-center point interaction this method was used in [11]. In this paper, we omit the details in the momentum representation for brevity and give the final results in the coordinate representation.

The Friedrichs extension of A is the free Hamiltonian $A_{\rm F}$, and $A_{\rm F}^{\frac{1}{2}} = (-\Delta)^{\frac{1}{2}}$ is a pseudodifferential operator of the form:

$$\begin{aligned}&\operatorname{dom}(A_{\rm F}^{\frac{1}{2}}) = {\rm D}[A_{\rm F}] = W_2^1(\mathbb{R}^2),\\ &A_{\rm F}^{\frac{1}{2}}f(x) = \frac{1}{4\pi^2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} |p|\exp\left(i(x-y)p\right)f(y)\,dy\,dp,\end{aligned}$$

where $W_2^k(\mathbb{R}^2)$, $k = 1, 2$, are the Sobolev spaces. Note that (see [2, p. 162]) for all $\lambda \in \mathbb{C}\setminus[0,+\infty)$, $\operatorname{Im}\sqrt{\lambda} > 0$, the resolvent is of the form

$$(A_{\rm F} - \lambda I)^{-1}f(x) = \frac{i}{4}\int_{\mathbb{R}^2} H_0^{(1)}(\sqrt{\lambda}|x-y|)f(y)\,dy, \quad f \in L_2(\mathbb{R}^2),$$

where $H_0^{(1)}(\cdot)$ denotes the Hankel function of the first kind and order zero [31, p. 217]. It is well known [2, p. 160] that

$$\mathfrak{N}_\lambda = \left\{\frac{\pi i}{2}\sum_{k=1}^{2} H_0^{(1)}\left(\sqrt{\lambda}|x-y_k|\right)c_k,\ c_1, c_2 \in \mathbb{C}\right\},\ \lambda \in \mathbb{C}\setminus[0,+\infty),\ \operatorname{Im}\sqrt{\lambda} > 0,$$

is the defect subspace of A, corresponding to λ.

Therefore, for the linear manifold $\mathfrak{L}$ defined by (1.6), we have

$$\mathfrak{L} = W_2^1(\mathbb{R}^2)\dot{+}\mathfrak{N}_\lambda$$

$$= \left\{ f(x) + \frac{\pi i}{2}\sum_{k=1}^{2} H_0^{(1)}(\sqrt{\lambda}|x-y_k|)c_k,\ f \in W_2^1(\mathbb{R}^2),\ c_1, c_2 \in \mathbb{C} \right\},$$

where λ is a number from $\mathbb{C} \setminus [0, +\infty)$. Now, let $\mathcal{H} = \mathbb{C}^2$ and set

$$\Gamma\left(f(x) + \frac{\pi i}{2}\sum_{k=1}^{2} H_0^{(1)}\left(\sqrt{\lambda}|x-y_k|\right)c_k \right) = \vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \in \mathbb{C}^2,\ f(x) \in W_2^1(\mathbb{R}^2).$$

Then from the equality $\overline{H_0^{(1)}\left(\sqrt{\lambda}|x|\right)} = H_0^{(2)}\left(\sqrt{\bar{\lambda}}|x|\right)$ [31, p. 226], it follows that

$$\gamma(\lambda)\vec{c} = \frac{\pi i}{2}\sum_{k=1}^{2} H_0^{(1)}\left(\sqrt{\lambda}|x-y_k|\right)c_k, \quad \vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \in \mathbb{C}^2,$$

$$\gamma(\bar{\lambda})^* h(x) = -\frac{\pi i}{2}\begin{bmatrix} \int_{\mathbb{R}^2} h(x) H_0^{(2)}\left(\sqrt{\lambda}|x-y_1|\right)dx \\ \int_{\mathbb{R}^2} h(x) H_0^{(2)}\left(\sqrt{\lambda}|x-y_2|\right)dx \end{bmatrix}.$$

Set $r = |y_1 - y_2|$ and $H(\lambda, r) = H_0^{(1)}(\sqrt{\lambda}r) - H_0^{(1)}(e^{3\pi i/4}r)$. From (2.5), using the unitarity of the Fourier transform, one can derive that the matrix $\mathcal{Q}(\lambda)$ in the standard basis is of the form

$$\mathcal{Q}(\lambda) = \frac{\lambda - i}{\lambda + i}\pi \begin{bmatrix} -\ln(\lambda i) & \pi i H(\lambda, r) \\ \pi i H(\lambda, r) & -\ln(\lambda i) \end{bmatrix}.$$

