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On the Class of Einstein Exponential-Type Finsler Metrics

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In this paper, a special class of Finsler metrics, the so-called (α, β)-metrics, which are defined by $F = \alpha\phi(s)$, where α is a Riemannian metric and β is a 1-form, is studied. First we show that the class of almost regular metrics obtained by Shen is Einstein if and only if it reduces to the class of Berwald metrics. In this case, the Riemannian metrics are Ricci-flat. Then we prove that an exponential metric is Einstein if and only if it is Ricci-flat.

Key words: Einstein metric, unicorn metric, exponential metric.

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1. Introduction

Let (M, F) be an n-dimensional Finsler manifold. Then F is called an Einstein metric if its Ricci curvature **Ric** is isotropic, $\mathbf{Ric} = (n-1)\lambda F^2$, where $\lambda = \lambda(x)$ is a scalar function on M. It is remarkable that every Riemannian surface is Einstein, but is not necessarily Ricci constant. In dimension $n = 3$, a Riemann metric is Einstein if and only if it is of constant sectional curvature. For the case of $n \geq 3$, a Schur-type lemma guarantees that every Riemannian Einstein metric is Ricci constant. In [7], Bao–Robles showed that every Einstein Randers metric $F = \alpha + \beta$ on a manifold M of dimension $n \geq 3$ is necessarily Ricci constant. In particular, if $n = 3$, then a Randers metric is Einstein if and only if it is of constant flag curvature. By now some progress has been achieved in studying Finsler Einstein metrics of Randers type (see [1, 2, 4, 7, 9–11, 14, 15, 19]).

In Finsler geometry, it is a long-existing open problem to find the *unicorns*, i.e., Landsberg metrics which are not Berwaldian [5, 6, 24]. Let us explain some details about the unicorns in Finsler geometry. For the sake of simpler prose, Bao referred to Landsberg metrics that are not of Berwald type as unicorns, by analogy with those mythical single-horned horse-like creatures for which no confirmed sighting is available. In [8], Bryant showed that in two dimensions, there is an abundance of the metrics depending on two families of functions of two variables; among them there is a subclass with zero flag curvature, depending on one family of functions of two variables. In [13], Shen proved the non-existence of the unicorn in the class of regular (α, β)-metrics. In [3], Asanov constructed a special family of unicorns which are non-regular (α, β)-metrics. Then Shen found a more complicated family of unicorns in the class of non-regular (α, β)-metrics which contains Asanov's metrics. Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric defined by

$$\phi(s) = \exp\left[\int_0^s \frac{kt + q\sqrt{b^2 - t^2}}{1 + kt^2 + qt\sqrt{b^2 - t^2}} dt\right], \tag{1}$$

where $q > 0$ and k are real constants. Suppose that β satisfies

$$r_{ij} = c(b^2 a_{ij} - b_i b_j), \quad s_{ij} = 0, \tag{2}$$

where $c = c(x)$ is a scalar function on M. If $c \neq 0$, then F is a Landsberg metric which is not Berwaldian. In this case, F is a unicorn metric [20]. If $c = 0$, then F reduces to a Berwald metric. If $k = 0$ and $c \neq 0$, then we obtain the family of unicorn metrics introduced by Asanov in [3].

Theorem 1.1. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on an n-dimensional manifold M defined by (1). Suppose that β satisfies (2) for some scalar function $c = c(x)$ on M. Then F is an Einstein metric if and only if it is a Berwald metric. In this case, the Riemannian metric α is Ricci-flat.*

The exponential metric is another (α, β)-metric which is given by $F = \alpha\exp(s)$, $s = \beta/\alpha$, (see [12, 25]). Then, for an exponential metric on a manifold M, we find the necessary and sufficient condition under which F is Einsteinian.

Theorem 1.2. *Let $F = \alpha\exp(s)$, $s = \beta/\alpha$, be an exponential metric on a manifold M. Then F is Einstein if and only if it is Ricci-flat.*

2. Preliminaries

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate system (x^i, y^i) for TM_0 is given by $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$, where $G^i(y)$ are local functions on TM given by

$$G^i := \frac{1}{4} g^{il}\left\{\frac{\partial^2[F^2]}{\partial x^k \partial y^l} y^k - \frac{\partial[F^2]}{\partial x^l}\right\}, \quad y \in T_xM, \tag{3}$$

where $g^{il} = (g_{il})^{-1}$. $\mathbf{G}$ is called the associated spray to (M, F). The projection of an integral curve of $\mathbf{G}$ is called a geodesic in M [17].

For a tangent vector $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ by

$$\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y) u^j v^k w^l \left.\frac{\partial}{\partial x^i}\right|_x,$$

where

$$B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad u = u^i \left.\frac{\partial}{\partial x^i}\right|_x, \quad v = v^i \left.\frac{\partial}{\partial x^i}\right|_x, \quad w = w^i \left.\frac{\partial}{\partial x^i}\right|_x.$$

$\mathbf{B}$ is called the Berwald curvature. A Finsler metric is called a Berwald metric if $\mathbf{B} = 0$ (see [16]).

For $y \in T_xM_0$, the Riemann curvature is a family of linear transformations $\mathbf{R}_y : T_xM \to T_xM$ with homogeneity $\mathbf{R}_{\lambda y} = \lambda^2 \mathbf{R}_y$, $\lambda > 0$, defined by $\mathbf{R}_y(u) := R^i_k(y) u^k \frac{\partial}{\partial x^i}$, where

$$R^i_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

The family $\mathbf{R} := \{\mathbf{R}_y\}_{y \in TM_0}$ is called the Riemann curvature [22].

The Ricci curvature $\mathbf{Ric}(x, y)$ is the trace of the Riemann curvature defined by

$$\mathbf{Ric}(x, y) := R^m_m(x, y).$$

A Finsler metric F on the n-dimensional manifold M is called an Einstein metric if the Ricci curvature satisfies

$$\mathbf{Ric} = (n-1)\lambda F^2,$$

where $\lambda = \lambda(x)$ is a scalar function on M.

An (α, β)-metric is a Finsler metric of the form $F := \alpha\phi(\beta/\alpha)$, where $\phi = \phi(s)$ is C^∞ on $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M [18, 20, 21, 23]. Among the (α, β)-metrics, the Randers metric $F = \alpha + \beta$ is a special and significant metric which has important applications in physics, biology, etc. For an (α, β)-metric $F := \alpha\phi(s)$, $s = \beta/\alpha$, let us define $b_{i|j}$ by

$$b_{i|j}\theta^j := db_i - b_j \theta_i^j,$$

where $\theta^i := dx^i$. Let $\theta_i^j := \Gamma^j_{ik} dx^k$ denote the Levi-Civita connection form of α and let

$$r_{ij} := \frac{1}{2}\Big[b_{i|j} + b_{j|i}\Big], \quad s_{ij} := \frac{1}{2}\Big[b_{i|j} - b_{j|i}\Big],$$
$$r_j := b^i r_{ij}, \quad s_j := b^i s_{ij}, \quad r_{i0} := r_{ij}y^j, \quad s_{i0} := s_{ij}y^j, \quad r_0 := r_j y^j, \quad s_0 := s_j y^j.$$
$$t_{ij} := s_i{}^m s_{mj}, \quad q_{ij} := r_i{}^m s_{mj}, \quad t_j := b^m t_{mj},$$

where $b^i := a^{ij} b_j$. It is obvious that β is closed if $s_{ij} = 0$, and it is parallel with respect to α if $s_{ij} = r_{ij} = 0$.

For the (α, β)-metric, put

$$Q = \frac{\phi'}{\phi - s\phi'},$$
$$\Delta = 1 + sQ + (b^2 - s^2)Q',$$
$$\Phi = -(Q - sQ')\{n\Delta + 1 + sQ\} - (b^2 - s^2)(1 + sQ)Q''.$$

3. Proof of Theorem 1.1

A Finsler metric $F = \alpha \exp p(s)$, $s = \beta/\alpha$, is called an exponential-type Finsler metric, where $p = p(s)$ is a C^∞ function on M. First, we obtain the following.

