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Comparison of Solutions of the Nonlinear Transfer Equation

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The paper considers the evolution of nonlinear equation describing the process of energy transfer by radiation. The comparison theorem for the solutions of this equation is formulated and proved.

Key words: nonlinear transfer equation, supersolutions, subsolutions, the comparison theorem.

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1. Introduction

Comparison of solutions of the Cauchy problem on the initial data is an effective means of qualitative analysis for quasi-linear differential equations of parabolic type [6–9, 15]. In particular, the equation describing the interaction of self-radiation with substance and radiation energy transfer process [16] is of this type. Currently there are not much works found on the subject. The difficulties in studying nonlinear equations describing these processes appear because of the complexity of nonlinear integral-differential equations. If the radiation path length is comparable with the size of the heated region of space, then the nonlocal nature of the interaction of radiation with substance should be taken into account. We consider a one-dimensional heat transfer process in the substance in the quasi-stationary approximation [16]. In this case, when the presence of heat sources and heat sinks in dimensionless variables is supposed, the process is described by the energy transfer equation.

2. Preliminary Notes

Let us consider the energy transfer equation (see [16])

$$\frac{\partial E}{\partial t} = -\frac{\partial S}{\partial x} + f. \tag{2.1}$$

Here $t \in \mathbb{R}_+$ is a time; $x \in \mathbb{R}$ is a straight line along which the heat is transferred; $E\,(T)$ is a specific internal energy of the substance, which is a continuous monotonically increasing function of the temperature substance, $E\,(0) = 0$; $T\,(x,t) \geqslant 0$ is a temperature of the substance; $f\,(T) \in C\,(\mathbb{R}_+)$ is a source (sink) function; $S\,(x,t)$ is a flux density of radiant energy, which is determined for the substance in the form of nonlinear integral operator $S\,(x,t) = S\,(T)$ [16],

$$S = \int_{-\infty}^{+\infty} T^4\,(\xi,t)\,K\,(T\,(\xi,t))\,\mathrm{sgn}\,P\,(x,\xi)\,W_2\,(|P\,(x,\xi)|)\,d\xi, \tag{2.2}$$

where

$$P\,(x,\xi) \equiv P\,(T) = \int_x^{\xi} K\,(T\,(\zeta,t))\,d\zeta, \quad (x,t) \in \Omega = \mathbb{R} \times \mathbb{R}_+;$$

$K\,(T) \in C\,(\mathbb{R}_+)$ is the coefficient of absorbtion of radiation by the substance, $K\,(T) > 0$, $T \geq 0$; $W_i\,(z) = \int_1^{+\infty} \exp\,(-z\mu)\mu^{-i} d\mu$, $i = 0, 1, 2, \ldots$ is the integral exponential function [4].

In physical meaning, $S\,(x,t)$ is a continuous function of its arguments. By (2.2), the continuity of S follows from the function $T\,(x,t)$ ($T\,(x,t) \in C\,(\Omega)$). In addition, the temperature rise $T\,(x,t)$ is limited as $|\,x\,| \to +\infty$ such that the improper integral in (2.2) is convergent (for example, $T\,(x,t) < A = \text{const}$, $(x,t) \in \Omega$).

3. Formulation of the Problem

From the physical considerations about the smoothness of the functions $T\,(x,t)$, $S\,(x,t)$, it follows that the derivatives in equation (2.1) generally do not exist in the classical sense. Therefore, for further analysis it is more convenient to consider the relation

$$L\,(T) \equiv \oint_{\Gamma} E\,(T)\,dx - \oint_{\Gamma} S\,(T)\,dt + \iint_{\omega} f\,(T)\,dxdt = 0 \tag{3.1}$$

instead of (2.1). It should be noticed that (3.1) does not contain derivatives [10–14]. Here $\Gamma \subset \Omega$ is the piecewise smooth contour bounding an arbitrary simply connected domain $\omega \subset \Omega$.

Bypassing the contour Γ is assumed to be single. However, the region remains on the left. Relation (3.1) can be obtained by applying Green's formula to (2.1). It can also be obtained independently from (2.1) as a consequence of the law of conservation of energy.