Hence,

$$\mathcal{Q}(\lambda)^* = \frac{\bar{\lambda} + i}{\bar{\lambda} - i}\pi \begin{bmatrix} -\ln\left(\frac{\bar{\lambda}}{i}\right) & -\pi i \bar{H}(\lambda, r) \\ -\pi i \bar{H}(\lambda, r) & -\ln\left(\frac{\bar{\lambda}}{i}\right) \end{bmatrix}.$$

Now we will find the subspace $\mathcal{D}_0$ and the sesquilinear form $\Omega_0[\cdot,\cdot]$ (see Theorem 3.3). We have

$$\begin{aligned}(\mathcal{Q}(\lambda)\vec{c}, \vec{d}) &= \frac{\lambda - i}{\lambda + i}\pi \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}^* \begin{bmatrix} -\ln(\lambda i) & \pi i H(\lambda, r) \\ \pi i H(\lambda, r) & -\ln(\lambda i) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \frac{\lambda - i}{\lambda + i}\pi \Big(-(c_1\bar{d}_1 + c_2\bar{d}_2)\ln(\lambda i) \\ &\qquad + (c_2\bar{d}_1 + c_1\bar{d}_2)\pi i \left(H_0^{(1)}(\sqrt{\lambda}r) - H_0^{(1)}(e^{3\pi i/4}r) \right) \Big).\end{aligned}$$

Taking into account the asymptotic behavior [31, p. 223]

$$H_0^{(1)}(\lambda) = 1 + \frac{2i}{\pi}\left(\ln\left(\frac{\lambda}{2}\right) + \gamma \right) + o(\lambda), \quad \lambda \to 0,$$

where γ is Euler's constant, we see that

$$\mathcal{D}_0 := \left\{ e \in \mathcal{H} : \varlimsup_{\substack{z\to 0,\\ z\in\mathbb{C}\setminus[0,+\infty)}} |(\mathcal{Q}(z)e,e)_{\mathcal{H}}| < \infty \right\} = \left\{ \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} \in \mathbb{C}^2 : \zeta \in \mathbb{C} \right\}.$$

Set $\vec{c}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then for the form Ω_0 given by (3.4), we have

$$\begin{aligned}
\Omega_0[\zeta\vec{c}_0, \eta\vec{c}_0] &= -\zeta\bar{\eta} \lim_{\substack{\lambda\to 0\\ \lambda<0}} (\mathcal{Q}(\lambda)\vec{c}_0, \vec{c}_0) \\
&= \pi\zeta\bar{\eta} \lim_{\substack{\lambda\to 0\\ \lambda<0}} \left(-2\ln(\lambda i) - 2\pi i \left(H_0^{(1)}\left(\sqrt{\lambda} r\right) - H_0^{(1)}\left(e^{3\pi i/4} r\right)\right)\right) \\
&= 2\pi\zeta\bar{\eta} \lim_{\substack{\lambda\to 0\\ \lambda<0}} \Bigg(-\ln(\lambda i) - \pi i \Bigg(1 + \frac{2i}{\pi}\left(\ln\left(\frac{\sqrt{\lambda} r}{2}\right) + \gamma \right) \\
&\qquad\qquad - H_0^{(1)}\left(e^{3\pi i/4} r\right)\Bigg)\Bigg) \\
&= 4\pi\zeta\bar{\eta} \left(\ln\frac{r}{2} - \frac{3\pi i}{4} + \gamma + \frac{\pi i}{2} H_0^{(1)}\left(e^{3\pi i/4} r\right)\right) = \omega_0\zeta\bar{\eta},
\end{aligned}$$

where $\omega_0 = 4\pi\left(\ln\frac{r}{2} - \frac{3\pi i}{4} + \gamma + \frac{\pi i}{2} H_0^{(1)}(e^{3\pi i/4} r)\right)$. From the latter equality one can obtain that

$$\operatorname{Re}\Omega_0[\zeta\vec{c}_0, \eta\vec{c}_0] = \operatorname{Re}\omega_0 \cdot \zeta\bar{\eta} = 4\pi\left(\ln\frac{r}{2} + \gamma + \mathbf{ker}(r)\right)\zeta\bar{\eta},$$

where the functions $\mathbf{ker}(\cdot)$ and $\mathbf{kei}(\cdot)$ are the Kelvin functions [31, p. 268], i.e., the real and the imaginary parts of the function $\frac{\pi i}{2} H_0^{(1)}(e^{3\pi i/4}(\cdot))$, respectively:

$$\mathbf{ker}(r) + i\,\mathbf{kei}(r) = \frac{\pi i}{2} H_0^{(1)}(e^{3\pi i/4} r).$$