Lemma 3.1. *Let $F = \alpha \exp p(s)$, $s = \beta/\alpha$, be an exponential-type Finsler metric on a manifold M. Suppose that β satisfies*

$$r_{ij} = c(b^2 a_{ij} - b_i b_j), \quad s_{ij} = 0, \tag{4}$$

where $c = c(x)$ is a scalar function on M. Then

$$\mathbf{Ric} = \overline{\mathbf{Ric}} + RT^m_m, \tag{5}$$

where $\mathbf{Ric}$ *and* $\overline{\mathbf{Ric}}$ *denote the Ricci curvatures of F and α, respectively, and*

$$\begin{aligned}
RT^m_m = \Upsilon\big[&- 2p''''(s)p'(s)^2b^4 - 66p''(s)^2p'(s)^4s^6 + 15p''(s)^2p'(s)^2s^4 \\
&+ 3p'(s)^2n - 18p''(s)^2p'(s)^2b^4 - 6p'''(s)p''(s)^2s^5 - 4p'''(s)p'(s)^3b^4 \\
&- 2p''''(s)p''(s)b^4 - 18p''(s)^3b^2s^2 - 2p'''(s)p'(s)^2s^3 + 2p'''(s)p'(s)^5s^6 \\
&+ 4p''''(s)p'(s)^2s^4 + 4p''(s)p'(s)^3s^3 - 6p'''(s)^2b^2s^2 + 4p'''(s)p'(s)s^2 \\
&- 52p''(s)p'(s)^6s^6 - 6p''''(s)p'(s)s^3 - 2p''''(s)p''(s)s^4 - 20p'''(s)p'(s)^3s^4 \\
&+ 24p''(s)p'(s)^4s^4 + 6p''(s)p'(s)^2s^2 + 24p''(s)^2p'(s)s^3 - 36p''(s)p'(s)^2b^2 \\
&- 10p'''(s)p'(s)b^2 + 22p'''(s)p''(s)s^3 - 2p''''(s)p'(s)^4s^6 - 60p''(s)^3p'(s)^2s^6 \\
&- 66p''(s)p'(s)^5s^5 + 3p'''(s)^2p'(s)^2s^6 - 18p''(s)^4b^4s^2 + 36p''(s)^4b^2s^4 \\
&- 96p''(s)^2p'(s)^3s^5 + 2p''''(s)p'(s)^3s^5 - 42p''(s)^3p'(s)s^5 - 28p''(s)p'(s)^4b^4 \\
&+ 12p'(s)sp''(s) - 6p'''(s)^2p'(s)b^4s + 14p'''(s)p'(s)^4s^5 - 6p'''(s)^2p'(s)s^5 \\
&- 6p'''(s)p''(s)p'(s)^2s^5 + 42p''(s)^3p'(s)b^2s^3 + 4p''''(s)p''(s)p'(s)s^5 \\
&+ 12p'''(s)^2p'(s)b^2s^3 - 36p'''(s)p''(s)p'(s)s^4 + 24p''(s)p'(s)^5ns^5 \\
&+ 12p''(s)^3p'(s)ns^5 - 6p'''(s)p''(s)^2b^4s + 12p'''(s)p''(s)^2b^2s^3 \\
&- 2p''''(s)p'(s)^2b^2s^2 + 8p'''(s)p''(s)p'(s)b^4 + 12p''(s)^3p'(s)^2ns^6 \\
&+ 4p''''(s)p''(s)b^2s^2 + 6p''''(s)p'(s)b^2s - 2p''(s)p'(s)^4b^2s^2 \\
&+ 22p'''(s)p'(s)^3b^2s^2 + 6p''(s)^2p'(s)^2b^2s^2 + 20p''(s)p'(s)^3b^2s \\
&+ 8p'''(s)p'(s)^2b^2s - 30p''(s)^2p'(s)b^2s + 2p'''(s)p'(s)nb^2 - 8p'''(s)p'(s)ns^2 \\
&- 22p'''(s)p''(s)b^2s - 4p'''(s)p'(s)^4ns^5 + 3p''(s)^4nb^4s^2 - 6p''(s)^4nb^2s^4 \\
&+ 18p''(s)^2p'(s)^3ns^5 + 14p''(s)p'(s)^4nb^4 - 48p''(s)p'(s)^4ns^4 \\
&+ 2p'''(s)p'(s)^3nb^4 - 2p'''(s)p'(s)^3ns^4 + 9p''(s)^2p'(s)^2nb^4 \\
&- 18p''(s)^2p'(s)ns^3 + 2p'''(s)p'(s)^5ns^6 + 2p'''(s)p'(s)^5b^4s^2 \\
&- 18p''(s)^2p'(s)^2ns^4 + 12p''(s)^3nb^2s^2 + 4p''(s)p'(s)^3ns^3 \\
&+ 10p'''(s)p'(s)^2ns^3 - 2p'''(s)p''(s)ns^3 + 12p''(s)p'(s)^2nb^2 \\
&+ 6p''(s)p'(s)^2ns^2 + 26p''(s)p'(s)^6ns^6 + 27p''(s)^2p'(s)^4ns^6
\end{aligned}$$

$$
\begin{aligned}
&- 52p''(s)p'(s)^6b^4s^2 + 104p''(s)p'(s)^6b^2s^4 - 4p'''(s)p'(s)^5b^2s^4\\
&- 66p''(s)^2p'(s)^4b^4s^2 + 132p''(s)^2p'(s)^4b^2s^4 - 2p''''(s)p'(s)^4b^4s^2\\
&+ 120p''(s)^3p'(s)^2b^2s^4 + 3p'''(s)^2p'(s)^2b^4s^2 + 4p''''(s)p'(s)^4b^2s^4\\
&+ 20p'''(s)p''(s)p'(s)^3s^6 - 60p''(s)^3p'(s)^2b^4s^2 + 8p''(s)p'(s)^5b^4s\\
&+ 58p''(s)p'(s)^5b^2s^3 - 2p''''(s)p''(s)p'(s)^2s^6 - 6p'''(s)^2p'(s)^2b^2s^4\\
&+ 6p'''(s)p''(s)^2p'(s)s^6 + 2p'''(s)p'(s)^4b^4s - 16p'''(s)p'(s)^4b^2s^3\\
&+ 9p''(s)^2ns^2 + 4p''''(s)p''(s)p'(s)^2b^2s^4 - 24p''(s)^2p'(s)^3b^4s\\
&+ 120p''(s)^2p'(s)^3b^2s^3 + 4p''''(s)p'(s)^3b^4s + 2p'''(s)ns + 3p''(s)^4ns^6\\
&- 12p''(s)^3ns^4 - 8p''''(s)p''(s)p'(s)b^2s^3 + 20p'''(s)p''(s)p'(s)^3b^4s^2\\
&- 40p'''(s)p''(s)p'(s)^3b^2s^4 - 2p''''(s)p''(s)p'(s)^2b^4s^2\\
&+ 6p'''(s)p''(s)^2p'(s)b^4s^2 - 12p'''(s)p''(s)^2p'(s)b^2s^4\\
&- 28p'''(s)p''(s)p'(s)^2b^4s + 34p'''(s)p''(s)p'(s)^2b^2s^3 + 2p'''(s)p'(s)^5nb^4s^2\\
&+ 2p'''(s)p''(s)p'(s)^3ns^6 + 4p''''(s)p''(s)p'(s)b^4s + 26p''(s)p'(s)^6nb^4s^2\\
&- 52p''(s)p'(s)^6nb^2s^4 - 4p'''(s)p'(s)^5nb^2s^4 + 27p''(s)^2p'(s)^4nb^4s^2\\
&- 54p''(s)^2p'(s)^4nb^2s^4 + 12p''(s)^3p'(s)^2nb^4s^2 - 24p''(s)^3p'(s)^2nb^2s^4\\
&- 28p''(s)p'(s)^5nb^4s + 4p''(s)p'(s)^5nb^2s^3 - 4p'''(s)p'(s)^4nb^4s\\
&+ 52p''(s)p'(s)^4nb^2s^2 + 5p'(s)^6nb^4 + 8p'(s)^8nb^4s^2 + 8p'''(s)p'(s)^4nb^2s^3\\
&- 18p''(s)^2p'(s)^3nb^4s - 6p'''(s)p''(s)p'(s)^2ns^5 - 12p''(s)^3p'(s)nb^2s^3\\
&- 2p'''(s)p'(s)^3nb^2s^2 + 18p''(s)^2p'(s)^2nb^2s^2 + 2p'''(s)p''(s)p'(s)nb^4\\
&+ 6p'''(s)p''(s)p'(s)ns^4 - 40p''(s)p'(s)^3nb^2s - 4p'''(s)p'(s)^2nb^2s\\
&+ 2p'''(s)p''(s)nb^2s + 31p'(s)^4ns^2 - 16p'(s)^3ns - 23p'(s)^6ns^4\\
&- 14p'(s)^5ns^3 + 8p'(s)^4nb^2 + 8p'(s)^8ns^6 + 34p'(s)^6nb^2s^2 - 28p'(s)^5nb^2s\\
&- 16p'(s)^8nb^2s^4 - 10p'(s)^7nb^4s - 4p'(s)^7nb^2s^3 - 16p'(s)^8b^4s^2\\
&+ 32p'(s)^8b^2s^4 + 8p'(s)^7b^4s - 2p''''(s)b^2 + 20p'(s)^3s\\
&+ 28p'''(s)p''(s)p'(s)b^2s^2 + 14p'(s)^7ns^5 + 20p'(s)^7b^2s^3 - 18p''(s)^4s^6\\
&- 12p''(s)^3b^4 + 3p'''(s)^2b^4 + 3p'''(s)^2s^4 + 30p''(s)^3s^4 + 2p''''(s)s^2\\
&- 12p''(s)^2b^2 + 15p''(s)^2s^2 + 2p'''(s)s - 10p'(s)^6b^4 - 16p'(s)^8s^6\\
&- 28p'(s)^7s^5 + 28p'(s)^6s^4 + 4p'(s)^5s^3 - 26p'(s)^6b^2s^2 + 26p'(s)^5b^2s\\
&- 16p'(s)^4b^2 - 23p'(s)^4s^2 - 3p'(s)^2 + 2p'''(s)p''(s)p'(s)^3nb^4s^2\\
&- 4p'''(s)p''(s)p'(s)^3nb^2s^4 - 4p'''(s)p''(s)p'(s)^2nb^4s\\
&+ 10p'''(s)p''(s)p'(s)^2nb^2s^3 - 8p'''(s)p''(s)p'(s)nb^2s^2 - 6p''''(s)p'(s)^3b^2s^3\big],
\end{aligned}
$$

where

$$\Upsilon := \frac{c^2(\alpha^2b^2 - \beta^2)^2}{4\alpha^2(p'(s)^2b^2 - p'(s)^2s^2 + p''(s)b^2 - p''(s)s^2 - p'(s)s + 1)^4}.$$

Now we restrict our attention on the special exponential (α, β)-metric $F =$

$\alpha \exp p(s)$, $s=\beta/\alpha$, which is defined by

$$p(s)=\int_0^s \frac{kt+q\sqrt{b^2-t^2}}{1+kt^2+qt\sqrt{b^2-t^2}}dt, \tag{6}$$

where $q>0$ and k are real constants.

