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# Asymptotic Properties of Integrals of Quotients when the Numerator Oscillates and the Denominator Degenerates

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*Dedicated to V.A. Marchenko on the occasion of his 95th birthday*

We study asymptotic expansion as $\nu \to 0$ for integrals over $\mathbb{R}^{2d} = \{(x,y)\}$ of quotients of the form $F(x,y)\cos(\lambda x \cdot y)/\big((x\cdot y)^2 + \nu^2\big)$, where $\lambda \geq 0$ and $F$ decays at infinity sufficiently fast. Integrals of this kind appear in the theory of wave turbulence.

*Key words:* asymptotic of integrals, oscillating integrals, four-waves interaction.

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## 1. Introduction

In the paper [2] we study asymptotic behaviour of integrals

$$I_\nu = \int_{\mathbb{R}^d \times \mathbb{R}^d} dx\, dy\, \frac{F(x,y)}{(x\cdot y)^2 + (\nu\Gamma(x,y))^2}, \quad d \geq 2,\ 0 < \nu \leq 1,$$

as $\nu \to 0$, where $F$ and $\Gamma$ are $C^2$-functions, $\Gamma$ is positive and the two satisfy certain conditions at infinity. In particular, if $\Gamma \equiv 1$, then

$$|\partial_z^\alpha F(z)| \leq C' \langle z \rangle^{-N-|\alpha|}, \qquad z = (x,y) \in \mathbb{R}^{2d},\ |\alpha| \leq 2, \tag{1.1}$$

where $C' > 0$ and $N > 2d - 2$. Denote by

$$\Sigma \subset \mathbb{R}^{2d} = \mathbb{R}^d_x \times \mathbb{R}^d_y \tag{1.2}$$

the quadric $\{(x,y) : x \cdot y = 0\}$, and by $\Sigma_*$ its regular part $\Sigma \setminus \{(0,0)\}$. It is proved in [2] (see [1] for related results) that

$$I_\nu = \pi\nu^{-1} \int_{\Sigma_*} \frac{F(z)}{|z|\,\Gamma(z)}\, d_{\Sigma_*} z + O(\chi_d(\nu)), \tag{1.3}$$

where

$$\chi_d(\nu) = \begin{cases} 1, & d \geq 3, \\ \max\left(\ln(\nu^{-1}), 1\right), & d = 2, \end{cases}$$

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$d_{\Sigma_*}$ is the volume element on $\Sigma_*$, induced from the standard Riemann structure in $\mathbb{R}^{2d}$, and the integral in (1.3) converges absolutely. Integrals of this form appear in the study of the four-waves interaction. The wave turbulence (WT) limit in systems with the four-waves interaction leads to oscillating versions of the integrals above with constant functions $\Gamma$. Re-denoting $\nu\Gamma$ back to $\nu$ we write the integrals in question as

$$J_\nu = \int_{\mathbb{R}^{2d}} dz\, \frac{F(z)\cos(\lambda x\cdot y)}{(x\cdot y)^2+\nu^2}, \quad d\ge 2,\ \ \lambda\ge 0,\ 0<\nu\le 1 \tag{1.4}$$

(as before, $z=(x,y)$). We assume that $F$ is a $C^2$-function, satisfying (1.1).

The aim of this work is to prove the following result, describing the asymptotic behaviour of $J_\nu$ when $\nu\to 0$, uniformly in $\lambda\ge 0$:

**Theorem 1.1.** *Let $0<\nu\le 1$ and $\lambda\ge 0$. Then the integral $J_\nu$ and the integral*

$$J_0 = \pi e^{-\nu\lambda}\int_{\Sigma_*} F(z)|z|^{-1}\, d_{\Sigma_*}z$$

*converge absolutely and*

$$\left|J_\nu - \nu^{-1}J_0\right| \le C\chi_d(\nu), \tag{1.5}$$

*where $C$ depends on $d$ and the constants $C', N$ in (1.1), but not on $\nu$ and $\lambda$.*

Note that since $C$ does not depend on $\lambda$, then relation (1.5) remains valid for integrals (1.4), where $\lambda=\lambda(\nu)$ is any function of $\nu$. Concerning the imposed restriction $d\ge 2$ see item iv) in Section 5.