For the functions $\Phi(\lambda)$, $\mathcal{G}(\lambda)$, $\mathcal{Q}(\bar{\lambda})^*$, and $q(\lambda)$ on $\mathcal{D}_0 = \operatorname{dom}(\Omega_0)$ defined in (2.5), we have

$$\begin{aligned}
\Phi(\lambda) X \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} &= \frac{\zeta}{4\pi^2} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|p|}{|p|^2 - \lambda} \exp\left(i(x-y)p\right) g(y)\, dy\, dp, \\
\mathcal{G}(\lambda) X \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} &= -\frac{\pi i(\lambda + i)\zeta}{2} \begin{bmatrix} \int_{\mathbb{R}^2} \Phi(\lambda)(f(x)) H_0^{(2)}\left(e^{3\pi i/4}|x - y_1|\right) dx \\ \int_{\mathbb{R}^2} \Phi(\lambda)(f(x)) H_0^{(2)}\left(e^{3\pi i/4}|x - y_2|\right) dx \end{bmatrix}, \\
\mathcal{Q}(\bar{\lambda})^* \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} &= \frac{\lambda + i}{\lambda - i}\pi \left(-\ln\left(\frac{\lambda}{i}\right) + \pi i \overline{H(\bar{\lambda}, |y_1 - y_2|)} \right) \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix},
\end{aligned}$$

$$q(\lambda)\begin{bmatrix}\zeta\\ -\zeta\end{bmatrix} = \frac{\pi i}{2}\frac{1}{i-\lambda}\zeta\left((i+\lambda)(H_0^{(1)}\left(\sqrt{\lambda}|x-y_2|\right) - H_0^{(1)}\left(\sqrt{\lambda}|x-y_1|\right)\right) \\ + 2i\left(H_0^{(1)}\left(e^{\pi i/4}|x-y_1|\right) - H_0^{(1)}\left(e^{\pi i/4}|x-y_2|\right)\right)\right).$$

Now we find the operator $X_0 e = i\widehat{A}_{\rm F}^{-\frac{1}{2}}\gamma(i)e$, $e\in\mathcal{D}_0$, from Theorem 3.3. As we have mentioned above, it is convenient to use the momentum representation. Let $\widehat{\gamma}(\lambda) = \mathcal{F}\gamma(\lambda)$. Then,

$$\widehat{\gamma}(\lambda)\vec{c} = \sum_{k=1}^{2} c_k \frac{e^{-ipy_k}}{|p|^2-\lambda}, \quad \forall \vec{c} = \begin{bmatrix}c_1\\ c_2\end{bmatrix}\in\mathbb{C}^2.$$

Hence,

$$\widehat{X}_0\begin{bmatrix}\zeta\\ -\zeta\end{bmatrix} = \mathcal{F}X_0\begin{bmatrix}\zeta\\ -\zeta\end{bmatrix} = i\widehat{A}_{\rm F}^{-\frac{1}{2}}\gamma(i)\begin{bmatrix}\zeta\\ -\zeta\end{bmatrix} = \frac{i\left(e^{-ipy_1}-e^{-ipy_2}\right)}{|p|\left(|p|^2-i\right)}\zeta.$$

So, $\widehat{X}_0\begin{bmatrix}\zeta\\ -\zeta\end{bmatrix} = \widehat{g}_0(p)\zeta$, where

$$\widehat{g}_0(p) = \frac{i\left(e^{-ipy_1}-e^{-ipy_2}\right)}{|p|\left(|p|^2-i\right)}. \tag{5.2}$$