Proof of Theorem 1.1. Let $F=\alpha\phi(s)$ be a unicorn metric on a manifold M. By (4), we get

$$G^i=G^i_\alpha+c\alpha^2(b^2-s^2)\left(\Theta\frac{y^i}{\alpha}+\Psi b^i\right). \tag{7}$$

For the Finsler metric F defined by (1), we have

$$Q=ks+q\sqrt{b^2-s^2}, \tag{8}$$

$$\Psi=\frac{k\sqrt{b^2-s^2}-qs}{2(1+kb^2)\sqrt{b^2-s^2}}. \tag{9}$$

Then the Ricci curvature of F related to the Ricci curvature $\overline{\mathbf{Ric}}$ of α is given by

$$\mathbf{Ric}=\overline{\mathbf{Ric}}+RT^m_m,$$

where

$$\begin{aligned}RT^m_m=&-\frac{1}{4}\frac{b^2qc^2}{\sqrt{(\alpha^2b^2-\beta^2)}(b^2k+1)^2}(2\alpha^2b^4\beta kn-\alpha^2b^4nq\sqrt{(\alpha^2b^2-\beta^2)}\\&-4\alpha^2b^4\beta k+2\alpha^2b^4q\sqrt{(\alpha^2b^2-\beta^2)}-2b^2\beta^3kn+b^2\beta^2nq\sqrt{(\alpha^2b^2-\beta^2)}\\&+2\alpha^2b^2\beta n+4b^2\beta^3k-2b^2\beta^2q\sqrt{(\alpha^2b^2-\beta^2)}-4\alpha^2b^2\beta-2\beta^3n+4\beta^3)\\=&-\frac{1}{2}\frac{(n-2)\sqrt{(\alpha^2b^2-\beta^2)}b^2qc^2\beta}{kb^2+1}+\frac{1}{4}\frac{(n-2)(\alpha^2b^2-\beta^2)b^4q^2c^2}{(kb^2+1)^2}.\end{aligned}$$

For more details, see Proposition 3.3 in [9]. By definition, F is Einstein if $\mathbf{Ric}=(n-1)\lambda F^2$, where $\lambda=\lambda(x)$ is a scalar function on M. Thus we have

$$\begin{aligned}\frac{1}{2}\frac{(n-2)\sqrt{b^2\alpha^2-\beta^2}b^2qc^2\beta}{kb^2+1}&+(n-1)\lambda F^2\\&=\frac{1}{4}\frac{(n-2)(b^2\alpha^2-\beta^2)b^4q^2c^2}{(kb^2+1)^2}+\overline{\mathbf{Ric}}.\end{aligned} \tag{10}$$

The left-hand side of (10) is an irrational polynomial in y while its right-hand side is a rational polynomial in y. Then (10) reduces to two relations:

$$-(n-2)\sqrt{b^2\alpha^2-\beta^2}b^2qc^2\beta=2(n-1)\lambda(kb^2+1)F^2 \tag{11}$$

and

$$\overline{\mathbf{Ric}}=\frac{(n-2)(\beta^2-\alpha^2b^2)b^4q^2c^2}{4(kb^2+1)^2}. \tag{12}$$

Since F is an irrational function in y, then by (11), we get $\lambda=0$. Putting it in (11) yields $c=0$. This implies that $r_{ij}=0$. By assumption, F reduces to a Berwald metric. On the other hand, by (12), we get $\overline{\mathbf{Ric}}=0$. □

4. Proof of Theorem 1.2

In this section, we are going to prove Theorem 1.2. By Proposition 3.3 from [9], we have the following.

Lemma 4.1. *Let $F = \alpha\exp(s)$, $s = \beta/\alpha$, be an exponential metric on a manifold M. Then the following holds:*

$$\mathbf{Ric} = \overline{\mathbf{Ric}} + RT^m_m, \tag{13}$$

where

$$RT^m_m = \frac{P_{13}\alpha^{13} + P_{12}\alpha^{12} + \cdots + P_1\alpha + P_0}{4\left[(1+b^2)\alpha^2 - \alpha\beta - \beta^2\right]^4(\alpha-\beta)^3}, \tag{14}$$

and

$$\begin{aligned}
P_0 &= 4(n-2)r_{00|0}\beta^{10} - 8(n-2)r_{00}^2\beta^9,\\
P_1 &= 2(2-n)r_{00|0}\beta^9 + 10(n-2)r_{00}^2\beta^8,\\
P_2 &= 42r_{00|0}\beta^8 - 24nr_{00|0}\beta^8 - 4r_{0m}r^m_0\beta^9 - 64r_{00}^2\beta^7 - 28r_{0m}s^m_0\beta^9 + 41nr_{00}^2\beta^7\\
&\quad - 32b^2r_{00}^2\beta^7 - 8r_{00}r_0\beta^8 + 24r_{00|0}b^2\beta^8 + 56r_{00}s_0\beta^8 - 24nr_{00}s_0\beta^8\\
&\quad - 48br_{00}s_0\beta^8 + 16nb^2r_{00}^2\beta^7 + 16nbr_0r_{00}\beta^8 + 16nbr_{00}s_0\beta^8 - 48br_{00}r_0\beta^8\\
&\quad + 8nr_{00}r_0\beta^8 - 12nb^2r_{00|0}\beta^8 + 16nr_{0m}s^m_0\beta^9 + 8ns_{0|0}\beta^9 + 8s^m_{0|m}\beta^10\\
&\quad + 4r_{00}r^m_m\beta^9 + 4b^mr_{00|m}\beta^9 - 4b^mr_{0m|0}\beta^9 + 8s_{0m}s^m_0\beta^9 - 20s_{0|0}\beta^9,\\
P_3 &= 148r_{00}^2\beta^6 - 40r_{00|0}\beta^7 + 16s_0^2\beta^8 - 16r_{0m}s^m_0\beta^8 - 8s_{0|0}\beta^8 - 89nr_{00}^2\beta^6\\
&\quad + 76b^2r_{00}^2\beta^6 + 20r_0r_{00}\beta^7 + 24nr_{00|0}\beta^7 - 32b^2r_{00|0}\beta^7 - 12s_0r_{00}\beta^7\\
&\quad - 16ns_0^2\beta^8 + 72br_{00}s_0\beta^7 - 44nb^2r_{00}^2\beta^6 - 4nr_{00}s_0\beta^7 + 32s_{0m}s^m_0\beta^8\\
&\quad - 24nbr_{00}r_0\beta^7 - 24nbr_{00}s_0\beta^7 - 12nr_{00}r_0\beta^7 + 72br_{00}r_0\beta^7 + 18nb^2r_{00|0}\beta^7\\
&\quad + 16r_0s_0\beta^8 + 8nr_{0m}s^m_0\beta^8 + 4ns_{0|0}\beta^8 + 16s^m_{0|m}\beta^9,\\
P_4 &= 4r_0^2\beta^7 - 4rr_{00}\beta^7 - 16br_0^2\beta^7 + 16brr_{00}\beta^7 - 12s_mr^m_0\beta^8 - 77r_{00}^2\beta^5\\
&\quad - 62r_{00|0}\beta^6 + 92s_0^2\beta^7 + 24r_{0m}r^m_0\beta^7 + 128r_{0m}s^m_0\beta^7 - 36s_ms^m_0\beta^8\\
&\quad - 8r_ms^m_0\beta^8 + 10nr_{00}^2\beta^5 - 10b^2r_{00}^2\beta^5 + 16b^4r_{00}^2\beta^5 + 8r_{00}r_0\beta^6 + 42r_{00|0}n\beta^6\\
&\quad - 76r_{00|0}\beta^6b^2 - 24r_{00|0}b^4\beta^6 - 272r_{00}s_0\beta^6 - 64s_0^2n\beta^7 - 128s_0^2b\beta^7\\
&\quad - 112r_{00}s_0\beta^6b^2 + 96r_{00}s_0\beta^6b^3 + 120r_{00}s_0\beta^6b + 2r_{00}^2n\beta^5b^2 - 8r_{00}^2nb^4\beta^5\\
&\quad + 144r_{00}s_0n\beta^6 - 56r_{00}r_0n\beta^6b - 16r_{00}r_0nb^2\beta^6 + 48r_{00}s_0nb^2\beta^6 + 100s_{0|0}\beta^7\\
&\quad - 32r_{00}s_0n\beta^6b^3 - 56nbr_{00}s_0\beta^6 - 32r_{00}r_0n\beta^6b^3 + 32r_0s_0n\beta^7b\\
&\quad - 48r_{0m}s^m_0nb^2\beta^7 - 24s_{0|0}nb^2\beta^7 + 32s_0^2n\beta^7b + 16r_0s_0n\beta^7 - 128r_0s_0b\beta^7\\
&\quad - 28r_{00}r_0n\beta^6 + 120br_{00}r_0\beta^6 + 96r_{00}r_0\beta^6b^3 + 16r_{00}r_0\beta^6b^2 + 42nb^2r_{00|0}\beta^6\\
&\quad + 12r_{00|0}nb^4\beta^6 + 24r_0s_0\beta^7 + 12r_{0m}r^m_0\beta^7b^2 - 88r_{0m}s^m_0n\beta^7 - 44s_{0|0}n\beta^7\\
&\quad + 60b^2s_{0|0}\beta^7 + 16s_ms^m_0n\beta^8 + 12r_{0m|0}b^m\beta^7b^2 - 32s_{0m}s^m_0\beta^7b^2\\
&\quad - 12r_{00}r^m_m\beta^7b^2 - 12b^2r_{00|m}b^m\beta^7 + 84r_{0m}s^m_0\beta^7b^2 - 40s^m_{0|m}\beta^8 + 8s_0r^m_m\beta^8
\end{aligned}$$