Let us consider the problem of the evolution of the initial heat pulse:

$$T(x,0) = T_0(x) \geq 0, \quad T_0(x) \in C(\mathbb{R}), \tag{3.2}$$
$$\int_{\mathbb{R}} E(T_0(x))\,dx = Q_0 < +\infty.$$

We assume that there exists a function $T(x,t) \in C(\Omega)$ satisfying conditions (3.2) and equation (3.1), and that the heat stream $S(x,t) \equiv S(T)$ vanishes at $|\,x\,| \to +\infty$,

$$S(x,t) \to 0, \quad |\,x\,| \to +\infty. \tag{3.3}$$

Let Γ be a contour projecting to the axis t and infinite with respect to x. Then from (2.1) it can be obtained that

$$Q(t) \equiv \int_{\mathbb{R}} E(T(x,t))\,dx = Q_0 + \int_0^t \int_{\mathbb{R}} f(T(x,t))\;dxdt. \tag{3.4}$$

Hence, physically natural existence of the energy integral $Q(t) < +\infty$ is equivalent to the existence of the integral on the right-hand side of (3.4). We define the function $T(x,t)$ satisfying (3.1)–(3.4) as a generalized solution to the Cauchy problem [3].

One of the main moments of the theory developed in this work is the possibility of estimating the integrals similar within the definition of the heat flux density (2.2). To do this, we assume that the function $K(z)$ satisfies simultaneously the inequalities:

$$K(z_1) \leq K(z_2), \quad K(z_1)\,z_1^4 \geq K(z_2)\,z_2^4, \tag{3.5}$$
$$z_1 \geq z_2, \quad z_i \in \mathbb{R}_+, \quad i = 1,2.$$

The first inequality in (3.5) means that the radiation path length is not decreasing with temperature increasing of the substance, and the second inequality means the obvious physical requirement of impossibility of reducing the radiant exitance of the substance when its temperature increases.

4. Super- and Subsolutions

Along with the solution $T(x,t)$, let us consider the functions

$$\theta(x,t) \geq 0, \quad \theta(x,t) \in C(\Omega), \quad L(\theta) \leq 0$$

which we call a supersolution of the Cauchy problem of equation (3.1) (see [6]), and

$$\tau(x,t) \geq 0, \quad \tau(x,t) \in C(\Omega), \quad L(\tau) \geq 0$$

which we call a subsolution of equation (3.1).

In this case, the introduction of super- and subsolutions is of essential importance since the class of functions $K(T)$, for which the simplest invariant solutions of equations can be constructed (2.1), is significantly poorer in functions than the class for the equation of radiative thermal conductivity [1].

The possibility of application of super- and subsolutions for qualitative estimate of the Cauchy problem (3.1)–(3.4) is determined by the lemma below.

Lemma 1. *Let the source function $f(z)$ satisfy at least one of the conditions:*

$$f(z_1) \geq f(z_2), \quad z_1 \leq z_2, \quad z_i \in \mathbb{R}_+, \quad i = 1, 2, \tag{4.1}$$

i.e., it is nonincreasing with respect to the temperature, $f(z) \in C(\mathbb{R}_+)$, $f(0) = 0$, or there exists such $a > 0$ that

$$\Phi(z) \leq 0, \quad z \in [0; a], \tag{4.2}$$

where $\Phi(z) = f(z) + \delta(z)$ for the supersolution, and $\Phi(z) = f(z) - \gamma(z)$ for the subsolution, and $\delta(z)$ and $\gamma(z)$, are defined by the inequalities

$$L(\theta) \leq \iint_\omega \delta(\theta)\, dxdt; \quad \delta(\theta) < 0,\ \theta > 0; \quad \delta(0) = 0; \quad \delta(\theta) \in C(\mathbb{R}_+), \tag{4.3}$$

$$L(\tau) \geq \iint_\omega \gamma(\tau)\, dxdt; \quad \gamma(\tau) > 0,\ \tau > 0; \quad \gamma(0) = 0; \quad \gamma(\tau) \in C(\mathbb{R}_+). \tag{4.4}$$

Thus, if inequalities (3.5) are fulfilled and

$$\tau(x, 0) \leq T_0(x) \leq \theta(x, 0), \quad x \in \mathbb{R}, \tag{4.5}$$

then

$$\tau(x, t) \leq T(x, t) \leq \theta(x, t), \quad (x, t) \in \Omega. \tag{4.6}$$

Proof. We prove the lemma for the supersolution $\theta(x, t)$. For the subsolution the proof is similar.

Following [2, 10–12] the lemma will be proved by contradiction. Suppose that in the domain Ω there is $\Omega_1 \subset \Omega$ such that

$$T(x, t) > \theta(x, t), \quad (x, t) \in \Omega_1 \backslash \partial\Omega_1 \quad \text{and} \quad (x, t) = \theta(x, t), \quad (x, t) \in \partial\Omega_1 = l.$$

As the area ω from (3.1), we choose the area ω_1 bounded by the contour Γ_1 composed of the arc of the curve l and the segment of the straight line m projecting to the time axis t (see Fig. 1).