Getting back to the coordinate representation, by using [35], [22, p.671], we obtain

$$\begin{aligned} g_0(x) = \mathcal{F}^{-1}\widehat{g}_0(p) &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{i\left(e^{ip(x-y_1)}-e^{ip(x-y_2)}\right)}{|p|\left(|p|^2-i\right)}\,dp \\ &= i\int_0^{+\infty}\frac{J_0\left(\rho|x-y_1|\right)-J_0\left(\rho|x-y_2|\right)}{\rho^2-i}\,d\rho \\ &= \frac{\pi i}{2\sqrt{-i}}\left(I_0\left(\sqrt{-i}|x-y_1|\right)-L_0\left(\sqrt{-i}|x-y_1|\right)\right) \\ &\quad - \frac{\pi i}{2\sqrt{-i}}\left(I_0\left(\sqrt{-i}|x-y_2|\right)-L_0\left(\sqrt{-i}|x-y_2|\right)\right) \\ &= -\frac{\pi}{2}e^{3\pi i/4}\left(\mathbf{M}_0\left(e^{-\pi i/4}|x-y_1|\right)-\mathbf{M}_0\left(e^{-\pi i/4}|x-y_2|\right)\right), \end{aligned}$$

where $I_0(\cdot)$ is the Bessel function and $L_0(\cdot),\mathbf{M}_0(\cdot)$ are the modified Struve functions [31, p. 288]. So,

$$X_0\begin{bmatrix}\zeta\\ -\zeta\end{bmatrix} = g_0(x)\zeta,$$

where $g_0(x) = -\frac{\pi}{2}e^{3\pi i/4}\left(\mathbf{M}_0\left(e^{-\pi i/4}|x-y_1|\right)-\mathbf{M}_0\left(e^{-\pi i/4}|x-y_2|\right)\right)$. According to (3.6), we have

$$\|g_0\|^2_{L_2(\mathbb{R}^2)} = \operatorname{Re}\omega_0 = 4\pi\left(\ln\frac{r}{2}+\gamma+\mathbf{ker}(r)\right).$$

Remark 5.1. Since $\|g_0(x)\|^2_{L_2(\mathbb{R}^2)} = \|\widehat{g}_0(p)\|^2_{L_2(\mathbb{R}^2)}$ (the unitarity of the Fourier transform), expression (5.2) for $\widehat{g}_0(p)$ gives $\|\widehat{g}_0(p)\|^2_{L_2(\mathbb{R}^2)} = 4\pi\int_0^\infty \frac{1-J_0(r\rho)}{\rho(\rho^4+1)}d\rho$. On the other hand, due to (3.6), we have $\|g_0(x)\|^2_{L_2(\mathbb{R}^2)} = \operatorname{Re}\omega_0$. This leads to the value of the improper integral $\int_0^\infty \frac{1-J_0(r\rho)}{\rho(\rho^4+1)}\,d\rho$:

$$\int_0^\infty \frac{1-J_0(r\rho)}{\rho(\rho^4+1)}\,d\rho = \frac{1}{4\pi}\operatorname{Re}\omega_0 = \left(\ln\frac{r}{2} + \gamma + \mathbf{ker}(r)\right).$$

To describe all m-sectorial extensions of A, we have to define the pairs $\langle \mathbf{Z}, X\rangle$ satisfying conditions 3), 4) from Theorem 2.4 and conditions 1)–3) from Theorem 3.9. Since $\mathbf{Z}$ is an m-accretive linear relation in $\mathbb{C}^2$ and $\operatorname{dom}(\mathbf{Z}) \subseteq \mathcal{D}_0$, there are only two possible cases:

1. $\mathbf{Z} = \left\langle \begin{bmatrix}\zeta\\-\zeta\end{bmatrix}, z\cdot\begin{bmatrix}\zeta\\-\zeta\end{bmatrix}\right\rangle \oplus \left\langle 0, \begin{bmatrix}\eta\\\eta\end{bmatrix}\right\rangle$, $\zeta, \eta, z \in \mathbb{C}$, $\operatorname{Re} z \geqslant 0$.
2. $\mathbf{Z} = \langle 0, \mathbb{C}^2\rangle$. As [11] says, this linear relation corresponds to the Friedrichs extension A_{F} of A.