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\begin{aligned}
&+ 8s_{0|m}b^m\beta^8 - 4s_{m|0}b^m\beta^8 + 4s^i_m s^m_i\beta^9 - 32s^m_{0|m}b^2\beta^8 - 24r_{00}r^m_m\beta^7\\
&- 24r_{00|m}b^m\beta^7 + 24r_{0m|0}b^m\beta^7 + 16s_{0m}s^m_0\beta^7,\\
P_5 = {}&100r_{00|0}\beta^5 - 12r_{0m}r^m_0\beta^6 - 73r^2_{00}\beta^4 + 12s^2_0\beta^6 - 16s_{0|0}\beta^6 - 52s_m s^m_0\beta^7\\
&- 12s_m r^m_0\beta^7 - 4r^2_0\beta^6 - 8r_m s^m_0\beta^7 + 128r^2_{00}n\beta^4 - 56r^2_{00}\beta^4 b^4 - 68r_{00}r_0\beta^5\\
&- 72r_{00|0}n\beta^5 + 148b^2 r_{00|0}\beta^5 + 32s^2_0\beta^6 b + 160r_{00}s_0 b^2\beta^5 + 52r_{00|0}\beta^5 b^4\\
&+ 316r_{00}s_0\beta^5 - 24s^2_0 n\beta^6 - 64s^2_0 b^2\beta^6 - 272r_{00}s_0 b\beta^5 + 4r_{0m}s^m_0\beta^6\\
&- 208r_{00}s_0 b^3\beta^5 + 126r^2_{00}n\beta^4 b^2 + 34r^2_{00}nb^4\beta^4 - 120r_{00}s_0 n\beta^5\\
&+ 40r_{00}r_0 n\beta^5 b^2 + 128r_{00}r_0 n\beta^5 b - 132b^2 r^2_{00}\beta^4 - 72r_{00}s_0 n\beta^5 b^2\\
&+ 128nbr_{00}s_0\beta^5 + 4rr_{00}\beta^6 + 80nb^3 r_{00}r_0\beta^5 - 16nbr_0 s_0\beta^6 + 24r_{0m}s^m_0 n\beta^6 b^2\\
&+ 12s_{0|0}n\beta^6 b^2 - 16rr_{00}b\beta^6 - 8nr_0 s_0\beta^6 + 64r_0 s_0\beta^6 b + 64r_{00}r_0 n\beta^5\\
&- 272r_{00}r_0 b\beta^5 - 48r_0 s_0 b^2\beta^6 - 208r_{00}r_0 b^3\beta^5 - 48r_{00}r_0 b^2\beta^5 - 96r_{00|0}n\beta^5 b^2\\
&+ 80nb^3 r_{00}s_0\beta^5 - 30r_{00|0}n\beta^5 b^4 + 64s^2_0 nb^2\beta^6 - 16s^2_0 n\beta^6 b + 4s_{0|0}n\beta^6\\
&+ 16r^2_0 b\beta^6 - 8r_0 s_0\beta^6 - 12r_{0m}r^m_0 b^2\beta^6 + 8r_{0m}s^m_0 n\beta^6 - 20r_{0m}s^m_0 b^2\beta^6\\
&- 64s^m_{0|m}\beta^7 - 28s_{0|0}b^2\beta^6 + 24s_m s^m_0 n\beta^7 - 12r_{0m|0}b^m b^2\beta^6 - 96s_{0m}s^m_0\beta^6 b^2\\
&+ 12r_{00}r^m_m b^2\beta^6 + 12r_{00|m}b^m b^2\beta^6 + 12r_{00}r^m_m\beta^6 + 12r_{00|m}b^m\beta^6\\
&- 12r_{0m|0}b^m\beta^6 + 12s^i_m s^m_i\beta^8 - 32s^m_{0|m}\beta^7 b^2 - 64s_{0m}s^m_0\beta^6 + 8s_0 r^m_m\beta^7\\
&+ 8s_{0|m}b^m\beta^7 - 4s_{m|0}b^m\beta^7,\\
P_6 = {}&60s_m r^m_0\beta^6 + 40r_m s^m_0\beta^6 + 136r^2_{00}\beta^3 - 8r_{00|0}\beta^4 - 352s^2_0\beta^5 - 48r_{0m}r^m_0\beta^5\\
&+ 4r_{00|0}n\beta^4 - 208r_{0m}s^m_0\beta^5 - 176s_{0|0}\beta^5 + 124s_m s^m_0\beta^6 + 192r^2_{00}b^2\beta^3\\
&+ 82r^2_{00}b^4\beta^3 + 60r_{00}r_0\beta^4 - 4r_{00|0}b^2\beta^4 + 10r_{00|0}\beta^4 b^4 + 8r_{00|0}b^6\beta^4\\
&+ 72r_{00}s_0\beta^4 + 188s^2_0 n\beta^5 - 224b^2 s^2_0\beta^5 + 400s^2_0 b\beta^5 + 272s^2_0 b^3\beta^5\\
&+ 16rr_{00}\beta^5 + 56r_{00}s_0\beta^4 b^4 - 48r_{00}s_0 b^5\beta^4 - 16r_{00}s_0 b^3\beta^4 - 158r^2_{00}n\beta^3\\
&+ 64r_{00}s_0 b\beta^4 - 198r^2_{00}nb^2\beta^3 - 59r^2_{00}nb^4\beta^3 - 16r^2_0\beta^5 - 152r_{00}s_0 n\beta^4\\
&+ 148r_{00}s_0 b^2\beta^4 + 16r_{00}s_0 nb^5\beta^4 + 8r_{00}r_0 nb^4\beta^4 - 16r_{00}r_0 n\beta^4 b\\
&- 24r_{00}s_0 nb^4\beta^4 - 16r_{00}s_0 n\beta^4 b - 104r_{00}s_0 n\beta^4 b^2 + 16r_{00}r_0 nb^5\beta^4\\
&- 32r_0 s_0 n\beta^5 b^2 - 128br_0 s_0 n\beta^5 - 64r_0 s_0 n\beta^5 b^3 + 64r_{00}r_0 b\beta^4\\
&- 48s_m s^m_0 nb^2\beta^6 + 192r_{0m}s^m_0 n\beta^5 b^2 + 48r_{0m}s^m_0 nb^4\beta^5 + 96s_{0|0}n\beta^5 b^2\\
&+ 24s_{0|0}nb^4\beta^5 + 32rs_0\beta^6 b - 64s^2_0 n\beta^5 b^3 - 64rr_{00}b\beta^5 + 8rr_{00}b^2\beta^5\\
&- 32rr_{00}b^3\beta^5 - 64r_0 s_0 n\beta^5 - 48r_0 s_0\beta^5 b^2 + 416r_0 s_0 b\beta^5 + 256r_0 s_0 b^3\beta^5\\
&- 8r^2_0 b^2\beta^5 - 8r_{00}r_0\beta^4 b^4 - 8r_{00}r_0 n\beta^4 - 16r_{00}r_0 b^3\beta^4 - 48r_{00}r_0 b^5\beta^4\\
&+ 28r_{00}r_0 b^2\beta^4 + 64r^2_0 b\beta^5 + 12r_{00|0}nb^2\beta^4 - 4r_{00|0}nb^6\beta^4 + 176s^2_0 n\beta^5 b^2\\
&- 24s_{0|m}b^m\beta^6 b^2 + 104s^m_{0|m}\beta^6 - 40s_0 r^m_m\beta^6 - 128s^2_0 n\beta^5 b + 24r_m s^m_0\beta^6 b^2\\
&+ 32r^2_0 b^3\beta^5 - 56r_0 s_0\beta^5 - 48r_{0m}r^m_0 b^2\beta^5 - 12b^4 r_{0m}r^m_0\beta^5 - 288r_{0m}s^m_0 b^2\beta^5\\
&- 84r_{0m}s^m_0\beta^5 b^4 + 88s_{0|0}n\beta^5 - 216s_{0|0}b^2\beta^5 + 12b^2 s_{m|0}b^m\beta^6 - 8rs_0\beta^6
\end{aligned}
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\begin{aligned}
& - 64s_m s_0^m n\beta^6 + 108s_m s_0^m b^2\beta^6 + 36s_m r_0^m \beta^6 b^2 - 12r_{0m|0} b^m \beta^5 b^4 \\
& + 48b^4 s_{0m} s_0^m \beta^5 + 48r_{00} r_m^m b^2 \beta^5 + 176 r_{0m} s_0^m n\beta^5 + 12r_{00} r_m^m \beta^5 b^4 \\
& + 48r_{00|m} b^m b^2 \beta^5 + 12b^4 r_{00|m} b^m \beta^5 - 60s_{0|0}\beta^5 b^4 - 48r_{0m|0} b^m b^2 \beta^5 \\
& - 16s_m^i s_i^m b^2 \beta^7 - 24s_0 r_m^m \beta^6 b^2 - 40s_{0|m} b^m \beta^6 + 20s_{m|0} b^m \beta^6 + 160s_{0|m}^m \beta^6 b^2 \\
& + 48s_{0|m}^m b^4 \beta^6 - 8s_m^i s_i^m \beta^7 - 8s_m s^m \beta^7 + 48r_{00} r_m^m \beta^5 + 48r_{00|m} b^m \beta^5 \\
& - 48r_{0m|0} b^m \beta^5 - 40s_{0m} s_0^m \beta^5, \\
P_7 = {} & 24r_0^2\beta^4 - 24rr_{00}\beta^4 - 96r_0^2 b\beta^4 + 96rr_{00} b\beta^4 - 92r_{00}^2\beta^2 - 64r_{00|0}\beta^3 \\
& + 112s_0^2\beta^4 + 48r_{0m} r_0^m \beta^4 + 96r_{0m} s_0^m \beta^4 + 16r_m s_0^m \beta^5 + 120s_{0|0}\beta^4 \\
& + 104s_m s_0^m \beta^5 + 24s_m r_0^m \beta^5 + 88r_{00}^2 n\beta^2 - 152r_{00}^2 b^2\beta^2 - 70r_{00}^2 b^4\beta^2 \\
& + 24r_{00} r_0 \beta^3 + 58r_{00|0} n\beta^3 - 144b^2 r_{00|0}\beta^3 - 104r_{00|0}\beta^3 b^4 - 24r_{00|0}\beta^3 b^6 \\
& - 336r_{00} s_0 \beta^3 + 28s_0^2 n\beta^4 + 168s_0^2\beta^4 b^2 + 96s_0^2 b^4\beta^4 - 272s_0^2 b\beta^4 - 240s_0^2 b^3\beta^4 \\
& - 148r_{00} s_0 \beta^3 b^4 + 136r_{00} s_0 \beta^3 b^5 + 336r_{00} s_0 b^3\beta^3 + 53r_{00}^2 n b^4 \beta^2 \\
& + 176r_{00} s_0 b\beta^3 + 296r_{00} s_0 n\beta^3 + 176r_{00} r_0 b\beta^3 + 142r_{00}^2 n b^2\beta^2 \\
& - 416r_{00} s_0 \beta^3 b^2 - 88r_{00} r_0 n\beta^3 b^2 - 28r_{00} r_0 n b^4 \beta^3 + 76r_{00} s_0 n b^4 \beta^3 \\
& - 176r_{00} s_0 n b^3 \beta^3 - 144r_{00} s_0 n b\beta^3 - 56r_{00} s_0 n b^5 \beta^3 + 312r_{00} s_0 n\beta^3 b^2 \\
& - 176r_{00} r_0 n b^3 \beta^3 + 48b^2 r_0 s_0 n\beta^4 - 144r_{00} r_0 n b\beta^3 - 56r_{00} r_0 n b^5 \beta^3 \\
& + 128r_0 s_0 n\beta^4 b + 96r_0 s_0 n\beta^4 b^3 - 24b^2 s_m s_0^m n\beta^5 - 192r_{0m} s_0^m n\beta^4 b^2 \\
& - 72r_{0m} s_0^m n b^4 \beta^4 - 96s_{0|0} n\beta^4 b^2 - 16rr_{00}\beta^4 b^2 + 64rr_{00} b^3\beta^4 + 64r_0 s_0 n\beta^4 \\
& - 36s_{0|0} n b^4 \beta^4 + 48r_0 s_0 \beta^4 b^2 + 48r_0 s_0 b^4 \beta^4 - 384r_0 s_0 b\beta^4 - 320r_0 s_0 b^3 \beta^4 \\
& - 72r_{00} r_0 n\beta^3 + 336r_{00} r_0 b^3\beta^3 + 136r_{00} r_0 \beta^3 b^5 + 56b^2 r_{00} r_0 \beta^3 + 28r_{00} r_0 \beta^3 b^4 \\
& + 108r_{00|0} n\beta^3 b^2 + 66r_{00|0} n\beta^3 b^4 + 14r_{00|0} n\beta^3 b^6 - 72nb^2 s_0^2\beta^4 - 96s_0^2 n b^4 \beta^4 \\
& + 128s_0^2 n\beta^4 b + 96s_0^2 n\beta^4 b^3 + 16r_0^2\beta^4 b^2 - 64r_0^2 b^3\beta^4 - 48r_0 s_0 \beta^4 \\
& + 72r_{0m} r_0^m \beta^4 b^2 + 24r_{0m} r_0^m \beta^4 b^4 + 184r_{0m} s_0^m \beta^4 b^2 + 88r_{0m} s_0^m \beta^4 b^4 \\
& - 56ns_{0|0}\beta^4 + 200s_{0|0}\beta^4 b^2 - 56s_m s_0^m n\beta^5 + 64s_m s_0^m \beta^5 b^2 + 80s_{0|m}^m \beta^5 \\
& - 16s_{0|m} b^m \beta^5 - 48r_{00|m} b^m \beta^4 + 64s_{0|m}^m \beta^5 b^2 - 16s_m s^m \beta^6 - 40s_m^i s_i^m \beta^6 \\
& - 48r_{00} r_m^m \beta^4 + 48r_{0m|0} b^m \beta^4 + 64s_{0m} s_0^m \beta^4 + 80s_{0|0}\beta^4 b^4 - 32s_m^i s_i^m \beta^6 b^2 \\
& + 160b^2 s_{0m} s_0^m \beta^4 + 96s_{0m} s_0^m \beta^4 b^4 - 112r_{0m} s_0^m n\beta^4 - 72r_{00} r_m^m \beta^4 b^2 \\
& - 24r_{00} r_m^m \beta^4 b^4 - 72b^2 r_{00|m} b^m \beta^4 + 8s_{m|0} b^m \beta^5 - 24r_{00|m} b^m \beta^4 b^4 \\
& + 72r_{0m|0} b^m \beta^4 b^2 + 24r_{0m|0} b^m \beta^4 b^4 - 16s_0 r_m^m \beta^5, \\
P_8 = {} & 50r_{00|0}\beta^2 + 324s_0^2\beta^3 + 8r_0^2\beta^3 + 20r_{0m} r_0^m \beta^3 + 76s_{0|0}\beta^3 - 200s_m s_0^m \beta^4 \\
& - 8rr_{00}\beta^3 - 80r_m s_0^m \beta^4 - 120s_m r_0^m \beta^4 + 29r_{00}^2\beta + 74r_{00}^2 b^2\beta + 38r_{00}^2 b^4\beta \\
& + 228r_{00} s_0 \beta^2 - 68r_{00} r_0 \beta^2 + 92r_{0m} s_0^m \beta^3 - 48r_{00|0} n\beta^2 + 116b^2 r_{00|0}\beta^2 \\
& + 92r_{00|0}\beta^2 b^4 + 24b^6 r_{00|0}\beta^2 - 220s_0^2 n\beta^3 + 432s_0^2\beta^3 b^2 + 172s_0^2\beta^3 b^4 \\
& - 432s_0^2 b^3\beta^3 - 25r_{00}^2 n\beta - 240s_0^2 b\beta^3 - 160s_0^2 b^5\beta^3 + 156r_{00} s_0 \beta^2 b^4
\end{aligned}
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\begin{aligned}
&- 120r_{00}s_0\beta^2b^5 - 272r_{00}s_0b^3\beta^2 - 160r_{00}s_0b\beta^2 - 25r_{00}^2nb^4\beta \\
&- 192r_{00}s_0n\beta^2 - 160r_{00}r_0b\beta^2 - 52r_{00}^2nb^2\beta + 344b^2r_{00}s_0\beta^2 \\
&+ 144r_{00}s_0nb\beta^2 + 104r_{00}r_0nb^2\beta^2 + 36r_{00}r_0nb^4\beta^2 + 208r_{00}r_0nb^3\beta^2 \\
&+ 48nb^2r_0s_0\beta^3 + 72r_{00}r_0nb^5\beta^2 + 144r_{00}r_0nb\beta^2 + 16r_0s_0n\beta^3b^4 \\
&+ 96r_0s_0nb^3\beta^3 + 96r_0s_0nb\beta^3 + 32r_0s_0nb^5\beta^3 - 96r_{00}s_0nb^4\beta^2 \\
&- 288r_{00}s_0nb^2\beta^2 + 208r_{00}s_0nb^3\beta^2 + 72r_{00}s_0nb^5\beta^2 + 168s_ms_0^mn\beta^4b^2 \\
&+ 48s_ms_0^mnb^4\beta^4 - 144r_{0m}s_0^mnb^2\beta^3 - 72r_{0m}s_0^mnb^4\beta^3 - 16nb^6r_{0m}s_0^m\beta^3 \\
&+ 16rs_0\beta^4b^2 - 72s_{0|0}nb^2\beta^3 - 36s_{0|0}nb^4\beta^3 - 8s_{0|0}n\beta^3b^6 - 64rs_0\beta^4b^3 \\
&- 128brs_0\beta^4 + 96s_0^2nb^3\beta^3 + 96s_0^2nb\beta^3 + 32s_0^2nb^5\beta^3 - 8rr_{00}b^2\beta^3 \\
&- 4rr_{00}b^4\beta^3 + 32rr_{00}b\beta^3 + 32rr_{00}b^3\beta^3 + 16rr_{00}b^5\beta^3 + 48r_0s_0n\beta^3 \\
&+ 72r_{00}r_0n\beta^2 + 120r_0s_0\beta^3b^2 + 24b^4r_0s_0\beta^3 - 272r_{00}r_0b^3\beta^2 - 120r_{00}r_0\beta^2b^5 \\
&- 352r_0s_0b^3\beta^3 - 224r_0s_0b\beta^3 - 128r_0s_0b^5\beta^3 - 96r_{00}r_0\beta^2b^2 - 36r_{00}r_0\beta^2b^4 \\
&- 108r_{00|0}n\beta^2b^2 - 78r_{00|0}n\beta^2b^4 - 18nb^6r_{00|0}\beta^2 - 328s_0^2n\beta^3b^2 \\
&- 160s_0^2n\beta^3b^4 - 96r_ms_0^mb^2\beta^4 - 24r_ms_0^m\beta^4b^4 + 8r_0^2b^2\beta^3 + 4r_0^2b^4\beta^3 \\
&+ 12r_{0m}r_0^mb^4\beta^3 - 32r_0^2b\beta^3 - 32r_0^2b^3\beta^3 - 16r_0^2b^5\beta^3 + 88r_0s_0\beta^3 \\
&+ 24r_{0m}r_0^mb^2\beta^3 + 4r_{0m}r_0^mb^6\beta^3 - 96r_{0m}s_0^mn\beta^3 + 28r_{0m}s_0^m\beta^3b^6 \\
&- 48s_{0|0}n\beta^3 + 144s_{0|0}b^2\beta^3 + 88s_{0|0}b^4\beta^3 + 32rs_0\beta^4 + 120s_ms_0^mn\beta^4 \\
&- 320s_ms_0^m\beta^4b^2 - 108s_ms_0^mb^4\beta^4 - 192s_{0|m}^m\beta^4b^4 - 152s_{0|m}^m\beta^4 \\
&+ 80s_{0|m}b^m\beta^4 - 40s_{m|0}b^m\beta^4 - 320b^2s_{0|m}^m\beta^4 - 20r_{00}r_m^m\beta^3 \\
&- 20r_{00|m}b^m\beta^3 - 32s_{0|m}^mb^6\beta^4 + 24s_ms^m\beta^5 + 12s_m^is_i^m\beta^5 + 80s_0r_m^m\beta^4 \\
&+ 48s_m^is_i^m\beta^5b^2 + 24s_m^is_i^mb^4\beta^5 + 20r_{0m|0}b^m\beta^3 + 16s_{0m}s_0^m\beta^3 \\
&+ 140r_{0m}s_0^mb^4\beta^3 + 96s_0r_m^mb^2\beta^4 + 24s_0r_m^m\beta^4b^4 + 96s_{0|m}b^mb^2\beta^4 \\
&+ 216r_{0m}s_0^mb^2\beta^3 + 24s_ms^m\beta^5b^2 + 24s_{0|m}b^m\beta^4b^4 - 48s_{m|0}b^mb^2\beta^4 \\
&- 12s_{m|0}b^m\beta^4b^4 + 20b^6s_{0|0}\beta^3 - 48s_{0m}s_0^m\beta^3b^4 - 32s_{0m}s_0^m\beta^3b^6 \\
&- 4r_{00|m}b^mb^6\beta^3 + 4r_{0m|0}b^mb^6\beta^3 - 144b^2s_mr_0^m\beta^4 - 24r_{00}r_m^mb^2\beta^3 \\
&- 36b^4s_mr_0^m\beta^4 - 12b^4r_{00}r_m^m\beta^3 - 4b^6r_{00}r_m^m\beta^3 - 24b^2b^mr_{00|m}\beta^3 \\
&- 12b^4b^mr_{00|m}\beta^3 + 24b^2b^mr_{0m|0}\beta^3 + 12b^4b^mr_{0m|0}\beta^3,
\end{aligned}
$$