If $\lambda=0$, the integral $J_\nu$ becomes a special case of $I_\nu$ (with $\Gamma=1$), and (1.5) follows from (1.3). Since $\sin^2(\frac{\lambda}{2}x\cdot y)=\frac12(1-\cos(\lambda x\cdot y))$, then combining (1.3) and (1.5) we get

**Corollary 1.2.** *As $\nu\to 0$,*

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} dx\,dy\, \frac{F(x,y)\sin^2(\frac{\lambda}{2}x\cdot y)}{(x\cdot y)^2+\nu^2} = \frac{\pi}{2}\nu^{-1}(1-e^{-\nu\lambda})\int_{\Sigma_*}\frac{F(z)}{|z|}\, d_{\Sigma_*}z + O(\chi_d(\nu)), \tag{1.6}$$

*uniformly in $\lambda\ge 0$.*

Classically the WT considers singular versions of the integral in the l.h.s. of (1.6):

$$\int dx\,dy\, \frac{F(x,y)\sin^2(\frac{\lambda}{2}x\cdot y)}{(x\cdot y)^2}. \tag{1.7}$$

The theory deals with these integrals by performing certain formal calculations, see Section 6 of [3] (e.g., note there equations (6.39)–(6.41)). Assertion (1.6) may be regarded as a regularisation of the integral (1.7). The factor $|z|^{-1}$ which it introduces in the limiting density is not present in the asymptotic description of integrals (1.7), used in the works on WT.

Theorem 1.1 is proved below in Sections 2–4, using the geometric approach of the paper [2], which also applies to various modifications of integrals $I_\nu$ and $J_\nu$. Some of these applications are discussed in the last Section 5.

**Notation.** As usual, we denote $\langle z\rangle = \sqrt{|z|^2+1}$. For an integral

$$I = \int_{\mathbb{R}^{2d}} f(z)\,dz$$

and a submanifold $M \subset \mathbb{R}^{2d}$, $\dim M = m \le 2d$, compact or not (if $m = 2d$, then $M$ is an open domain in $\mathbb{R}^{2d}$) we write

$$\langle I, M\rangle = \int_M f(z)\,d_M(z),$$

where $d_M(z)$ is the volume-element on $M$, induced from $\mathbb{R}^{2d}$. Similar $\langle |I|, M\rangle$ stands for the integral $\int_M |f(z)|\,d_M(z)$.

## 2. Geometry of the quadric $\{x \cdot y = 0\}$ and its vicinity

**2.1. The geometry of the quadric.** The difficulty in studying the integral $J_\nu$ with small $\nu$ comes from the vicinity of the quadric $\Sigma = \{x \cdot y = 0\}$. To examine the integral's behaviour there we first analyse the geometry of the vicinity of the regular part of the quadric $\Sigma_* = \Sigma \setminus \{(0,0)\}$, following [2]. Example 5.1 at the end of the paper provides an elementary illustration to the objects, involved in this analysis.

The set $\Sigma_*$ is a smooth submanifold of $\mathbb{R}^{2d}$ of dimension $2d-1$. We denote by $\xi$ a local coordinate on $\Sigma_*$ with the coordinate mapping $\xi \mapsto (x_\xi, y_\xi) = z_\xi \in \Sigma_*$, denote $|\xi| = |(x_\xi, y_\xi)|$ and denote $N(\xi) = (y_\xi, x_\xi)$. The latter is the normal to $\Sigma_*$ at $\xi$ of length $|\xi|$. For any $0 \le R_1 < R_2$ we set

$$S^{R_1} = \{z \in \mathbb{R}^{2d} : |z| = R_1\}, \qquad \Sigma^{R_1} = \Sigma \cap S^{R_1},$$
$$S^{R_2}_{R_1} = \{z : R_1 < |z| < R_2\}, \qquad \Sigma^{R_2}_{R_1} = \Sigma \cap S^{R_2}_{R_1},$$

and for $t > 0$ denote by $D_t$ the dilation operator

$$D_t : \mathbb{R}^{2d} \to \mathbb{R}^{2d}, \quad z \mapsto tz.$$

For $z = (x, y)$ we write $\omega(z) = x \cdot y$.