In the first case, the operator X, acting from $\operatorname{dom}(\mathbf{Z})$ into $L_2(\mathbb{R}^2)$, takes the form $X\begin{bmatrix}\zeta\\-\zeta\end{bmatrix} = \zeta g(x)$, where a function $g \in L_2(\mathbb{R}^2)$ satisfies the condition

$$\|g\|^2_{L_2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |g|^2 dx \leqslant 2\operatorname{Re} z. \tag{5.3}$$

For the form $\omega[\cdot,\cdot]$ defined by (3.12), we have

$$\begin{aligned}\omega[\zeta\vec{c}_0, \eta\vec{c}_0] &= (\mathbf{Z}\zeta\vec{c}_0, \eta\vec{c}_0) - \Omega_0[\zeta\vec{c}_0, \eta\vec{c}_0] - 2((X - X_0)\zeta\vec{c}_0, X_0\eta\vec{c}_0)\\ &= \left(2z - \omega_0 - 2\int_{\mathbb{R}^2}(g(x) - g_0(x))\overline{g_0(x)}\,dx\right)\zeta\bar{\eta},\\ \operatorname{Re}\omega[\zeta\vec{c}_0] &= \left(2\operatorname{Re} z + \int_{\mathbb{R}^2}|g(x) - g_0(x)|^2\,dx - \int_{\mathbb{R}^2}|g(x)|^2\,dx\right)|\zeta|^2.\end{aligned}$$

Thus, the form $\omega[\cdot,\cdot]$ is determined by the number

$$w_{\langle z,g\rangle} = 2z - \omega_0 - 2\int_{\mathbb{R}^2}(g(x) - g_0(x))\overline{g_0(x)}\,dx. \tag{5.4}$$

Clearly, the form $\omega[\cdot,\cdot]$ is sectorial iff

$$\begin{aligned}\operatorname{Re} w_{\langle z,g\rangle} = 2\operatorname{Re} z &+ \int_{\mathbb{R}^2}|g(x) - g_0(x)|^2\,dx\\ &- \int_{\mathbb{R}^2}|g(x)|^2\,dx > 0 \quad \text{or} \quad w_{\langle z,g\rangle} = 0.\end{aligned} \tag{5.5}$$

Remark 5.2. Due to $2\operatorname{Re} z - \int_{\mathbb{R}^2} |g(x)|^2\,dx \geq 0$, the equality $w_{\langle z,g\rangle} = 0$ implies that $g(x) = g_0(x)$ almost everywhere and $z = \omega_0/2$.

Further, condition 3) from Theorem 3.9 takes the form

$$M\int_{\mathbb{R}^2} |g(x) - g_0(x)|^2\,dx \leqslant 2\operatorname{Re} z - \int_{\mathbb{R}^2} |g(x)|^2\,dx,$$

where $M > 0$. The latter inequality can be simplified as follows:

$$2\operatorname{Re} z - \int_{\mathbb{R}^2} |g(x)|^2\,dx > 0. \tag{5.6}$$

So, conditions (5.3), (5.5) are satisfied. Note that in this case, the linear relation $\mathbf{W}(\lambda)$, see (2.10), is of the form

$$\mathbf{W}(\lambda) = \left\langle \begin{bmatrix}\zeta\\ -\zeta\end{bmatrix}, \left(z - \frac{\lambda+i}{\lambda-i}\pi\left(-\ln\left(\frac{\lambda}{i}\right) + \pi i\overline{H\left(\bar{\lambda}, |y_1 - y_2|\right)}\right)\right)\begin{bmatrix}\zeta\\ -\zeta\end{bmatrix} \right.$$
$$\left. - \pi i(\lambda+i)\zeta\begin{bmatrix}\int_{\mathbb{R}^2}\Phi(\lambda)(g(x))H_0^{(2)}\left(e^{3\pi i/4}|x-y_1|\right)\,dx\\ \int_{\mathbb{R}^2}\Phi(\lambda)(g(x))H_0^{(2)}\left(e^{3\pi i/4}|x-y_2|\right)\,dx\end{bmatrix} + \begin{bmatrix}\eta\\ \eta\end{bmatrix}\right\rangle$$

for all $\lambda \in \rho(A_{\mathrm{F}}) = \mathbb{C} \setminus [0, +\infty)$. Then

$$\mathbf{W}(\lambda)^{-1} = \left\langle \begin{bmatrix}\zeta\\ \eta\end{bmatrix}, \frac{1}{w_{\langle z,g\rangle}(\lambda)}\begin{bmatrix}\zeta - \eta\\ -\zeta + \eta\end{bmatrix}\right\rangle,$$

where

$$w_{\langle z,g\rangle}(\lambda) = 2\left(z - \frac{\lambda+i}{\lambda-i}\pi\left(-\ln\left(\frac{\lambda}{i}\right) + \pi i\overline{H\left(\bar{\lambda}, |y_1-y_2|\right)}\right)\right)$$
$$- \pi i(\lambda+i)\int_{\mathbb{R}^2}\Phi(\lambda)(g(x))\left(H_0^{(2)}\left(e^{3\pi i/4}|x-y_1|\right) - H_0^{(2)}\left(e^{3\pi i/4}|x-y_2|\right)\right)dx.$$