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\begin{aligned}
P_9 = {}& 3r_{00}^2n - 16r_{00}^2b^2 - 10r_{00}^2b^4 - 16r_{00|0}\beta - 220s_0^2\beta^2 - 32r_0^2\beta^2 - 48r_{0m}r_0^m\beta^2 \\
&- 96r_{0m}s_0^m\beta^2 - 36r_{00|0}\beta b^2 - 48r_{00}s_0\beta + 40r_{00}r_0\beta + 16r_{00|0}n\beta \\
&- 8s_ms_0^m\beta^3 + 24s_mr_0^m\beta^3 - 120s_{0|0}\beta^2 - 28r_{00|0}\beta b^4 - 8r_{00|0}\beta b^6 + 108s_0^2n\beta^2 \\
&- 360s_0^2b^2\beta^2 - 236s_0^2\beta^2b^4 + 16r_ms_0^m\beta^3 - 3r_{00}^2 - 160r_{00}s_0\beta b^2 + 24r_{00}s_0\beta b^5 \\
&- 64s_0^2b^6\beta^2 + 528s_0^2b^3\beta^2 + 304s_0^2b\beta^2 + 224s_0^2b^5\beta^2 + 64r_{00}s_0b^3\beta + 32rr_{00}\beta^2 \\
&+ 56r_{00}s_0b\beta + 60r_{00}s_0n\beta + 56r_{00}r_0b\beta - 92b^4r_{00}s_0\beta + 8r_{00}^2nb^2 \\
&- 48r_{00}r_0nb^2\beta - 20r_{00}r_0nb^4\beta - 96r_{00}r_0nb^3\beta - 40r_{00}r_0nb^5\beta - 56nbr_0r_{00}\beta
\end{aligned}
$$