The following result is Lemma 3.1 from [2]:

**Lemma 2.1.**

1) *There exists $\theta_0 \in (0,1]$ such that a suitable neighbourhood $\Sigma^{nbh} = \Sigma^{nbh}(\theta_0)$ of $\Sigma_*$ in $\mathbb{R}^{2d} \setminus \{0\}$, is invariant with respect to the dilations $D_t$, $t > 0$, and may be uniquely parametrized as*

$$\Sigma^{nbh} = \{\pi(\xi, \theta) : \xi \in \Sigma_*,\ |\theta| < \theta_0\},$$

*where $\pi(\xi,\theta) = (x_\xi, y_\xi) + \theta N_\xi = (x_\xi, y_\xi) + \theta(y_\xi, x_\xi)$. In particular, $|\pi(\xi,\theta)|^2 = |\xi|^2(1+\theta^2)$.*

2) *If $\pi(\xi,\theta) \in \Sigma^{nbh}$, then*

$$\omega\big(\pi(\xi,\theta)\big) = |\xi|^2\theta. \tag{2.1}$$

3) *If* $(x,y) \in S^R \setminus \Sigma^{nbh}$, *then* $|x \cdot y| \geq cR^2$ *for some* $c = c(\theta_0) > 0$.

For $0 \leq R_1 < R_2$ we will denote

$$(\Sigma^{nbh})_{R_1}^{R_2} = \pi(\Sigma_{R_1}^{R_2} \times (-\theta_0, \theta_0)).$$

Now we discuss the Riemann geometry of the domain $\Sigma^{nbh} = \Sigma^{nbh}(\theta_0)$, following [2].

The set $\Sigma$ is a cone with the vertex in the origin, and $\Sigma_* = \{tz : t > 0, z \in \Sigma^1\}$. The set $\Sigma^1$ is a closed manifold of dimension $2d-2$. Let us cover it by a finite system of charts $\mathcal{N}_1, \dots, \mathcal{N}_{\tilde{n}}$, $\mathcal{N}_j = \{\eta^j = (\eta_1^j, \dots, \eta_{2d-2}^j)\}$, and for any chart $\mathcal{N}_j$ denote by $m(d\eta^j)$ the volume element on $\Sigma^1$, induced from $\mathbb{R}^{2d}$. Below we write points in any chart $\mathcal{N}_j$ as $\eta$, and the volume element — as $m(d\eta)$.

The mapping

$$\Sigma^1 \times \mathbb{R}^+ \to \Sigma_*, \quad ((x_\eta, y_\eta), t) \to D_t(x_\eta, y_\eta)$$

is a diffeomorphism. Accordingly, we can cover $\Sigma_*$ by the $\tilde{n}$ charts $\mathcal{N}_j \times \mathbb{R}_+$ with the coordinates $(\eta^j, t) =: (\eta, t)$. The coordinates $(\eta, t, \theta)$, where $\eta \in \mathcal{N}_j, t > 0$ and $|\theta| < \theta_0$, $1 \leq j \leq \tilde{n}$, make coordinate systems on $\Sigma^{nbh} = \Sigma^{nbh}(\theta_0)$. In the coordinates $(\eta, t)$ the volume element on $\Sigma_*$ is

$$d_{\Sigma_*} = t^{2d-2} m\,(d\eta)\,dt. \tag{2.2}$$

In the coordinates $(\eta, t, \theta)$ the volume elements in $\mathbb{R}^{2d}$ reeds

$$dx\,dy = t^{2d-1}\mu(\eta,\theta) m\,(d\eta)\,dt\,d\theta, \quad \text{where } \mu(\eta, 0) = 1 \tag{2.3}$$

(see [2]), a dilation map $D_r$, $r > 0$, reeds $D_r(\eta, t, \theta) = (\eta, rt, \theta)$, and by (2.1)

$$\omega(\eta, t, \theta) = t^2\theta. \tag{2.4}$$

Finally, since at a point $z = \pi(\xi, \theta) \in \Sigma^{nbh}$ we have $\frac{\partial}{\partial\theta} = \nabla_z \cdot (y_\xi, x_\xi)$, then in view of (1.1) for any $(\eta, t, \theta)$ and any $k \leq 2$,

$$\left| \frac{\partial^k}{\partial \theta^k} F(\eta, t, \theta) \right| \leq C \langle t \rangle^{-N}, \quad N > 2d - 4. \tag{2.5}$$

**2.2. The volume element $d_{\Sigma_*}$ and the measure $|z|^{-1} d_{\Sigma_*}$.** Theorem 1.1 and the result of [2] (see (1.3)) show that the manifold $\Sigma_*$, equipped with the measure $|z|^{-1} d_{\Sigma_*}$, is crucial to study asymptotic of integrals $I_\nu$, $J_\nu$ and their similarities (cf. Section 6 of [2] and Section 5 below). The coordinates $(\eta, t)$ and the presentation (2.2) for the volume element are sufficient for the purposes of this work. But the quadric $\Sigma$ is reach in structures and admits more instrumental coordinate systems. In particular, if $d = 2$, we can introduce in the space $\mathbb{R}_x^2$ in (1.2) the polar coordinates $(r, \varphi)$. Then for any fixed non-zero vector $x = (r, \varphi) \in \mathbb{R}_x^2$ the set $\{y \in \mathbb{R}_y^2 : (x, y) \in \Sigma_*\}$ is the line in $\mathbb{R}_y^2$, perpendicular to $x$, and having the angle $\varphi + \pi/2$ with the horizontal axis. Parametrizing it by the length-coordinate $l$ we get on $\Sigma_*$ the coordinates $(r, l, \varphi) \in \mathbb{R}^+ \times \mathbb{R} \times S^1$, $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, with the coordinate mapping