Clearly, $\ker(\mathbf{W}(\lambda)) \neq \{0\}$ iff $w_{\langle z,g\rangle}(\lambda) = 0$ and

$$\ker(\mathbf{W}(\lambda)) = \operatorname{dom}(\mathbf{W}(\lambda)) = \begin{bmatrix}\eta\\ -\eta\end{bmatrix}, \quad \eta \in \mathbb{C}.$$

Let an m-sectorial extension $\widetilde{A}$ of A be defined by a pair $\langle z, g(x)\rangle$ satisfying (5.6) (see Theorem (3.9)). Since $\widetilde{A}$ is an m-sectorial extension, and

$$G(-i) = 0, \quad \mathcal{Q}(-i)^* = 0, \quad q(-i) = \gamma(i),$$

it is suitable to take $\lambda = -i$ and apply Theorem 2.5, Remark 2.6, and equalities (2.12), (2.13), (2.14). Then,

$$\mathbf{W} = \mathbf{W}(-i) = \mathbf{Z} = \left\langle \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix}, z \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} \right\rangle \oplus \left\langle \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \eta \\ \eta \end{bmatrix} \right\rangle,$$
$$\mathbf{W}^{-1} = \left\langle \begin{bmatrix} \zeta \\ \eta \end{bmatrix}, \frac{1}{2z} \begin{bmatrix} \zeta - \eta \\ -\zeta + \eta \end{bmatrix} \right\rangle.$$

By (2.12),

$$\operatorname{dom}(\widetilde{A}) = \Big(I + (q(-i) - 2\Phi(-i)X)\mathbf{W}(-i)^{-1}\gamma(-i)^*(A_{\mathrm{F}} + iI)\Big) \operatorname{dom}(A_{\mathrm{F}}).$$

Further, let $\delta(x)$, $x = (x_1, x_2)$ be the Dirac delta-function. Then $\delta(x) \in W_2^{-2}(\mathbb{R}^2)$ [2]. Since $\mathcal{F}(\delta(x)) = 1/2\pi$, then $\mathcal{F}^{-1}(1) = 2\pi\delta(x)$. So, if $\mathcal{F}(h(x)) = \widehat{h}(p)$ and $h(x) \in \operatorname{dom}(A_{\mathrm{F}}) = W_2^2(\mathbb{R}^2)$, then

$$\int_{\mathbb{R}^2} \left(e^{ipy_1} - e^{ipy_2}\right) \widehat{h}(p)\, dp = 2\pi(h(y_1) - h(y_2)).$$

Using the latter equality and the Fourier transform, we obtain

$$\mathbf{W}(-i)^{-1}\gamma(-i)^*(A_{\mathrm{F}} + iI)h(x) = \frac{\pi(h(y_1) - h(y_2))}{z}.$$

If $h \in \operatorname{dom}(A_{\mathrm{F}})$, then

$$\operatorname{dom}(\widetilde{A}) = \left\{ u(x) = h(x) + \frac{\pi(h(y_1) - h(y_2))}{z} \left(\frac{\pi i}{2} \left(H_0^{(1)}\left(e^{3\pi i/4}|x - y_1|\right) - H_0^{(1)}\left(e^{3\pi i/4}|x - y_2|\right) \right) - 2\Phi(-i)(g(x)) \right) \right\}. \quad (5.7)$$

Then, applying Theorems 2.5, 3.9, we arrive at the following statement.

Theorem 5.3. *There is a bijective correspondence between all m-sectorial extensions $\widetilde{A}$ (except Friedrichs and Kreĭn–von Neumann extensions) of A given by* (5.1) *and all pairs $\langle z, g\rangle$, where $z \in \mathbb{C}$ and a function $g \in L_2(\mathbb{R}^2)$ are such that*

$$\|g\|^2_{L_2(\mathbb{R}^2)} < 2 \operatorname{Re} z.$$

This correspondence is given by (5.7)*, where $h \in W_2^2(\mathbb{R}^2)$, and by the relation*

$$\widetilde{A}u(x) = -\Delta h(x) - i\frac{\pi(h(y_1) - h(y_2))}{z} \times \left(\frac{\pi i}{2} \left(H_0^{(1)}(e^{3\pi i/4}|x - y_1|) - H_0^{(1)}(e^{3\pi i/4}|x - y_2|) \right) - 2\Phi(-i)(g(x)) \right).$$