$$
\begin{aligned}
&- 192r_0s_0nb\beta^2 - 80r_0s_0nb^5\beta^2 - 128r_0s_0nb^2\beta^2 - 40r_0s_0n\beta^2b^4\\
&- 256r_0s_0nb^3\beta^2 + 60r_{00}s_0nb^4\beta + 96s_{0|0}nb^4\beta^2 + 128r_{00}s_0nb^2\beta\\
&- 96r_{00}s_0nb^3\beta - 40r_{00}s_0nb^5\beta + 192r_{0m}s_0^mnb^4\beta^2 - 56r_{00}s_0nb\beta + 5r_{00}^2nb^4\\
&- 24s_ms_0^mn\beta^3b^4 + 288r_{0m}s_0^mnb^2\beta^2 + 40r_{0m}s_0^mn\beta^2b^6 + 144s_{0|0}nb^2\beta^2\\
&+ 20s_{0|0}n\beta^2b^6 + 64rs_0b\beta^3 + 64rs_0b^3\beta^3 - 16rs_0b^2\beta^3 - 128rr_{00}b\beta^2\\
&- 160rr_{00}b^3\beta^2 - 24s_ms_0^mn\beta^3b^2 - 80s_0^2nb^5\beta^2 + 40rr_{00}b^2\beta^2 + 12rr_{00}b^4\beta^2\\
&- 48rr_{00}b^5\beta^2 - 96r_0s_0n\beta^2 - 8r_0s_0b^2\beta^2 - 56r_0s_0\beta^2b^4 - 16r_0s_0b^6\beta^2\\
&+ 672r_0s_0b^3\beta^2 + 24r_{00}r_0\beta b^5 + 56r_{00}r_0\beta b^2 + 416r_0s_0b\beta^2 + 256r_0s_0b^5\beta^2\\
&- 28r_{00}r_0n\beta + 64r_{00}r_0b^3\beta + 20r_{00}r_0\beta b^4 + 42r_{00|0}n\beta b^2 + 36r_{00|0}n\beta b^4\\
&+ 10r_{00|0}n\beta b^6 + 192s_0^2nb^2\beta^2 + 152nb^4s_0^2\beta^2 + 24r_ms_0^m\beta^3b^4 - 40r_0^2b^2\beta^2\\
&+ 64s_0^2nb^6\beta^2 - 256s_0^2nb^3\beta^2 - 192s_0^2nb\beta^2 + 48b^2r_ms_0^m\beta^3 - 12r_0^2b^4\beta^2\\
&+ 128r_0^2b\beta^2 + 160r_0^2b^3\beta^2 + 48r_0^2b^5\beta^2 + 24r_0s_0\beta^2 - 96r_{0m}r_0^mb^2\beta^2\\
&- 52r_{0m}s_0^m\beta^2b^6 + 68s_{0|0}n\beta^2 - 60r_{0m}r_0^mb^4\beta^2 - 12r_{0m}r_0^mb^6\beta^2\\
&+ 136r_{0m}s_0^mn\beta^2 - 264s_{0|0}b^2\beta^2 - 192s_{0|0}b^4\beta^2 - 16rs_0\beta^3 + 8s_ms_0^mn\beta^3\\
&+ 8s_ms_0^m\beta^3b^2 + 28b^4s_ms_0^m\beta^3 + 32s_{0|m}^m\beta^3b^6 + 40s_ms^m\beta^4 - 16s_{0|m}b^m\beta^3\\
&+ 8s_{m|0}b^m\beta^3 + 64s_{0|m}^m\beta^3b^2 + 96b^4s_{0|m}^m\beta^3 + 52s_m^is_i^m\beta^4 + 48r_{00}r_m^m\beta^2\\
&+ 48r_{00|m}b^m\beta^2 - 48r_{0m|0}b^m\beta^2 - 32s_{0m}s_0^m\beta^2 - 16s_0r_m^m\beta^3 + 60r_{00|m}b^m\beta^2b^4\\
&+ 60r_{00}r_m^mb^4\beta^2 + 12r_{00}r_m^mb^6\beta^2 + 96r_{00|m}b^m\beta^2b^2 - 44b^6s_{0|0}\beta^2\\
&+ 80s_m^is_i^mb^2\beta^4 + 24s_m^is_i^m\beta^4b^4 - 60r_{0m|0}b^m\beta^2b^4 + 72s_mr_0^m\beta^3b^2\\
&- 96b^2r_{0m|0}b^m\beta^2 - 24s_0r_m^m\beta^3b^4 - 48s_{0|m}b^m\beta^3b^2 - 24s_{0|m}b^m\beta^3b^4\\
&- 48s_0r_m^m\beta^3b^2 + 36b^4s_mr_0^m\beta^3 - 204r_{0m}s_0^mb^4\beta^2 + 24s_{m|0}b^m\beta^3b^2\\
&+ 12s_{m|0}b^m\beta^3b^4 + 24s_ms^mb^2\beta^4 - 240b^2r_{0m}s_0^m\beta^2 - 96s_{0m}s_0^m\beta^2b^4\\
&- 32s_{0m}s_0^m\beta^2b^6 + 12r_{00|m}b^mb^6\beta^2 - 12r_{0m|0}b^mb^6\beta^2 - 96b^2s_{0m}s_0^m\beta^2\\
&+ 96r_{00}r_m^mb^2\beta^2,
\end{aligned}
$$