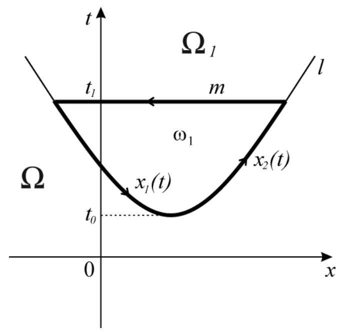

Fig. 1: Contour Γ_1.

Then, using (3.1) and the definition of supersolution, we can get the residual

$$L(\theta) - L(T) \equiv \oint_{\Gamma_1} (E(\theta) - E(T))\,dx - \oint_{\Gamma_1} (S(\theta) - S(T))\,dx + \iint_{\omega_1} (f(\theta) - f(T))\,dxdt \le 0. \quad (4.7)$$

Let us estimate each integral from (4.7). It is easy to estimate the first integral

$$\oint_{\Gamma_1} (E(\theta) - E(T))\,dx = \oint_{m} (E(\theta) - E(T))\,dx > 0, \quad (4.8)$$

because the direction of integration coincides with the direction of the contour Γ_1 bypass. The second integral from (4.7) can be reduced to the form

$$\oint_{\Gamma_1} (S(\theta) - S(T))\,dt = \int_{t_0}^{t_1}\int_0^1 ((q_1-1)(q_{11}+q_{12}) + \Delta q_1)\,d\mu dt - \int_{t_0}^{t_1}\int_0^1 ((q_2-1)(q_{21}+q_{22}) + \Delta q_2)\,d\mu dt,$$

where

$$q_i(\mu,t) = \exp\left(-\frac{1}{\mu}\left|p_i(x_1(t),\, x_2(t))\right|\right),$$

$$q_{i1}(\mu,t) = \int_{-\infty}^{x_1(t)} T_i^4(\xi,t)\,K(T_i(\xi,t))\exp\left(-\frac{1}{\mu}\left|p_i(x_1(t),\xi)\right|\right)d\xi,$$

$$q_{i2}(\mu,t) = \int_{x_2(t)}^{+\infty} T_i^4(\xi,t)\,K(T_i(\xi,t))\exp\left(-\frac{1}{\mu}\left|p_i(x_2(t),\xi)\right|\right)d\xi,$$

$$\Delta q_i(\mu,t) = \int_{x_1(t)}^{x_2(t)} T_i^4(\xi,t)\,K(T_i(\xi,t))\left(\exp\left(-\frac{1}{\mu}\left|p_i(x_2(t),\xi)\right|\right)\right.$$

$$+\exp\left(-\frac{1}{\mu}\left|p_i\left(x_1\left(t\right),\xi\right)\right|\right)\right)d\xi,$$

$$T_1\left(x,t\right)=\theta\left(x,t\right),\quad T_2\left(x,t\right)=T\left(x,t\right),\ p_i=p\left(T_i\right),\quad i=1,2.$$

By (3.5), the integrands in (2.2), taken in absolute value, are nondecreasing with respect to temperature. Then, using the definitions, we can obtain the following inequalities:

$$0<q_1\left(\mu,t\right)\le q_2\left(\mu,t\right)<1,\qquad q_{11}\left(\mu,t\right)\ge q_{21}\left(\mu,t\right),$$
$$q_{12}\left(\mu,t\right)\ge q_{22}\left(\mu,t\right),\qquad \Delta q_1\left(\mu,t\right)>\Delta q_2\left(\mu,t\right).$$

Hence, we have the estimate

$$\oint_{\Gamma_1}\left(S\left(\theta\right)-S\left(T\right)\right)dt<0. \tag{4.9}$$

The estimate of the last integral from (4.7) is trivial if the following inequality holds:

$$\iint_{\omega}\left(f\left(\theta\right)-f\left(T\right)\right)dxdt\ge 0, \tag{4.10}$$

since $f\left(\theta\right)\ge f\left(T\right)$, $\theta<T$, $(x,t)\in\omega_1$.