Moreover,

1) *a number* $\lambda \in \mathbb{C} \setminus [0, +\infty)$ *is a regular point of* $\widetilde{A}$ *iff* $w_{\langle z, g(x)\rangle}(\lambda) \neq 0$, *and*

$$
\begin{aligned}
(\widetilde{A} - \lambda I)^{-1} h(x) = & \frac{i}{4} \int_{\mathbb{R}^2} H_0^{(1)}(\sqrt{\lambda}|x-y|) f(y)\, dy + \frac{1}{w_{\langle z, g(x)\rangle}} \\
& \times \left(\frac{\pi i}{2} \frac{1}{i - \lambda} \Big((i+\lambda)(H_0^{(1)}(\sqrt{\lambda}|x-y_2|) - H_0^{(1)}(\sqrt{\lambda}|x-y_1|)) \right. \\
& \left. + 2i \, (H_0^{(1)}(e^{\pi i/4}|x-y_1|) - H_0^{(1)}(e^{\pi i/4}|x-y_2|)) \Big) - 2\Phi(\lambda)(g(x)) \right) \\
& \times \left(-\frac{\pi i}{2} \right) \int_{\mathbb{R}^2} \Big(H_0^{(2)}(\sqrt{\lambda}|x-y_1|) - H_0^{(2)}(\sqrt{\lambda}|x-y_2|) \Big) h(x) dx;
\end{aligned}
$$

2) *a number* $\lambda \in \rho(A_{\rm F})$ *is an eigenvalue of* $\widetilde{A}$ *iff* $w_{\langle z, g(x)\rangle}(\lambda) = 0$, *and*

$$
\begin{aligned}
\ker(\widetilde{A} - \lambda I) = & \left(\frac{\pi i}{2} \frac{1}{i-\lambda} \Big((i+\lambda)(H_0^{(1)}(\sqrt{\lambda}|x-y_2|) - H_0^{(1)}(\sqrt{\lambda}|x-y_1|)) \right. \\
& + 2i \, (H_0^{(1)}(e^{\pi i/4}|x-y_1|) - H_0^{(1)}(e^{\pi i/4}|x-y_2|)) \Big) \\
& \left. - 2\Phi(\lambda)(g(x)) \right) \eta, \ \eta \in \mathbb{C}.
\end{aligned}
$$

Corollary 5.4. *Let* A *be given by* (5.1). *Then there is a bijective correspondence between all m-accretive quasi-selfadjoint extensions* $\widetilde{A}$ *of* A *(except Friedrichs and Kreĭn–von Neumann extensions) and all complex numbers* $z \in \mathbb{C}$ *such that*

$$
\operatorname{Re} z \geq 2\pi \left(\ln \frac{|y_1 - y_2|}{2} + \gamma + \mathbf{ker}(|y_1 - y_2|) \right).
$$

Moreover, an extension $\widetilde{A}$ *is m-sectorial iff*

$$
\operatorname{Re} z > 2\pi \left(\ln \frac{|y_1 - y_2|}{2} + \gamma + \mathbf{ker}(|y_1 - y_2|) \right),
$$

and nonnegative selfadjoint iff

$$
\operatorname{Im} z = \pi \left(-3\pi + 4\, \mathbf{kei}(|y_1 - y_2|) \right).
$$

The correspondence is given by the relations

$$
\begin{aligned}
\operatorname{dom}(\widetilde{A}) = & \left\{ u(x) = h(x) + \frac{\pi (h(y_1) - h(y_2))}{z} \left(\frac{\pi i}{2} \Big(H_0^{(1)}(e^{3\pi i/4}|x-y_1|) \right. \right. \\
& \left. \left. - H_0^{(1)}(e^{3\pi i/4}|x-y_2|) \Big) \right), \ h(x) \in W_2^2(\mathbb{R}^2) \right\}, \qquad (5.8) \\
\widetilde{A} u(x) = & - \Delta h(x) + \frac{\pi^2 (h(y_1) - h(y_2))}{2z} \Big(H_0^{(1)} \Big(e^{3\pi i/4}|x-y_1| \Big)
\end{aligned}
$$