$$
\begin{aligned}
P_{10} = {}& 52s_{0|0}\beta + 124s_ms_0^m\beta^2 - 2nr_{00|0} + 4b^2r_{00|0} + 2b^4r_{00|0} - 16s_0^2\beta + 20r_0^2\beta\\
&+ 24r_{0m}r_0^m\beta + 28r_{00}s_0b^4 - 8r_{00}s_0n + 8r_{00}s_0b^5 - 8r_{00}s_0b + 84s_mr_0^m\beta^2\\
&+ 56r_ms_0^m\beta^2 + 16b^7s_0^2\beta - 4r_{00}r_0b^4 + 4r_{00}r_0n + 8r_{00}r_0b^5 - 8r_{00}r_0b\\
&- 6r_{00|0}nb^2 - 12r_{00}r_0b^2 + 32r_{0m}s_0^m\beta - 44s_0^2\beta b^4 - 20rr_{00}\beta - 6r_{00|0}nb^4\\
&- 2r_{00|0}nb^6 + 36r_{00}s_0b^2 - 4s_0^2n\beta - 8b^2s_0^2\beta - 40s_0^2\beta b^6 - 112s_0^2b^3\beta\\
&- 16s_0^2b^5\beta - 112s_0^2b\beta + 16r_{00}s_0nb^3 + 8r_{00}s_0nb^5 + 8nbs_0r_{00} + 80r_0s_0n\beta b^2\\
&+ 32r_0s_0n\beta b^4 - 16r_{00}s_0nb^4 - 24r_{00}s_0nb^2 - 8r_{00}r_0 + 2r_{00|0} - 4s_0r_{00}\\
&+ 160r_0s_0nb^3\beta + 96r_0s_0nb\beta + 64r_0s_0nb^5\beta - 96s_ms_0^mn\beta^2b^4\\
&- 16s_ms_0^mnb^6\beta^2 - 16s_{0|0}n\beta b^6 - 144r_{0m}s_0^mnb^2\beta - 120r_{0m}s_0^mnb^4\beta\\
&- 32r_{0m}s_0^mn\beta b^6 - 72s_{0|0}nb^2\beta - 12rr_{00}b^4\beta + 80rr_{00}b\beta + 128rr_{00}b^3\beta
\end{aligned}
$$

$$
\begin{aligned}
&+128rs_0b\beta^2+128rs_0b^3\beta^2+32rs_0b^5\beta^2+48rr_{00}b^5\beta-32rr_{00}b^2\beta\\
&-32rs_0b^2\beta^2-8rs_0\beta^2b^4-168s_ms_0^mnb^2\beta^2+64s_0^2nb^5\beta-192r_0s_0b\beta\\
&-128r_0s_0b^5\beta+48r_0s_0n\beta-104r_0s_0\beta b^2-56r_0s_0\beta b^4-288r_0s_0b^3\beta\\
&+8r_{00}r_0nb^2+16r_{00}r_0nb^3+8r_{00}r_0nb^5+8r_{00}r_0nb+32s_0^2n\beta b^2\\
&-60s_{0|0}nb^4\beta+4r_{00}r_0nb^4+92s_0^2n\beta b^4+48s_0^2n\beta b^6+160s_0^2nb^3\beta+96s_0^2nb\beta\\
&+68r_{0m}s_0^mb^2\beta+60r_{0m}r_0^mb^2\beta-56r_0s_0\beta+32r_0^2b^2\beta+12r_0^2b^4\beta-80r_0^2b\beta\\
&-128r_0^2b^3\beta-48r_0^2b^5\beta+56r_{0m}s_0^mb^4\beta+20r_{0m}s_0^m\beta b^6+48r_{0m}r_0^mb^4\beta\\
&+12r_{0m}r_0^mb^6\beta-56r_{0m}s_0^mn\beta-28s_{0|0}n\beta+124s_{0|0}b^2\beta+100s_{0|0}b^4\beta\\
&+28s_{0|0}\beta b^6-32rs_0\beta^2-88s_ms_0^mn\beta^2+272s_ms_0^mb^2\beta^2+192s_ms_0^m\beta^2b^4\\
&+36s_ms_0^mb^6\beta^2+144s_mr_0^mb^2\beta^2+72s_mr_0^mb^4\beta^2+12s_mr_0^mb^6\beta^2\\
&+96r_ms_0^mb^2\beta^2+48r_ms_0^mb^4\beta^2+8r_ms_0^mb^6\beta^2+88s_{0|m}^m\beta^2+224s_{0|m}^mb^2\beta^2\\
&+192s_{0|m}^mb^4\beta^2+64s_{0|m}^mb^6\beta^2+8s_{0|m}^mb^8\beta^2-40s_ms^m\beta^3-24s_m^is_i^m\beta^3\\
&-24r_{00}r_m^m\beta-24r_{00|m}b^m\beta+24r_{0m|0}b^m\beta+8s_{0m}s_0^m\beta-56s_0r_m^m\beta^2\\
&-56s_{0|m}b^m\beta^2+28s_{m|0}b^m\beta^2-72s_m^is_i^m\beta^3b^4-16s_m^is_i^mb^6\beta^3-96s_0r_m^m\beta^2b^2\\
&-48b^4s_0r_m^m\beta^2-96s_{0|m}b^m\beta^2b^2-48s_{0|m}b^m\beta^2b^4+48s_{m|0}b^m\beta^2b^2\\
&+24b^4s_{m|0}b^m\beta^2-8s_0r_m^mb^6\beta^2-8s_{0|m}b^mb^6\beta^2+4s_{m|0}b^mb^6\beta^2\\
&-72s_ms^mb^2\beta^3-24s_ms^m\beta^3b^4-80s_m^is_i^mb^2\beta^3-12r_{00|m}b^mb^6\beta\\
&+12r_{0m|0}b^mb^6\beta+32s_{0m}s_0^m\beta b^2+48s_{0m}s_0^m\beta b^4+32b^6s_{0m}s_0^m\beta\\
&+8s_{0m}s_0^m\beta b^8-60r_{00}r_m^mb^2\beta-48r_{00}r_m^mb^4\beta-12r_{00}r_m^mb^6\beta-60b^2r_{00|m}b^m\beta\\
&-48r_{00|m}b^m\beta b^4+60r_{0m|0}b^m\beta b^2+48b^4r_{0m|0}b^m\beta,
\end{aligned}
$$