If (4.2), (4.3) are satisfied instead of (4.1), then we can define the domain

$$\Omega_2=\left\{(x,t)\mid F\left(\theta,T\right)=f\left(\theta\right)-f\left(T\right)-\delta\left(\theta\right)\ge 0\right\}.$$

Due to the conditions (4.2), (4.3), the continuity of function $F\left(\theta,T\right)\in C\left(\Omega\right)$, and inequality $F\left(\theta,T\right)\ge 0$, $(x,t)\in l$, the strip $\Omega_3=\Omega_2\cap\Omega_1$ closes nowhere and as a boundary it has the line $l=\partial\Omega_1$. Assuming $\omega_1\subset\Omega_3$, we arrive at the inequality

$$\iint_{\omega_1}\left(\Phi\left(\theta\right)-f\left(T\right)\right)\,dxdt\ge 0. \tag{4.11}$$

The estimates of (4.8)–(4.11) show that inequality (4.7) does not hold for an arbitrary domain, which proves the lemma. □

Remark 1. The lemma remains valid if we set $\Phi\left(z\right)=f\left(z\right)$ for the super- and subsolutions in (4.2) and consider $\delta=\delta\left(x,t\right)$, $\gamma=\gamma\left(x,t\right)$ in (4.3) and (4.4). The resulting lemma can be applied not only directly. It can also be used as a basis for the proofs of various comparison theorems for the Cauchy problem (3.1)–(3.4). In particular, we can extend the class of source functions if we do not take into account conditions (4.1), (4.2).

5. The Main Result

Let us consider two solutions of the Cauchy problem (3.1)–(3.4), $T_i(x,t) \in C(\Omega) \cap L_1(\Omega)$, determined by the initial conditions $T_i(x,0) = T_{oi}(x)$, $i = 1,2$. The following theorem holds.

Theorem 1. *Let $E = T$, and the source function $f(z) \in C(\mathbb{R}_+)$ satisfy the Lipschitz condition*

$$|f(z_1) - f(z_2)| \le M|z_1 - z_2|, \quad z_i > 0, \quad i = 1,2, \quad M = \text{const} > 0. \tag{5.1}$$

Suppose also that there exist positive constants $\beta > 1$, a, M_1 such that

$$f(z) \le M_1 z^{\beta}, \quad z \in [0;a]. \tag{5.2}$$

Thus, if $T_{01}(x) \ge T_{02}(x)$, $x \in \mathbb{R}$, then

$$T_1(x,t) \le T_2(x,t), \quad (x,t) \in \Omega. \tag{5.3}$$

Proof. Consider the auxiliary Cauchy problem, which is obtained by substituting $f(T)$ in (3.1) by $f_{ci} = f(T_{ci}) - CT_{ci}$, $i = 1,2$, $C = \text{const} > 0$. From condition (19) and the lemma, we get the inequalities:

$$T_i(x,t) \ge T_{ci}(x,t), \quad T_{c1}(x,t) \ge T_{c2}(x,t), \quad (x,t) \in \Omega, \quad i = 1,2. \tag{5.4}$$

By taking Γ as a contour, normal over t and infinite over x and reducing the double integrals to the repeated integral, from (3.1), one can obtain

$$\int_{\mathbb{R}} (T_i - T_{ci})\, dx = \int_0^t \int_{\mathbb{R}} (f(T_i) - f(T_{ci}))\, dxdt + C\int_0^t \int_{\mathbb{R}} T_{ci}\, dxdt.$$

Applying formulas (5.1), (5.4) to the equality, we obtain the inequality

$$\int_{\mathbb{R}} (T_i - T_{ci})\, dx \le M\int_0^t \int_{\mathbb{R}} (T_i - T_{ci})\, dxdt + CN_i t,$$

where $N_i = \sup\limits_t \int_{\mathbb{R}} T_i dx$. Hence, by the Gronwall inequality [5] and formulas (5.4), we obtain

$$0 \le \int_{\mathbb{R}} (T_i - T_{ci})\, dx \le CN_i t \exp(Mt). \tag{5.5}$$

Passing in (5.5) to the limit at $C \to 0$, by (5.4), we obtain the desired inequality (5.3) for the continuous functions $T_i(x,t) \in C(\Omega)$, $i = 1,2$. This proves the theorem. □

Under the conditions of Theorem 1, the uniqueness of the solution of the Cauchy problem (3.1)–(3.4) in the class of continuous functions follows immediately.

Theorem 2. *Let the conditions of Theorem 1 be satisfied. Then the solution of the Cauchy problem (3.1)–(3.4) is unique.*

If there are two solutions of the Cauchy problem (3.1)–(3.4), then by (5.3) these solutions coincide.

6. Comment

All conclusions of the theory developed here remain true if in the definition of the radiant flux (2.2) and conditions (3.5) if the function T^4 is substituted by an arbitrary monotonically increasing function $\varphi(T)$.

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