$$-H_0^{(1)}\left(e^{3\pi i/4}|x-y_2|\right)\Big). \tag{5.9}$$

Moreover,

1) *a number* $\lambda \in \mathbb{C}\setminus[0,+\infty)$ *is a regular point of* $\widetilde{A}$ *iff*

$$w(z,\lambda) = z - \pi\ln(\lambda i) - \pi^2 i(H_0^{(1)}(\sqrt{\lambda}|y_1-y_2|) - H_0^{(1)}(e^{3\pi i/4}|y_1-y_2|)) \neq 0,$$

and

$$\begin{aligned}(\widetilde{A}-\lambda I)^{-1}h(x) =& \frac{i}{4}\int_{\mathbb{R}^2} H_0^{(1)}(\sqrt{\lambda}|x-y|)f(y)dy \\ &+ \frac{\pi^2}{8w(z,\lambda)}\left(H_0^{(1)}(\sqrt{\lambda}|x-y_1|) - H_0^{(1)}(\sqrt{\lambda}|x-y_2|)\right) \\ &\times \int_{\mathbb{R}^2}\left(H_0^{(2)}(\sqrt{\lambda}|x-y_1|) - H_0^{(2)}(\sqrt{\lambda}|x-y_2|)\right)h(x)\,dx;\end{aligned}$$

2) *a number* $\lambda \in \mathbb{C}\setminus[0,+\infty)$ *is an eigenvalue of* $\widetilde{A}$ *iff* $w(z,\lambda)=0$, *and*

$$\ker(\widetilde{A}-\lambda I) = \left(H_0^{(1)}(\sqrt{\lambda}|x-y_1|) - H_0^{(1)}(\sqrt{\lambda}|x-y_2|)\right)\eta, \quad \eta\in\mathbb{C}.$$

Remark 5.5. One can obtain a description of the Kreĭn–von Neumann extension $A_{\rm N}$ of A from relations (5.8), (5.9) by substituting

$$2z = \omega_0 = 4\pi\left(\ln\frac{|y_1-y_2|}{2} - \frac{3\pi i}{4} + \gamma + \frac{\pi i}{2}H_0^{(1)}(e^{3\pi i/4}|y_1-y_2|)\right).$$

It follows from (4.1) that the form $A_{\rm N}[u,v]$, associated with the Kreĭn–von Neumann extension $A_{\rm N}$, can be given by

$$\begin{aligned}{\rm D}[A_{\rm N}] = \Big\{u(x) = h(x) + \frac{\pi i}{2}\left(H_0^{(1)}(e^{\pi i/4}|x-y_1|) - H_0^{(1)}(e^{\pi i/4}|x-y_2|)\right)\omega, \\ h(x)\in W_2^1(\mathbb{R}^2),\ \omega\in\mathbb{C}\Big\},\end{aligned}$$

and if

$$\begin{aligned}u(x) &= h_1(x) + \frac{\pi i}{2}\left(H_0^{(1)}(e^{\pi i/4}|x-y_1|) - H_0^{(1)}(e^{\pi i/4}|x-y_2|)\right)\omega_1, \\ v(x) &= h_2(x) + \frac{\pi i}{2}\left(H_0^{(1)}(e^{\pi i/4}|x-y_1|) - H_0^{(1)}(e^{\pi i/4}|x-y_2|)\right)\omega_2,\end{aligned}$$

where $h_1(x), h_2(x)\in W_2^1(\mathbb{R}^2), \quad \omega_1,\omega_2\in\mathbb{C}$, then

$$A_{\rm N}[u,v] = \int_{\mathbb{R}^2}\nabla h_1(x)\overline{\nabla h_2(x)}dx$$

$$
\begin{aligned}
&-\frac{\pi\bar{\omega}_2}{2}\int_{\mathbb{R}^2} h_1(x)\overline{\left(H_0^{(1)}(e^{\pi i/4}|x-y_1|)-H_0^{(1)}(e^{\pi i/4}|x-y_2|)\right)}dx\\
&-\frac{\pi\omega_1}{2}\int_{\mathbb{R}^2}\left(H_0^{(1)}(e^{\pi i/4}|x-y_1|)-H_0^{(1)}(e^{\pi i/4}|x-y_2|)\right)\overline{h_2(x)}dx\\
&+4\pi\left(\ln\frac{|y_1-y_2|}{2}+\gamma+\mathbf{ker}\,|y_1-y_2|\right)\omega_1\bar{\omega}_2.
\end{aligned}
$$

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