$$
\begin{aligned}
P_{11}=\;&16b^8s_0^2-8s_{0|0}+16br_0^2-16brr_{00}-4ns_0^2+40b^2s_0^2+60b^4s_0^2+56b^6s_0^2\\
&-16b^3s_0^2+16s_0^2b+4r_{0m}s_0^mb^4-48s_0^2b^5-16s_0^2b^7-8r_0^2b^2-4r_0^2b^4+32r_0^2b^3\\
&+16r_0^2b^5-40r_ms_0^m\beta-12r_{0m}r_0^mb^2-12r_{0m}r_0^mb^4-4r_{0m}r_0^mb^6+4r_{0m}s_0^mb^6\\
&-4r_{0m}s_0^mb^2-4s_{0|0}b^6-20s_{0|0}b^2-16s_{0|0}b^4+4s_{0|0}n-68s_ms_0^m\beta\\
&-60s_mr_0^m\beta-24s_0^2nb^2-60s_0^2nb^4-56s_0^2nb^6-32s_0^2nb^3-16s_0^2nb^5\\
&+8r_{0m}s_0^mn-16s_0^2nb-16s_0^2nb^8+4rr_{00}+16r_0s_0+4r_{00}r_m^m+4r_{00|m}b^m\\
&-4r_{0m|0}b^m+32s_0^2-4r_0^2-4r_{0m}r_0^m+24s_ms_0^mn\beta b^6-192rs_0b^3\beta\\
&-4r_{0m}s_0^m-128rs_0b\beta-64rs_0\beta b^5+48b^2rs_0\beta+120s_ms_0^mn\beta b^2\\
&+96s_ms_0^mn\beta b^4-16r_0s_0nb^2-8r_0s_0nb^4-32r_0s_0nb^3\\
&+16rs_0\beta b^4+12s_{0|0}nb^2-16r_0s_0nb^5-16r_0s_0nb+8rr_{00}b^2+4rr_{00}b^4\\
&-16rr_{00}b^5-8r_0s_0n+40r_0s_0b^2+40r_0s_0b^4+16r_0s_0b^6-132b^4s_ms_0^m\beta\\
&+32r_0s_0b^3+32r_0s_0b+24r_{0m}s_0^mnb^2+24r_{0m}s_0^mnb^4+8r_{0m}s_0^mnb^6\\
&+12s_{0|0}nb^4+4s_{0|0}nb^6+32rs_0\beta+48s_ms_0^mn\beta-160b^2s_ms_0^m\beta\\
&-40s_ms_0^m\beta b^6-144s_mr_0^mb^2\beta-108s_mr_0^mb^4\beta-24s_mr_0^mb^6\beta-96r_ms_0^mb^2\beta
\end{aligned}
$$

$$
\begin{aligned}
&- 16r_m s_0^m b^6\beta - 48s^m_{0|m}\beta - 24s^i_m s^m_i\beta^2 - 160s^m_{0|m}b^2\beta - 192s^m_{0|m}b^4\beta\\
&- 96s^m_{0|m}b^6\beta - 16s^m_{0|m}b^8\beta - 24s_m s^m\beta^2 + 12r_{00}r^m_m b^2 + 12r_{00}r^m_m b^4\\
&+ 4r_{00}r^m_m b^6 + 12r_{00|m}b^m b^2 + 12r_{00|m}b^m b^4 + 4r_{00|m}b^m b^6 - 72r_m s_0^m b^4\beta\\
&- 12r_{0m|0}b^m b^2 - 12r_{0m|0}b^m b^4 - 4r_{0m|0}b^m b^6 + 40s_0 r^m_m\beta + 40s_{0|m}b^m\beta\\
&- 20s_{m|0}b^m\beta + 72s_0 r^m_m\beta b^4 + 96s_{0|m}b^m\beta b^2 + 72s_{0|m}b^m\beta b^4 - 48b^2 s_{m|0}b^m\beta\\
&- 36s_{m|0}b^m\beta b^4 + 16s_0 r^m_m b^6\beta + 16s_{0|m}b^m b^6\beta - 8s_{m|0}b^m b^6\beta - 24b^2 s_m s^m\beta^2\\
&- 48s^i_m s^m_i\beta^2 b^2 - 24s^i_m s^m_i\beta^2 b^4 + 96s_0 r^m_m\beta b^2 - 32rr_{00}b^3,
\end{aligned}
$$

$$
\begin{aligned}
P_{12} = {}& 12s_m s_0^m + 8r_m s_0^m - 24s_0 r^m_m b^2 - 24s_0 r^m_m b^4 - 8s_0 r^m_m b^6 - 24s_{0|m}b^m b^4\\
&- 8s_{0|m}b^m b^6 + 12s_{m|0}b^m b^2 + 12s_{m|0}b^m b^4 - 16rs_0 b^2 + 4s_{m|0}b^m b^6\\
&+ 32s_m s^m\beta + 20s^i_m s^m_i\beta - 8rs_0 b^4 + 64rs_0 b^3 - 8s_0 r^m_m + 32rs_0 b^5 + 32rs_0 b\\
&- 24s_m s_0^m nb^2 - 24s_m s_0^m nb^4 - 8s_m s_0^m nb^6 + 12s_m r_0^m + 4s_{m|0}b^m + 8s^m_{0|m}\\
&+ 32s^m_{0|m}b^2 + 48s^m_{0|m}b^4 + 32s^m_{0|m}b^6 + 24r_m s_0^m b^2 + 24r_m s_0^m b^4 + 8r_m s_0^m b^6\\
&+ 36s_m r_0^m b^2 + 36s_m r_0^m b^4 + 8s^m_{0|m}b^8 - 8rs_0 - 8s_m s_0^m n + 28s_m s_0^m b^2\\
&+ 20s_m s_0^m b^4 + 4s_m s_0^m b^6 + 72b^2 s_m s^m\beta + 64s^i_m s^m_i\beta b^2 + 72s^i_m s^m_i\beta b^4\\
&+ 32s^i_m s^m_i\beta b^6 + 8s_m s^m b^6\beta + 48s_m s^m\beta b^4 - 24s_{0|m}b^m b^2 - 8s_{0|m}b^m\\
&+ 4s^i_m s^m_i b^8\beta + 12s_m r_0^m b^6,
\end{aligned}
$$

$$
\begin{aligned}
P_{13} = {}& -16s^i_m s^m_i b^6 - 24s^i_m s^m_i b^4 - 4s^i_m s^m_i - 8s_m s^m - 24s_m s^m b^2 - 24s_m s^m b^4\\
&- 8s_m s^m b^6 - 4s^i_m s^m_i b^8 - 16s^i_m s^m_i b^2.
\end{aligned}
$$

Now we are going to prove Theorem 1.2.

Proof of Theorem 1.2. Let F be an exponential metric on a manifold M. By (13) and (14), F is Einsteinian if and only if the following holds:

$$
\begin{aligned}
P_{13}\alpha^{13} + P_{12}\alpha^{12} + \Big(M_{11}\overline{\mathbf{Ric}} + P_{11} - (n-1)\lambda M_{11}F^2\Big)\alpha^{11} + \cdots&\\
+ \Big(M_0\overline{\mathbf{Ric}} + P_0 - (n-1)\lambda M_0 F^2\Big) = 0,& \qquad (15)
\end{aligned}
$$

where M_i, $i = 0, \ldots, 11$, are obtained by the relation

$$
M_{11}\alpha^{11} + M_{10}\alpha^{10} + \cdots + M_1\alpha + M_0 = 4\Big[(1+b^2)\alpha^2 - \alpha\beta - \beta^2\Big]^4(\alpha-\beta)^3. \quad (16)
$$

By (15), the only term that is rational is $P_{12}\alpha^{12}$, so we get $P_{12} = 0$. Then the only term that is algebraic is $P_{13}\alpha^{13}$. Thus $P_{13} = 0$. This implies that

$$
\begin{aligned}
(M_{11}\overline{\mathbf{Ric}} + P_{11})\alpha^{11} + (M_{10}\overline{\mathbf{Ric}} + P_{10})\alpha^{10} + \cdots + (M_0\overline{\mathbf{Ric}} + P_0)&\\
= (n-1)\lambda(M_{11} + M_{10} + \ldots + M_0)F^2.& \quad (17)
\end{aligned}
$$

The left-hand side of (17) is polynomial algebraic while the right-hand side is not. It results that $\lambda = 0$. □

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Про клас ейнштейнових фінслерових метрик експоненціального типу

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У статті вивчається спеціальний клас фінслерових метрик, що називаються (α, β)-метриками, які визначаються формулою $F = \alpha\phi(s)$, де α — ріманова метрика, а β — 1-форма. Спочатку ми показуємо, що клас майже регулярних метрик, отриманий Шеном, є ейнштейновим тоді і тільки тоді, коли він зводиться до класу метрик Бервальда. В цьому випадку метрики є Річчі-пласкими. Потім ми доводимо, що експоненціальна метрика є ейнштейновою тоді і тільки тоді, коли вона Річчі-пласка.

Ключові слова: ейнштейнова метрика, метрика unicorn, експоненціальна метрика.