$$\Phi : (r, l, \varphi) \mapsto \big(x = (r\cos\varphi, r\sin\varphi), y = (-l\sin\varphi, l\cos\varphi)\big)$$

(this map is singular at $r = 0$). Since

$$|\partial\Phi/\partial r|^2 = 1, \qquad |\partial\Phi/\partial l|^2 = 1, \qquad |\partial\Phi/\partial\varphi|^2 = r^2 + l^2,$$
$$\langle\partial\Phi/\partial r, \partial\Phi/\partial l\rangle = \langle\partial\Phi/\partial r, \partial\Phi/\partial\varphi\rangle = \langle\partial\Phi/\partial l, \partial\Phi/\partial\varphi\rangle = 0,$$

then in these coordinates the volume element on $\Sigma_*$ reeds as $\sqrt{r^2 + l^2}\, dr\, dl\, d\varphi$, and the measure $|z|^{-1} d_{\Sigma_*}$ — as $dr\, dl\, d\varphi$. Consider the fibre

$$\Pi : \mathbb{R}^2_x \times \mathbb{R}^2_y \supset \Sigma_* \to \mathbb{R}^2_x, \quad (x, y) \mapsto x.$$

It has a singular fibre $\Pi^{-1}0 = \{0\} \times \mathbb{R}^2_y$, and for any non-zero $x$ the fibre $\Pi^{-1}x$ equals $\{x\} \times x^\perp$, where $x^\perp$ is the line in $\mathbb{R}^2_y$, perpendicular to $x$. Since $dx = r\, dr\, d\varphi$, then the given above presentation for the measure $|z|^{-1} d_{\Sigma_*}$ implies that its restriction to the regular part $\Sigma^+_*$ of the fibred manifold $\Sigma_*$, $\Sigma^+_* = \Sigma_* \setminus (\{0\} \times \mathbb{R}^2_y)$, disintegrates by the foliation $\Pi$ as

$$(|z|^{-1} d_{\Sigma_*})\,|_{\Sigma^+_*} = |x|^{-1}\, dx\, d_{x^\perp} y, \quad x \neq 0,\ y \in x^\perp, \tag{2.6}$$

where $d_{x^\perp}$ is the length on the euclidean line $x^\perp \subset \mathbb{R}^2_y$.

We do not undertake the job of getting a right analogy of this result for the multidimensional case $d > 2$, but note that a straightforward modification of the construction above leads to the observation that for any $d \geq 2$ the measure $|z|^{-1} d_{\Sigma_*}$, restricted to $\Sigma^+_*$, disintegrates as

$$p_d(x, y)\, dx\, d_{x^\perp} y, \quad x \in \mathbb{R}^d \setminus \{0\},\ y \in x^\perp, \tag{2.7}$$

where $x^\perp$ is the orthogonal complement to $x$ in $R^d_y$, , $d_{x^\perp}$ is the volume element on this Euclidean space, and the function $p_d$ satisfies the estimate $p_d \leq C(|x| + |y|)^{d-2}|x|^{1-d}$.

## 3. Integral over the vicinity of $\Sigma$

To study the behaviour of the integral over a neighbourhood of $\Sigma$ we first prove that the integral, evaluated over the vicinity of the singular point $(0, 0)$ is small, and next study the integral over the vicinity of the regular part $\Sigma_*$ of the quadric.

For $0 < \delta \leq 1$ denote

$$K_\delta = \{z = (x, y) : |x| \leq \delta,\ |y| \leq \delta\} \subset \mathbb{R}^d \times \mathbb{R}^d.$$

An upper bound for the integral over $K_\delta$ follows from Lemma 2.1 of [2]:

$$|\,\langle |J_\nu|, K_\delta\rangle\,| \leq \int_{K_\delta} \frac{|F(z)|\, dz}{(x \cdot y)^2 + \nu^2} \leq C\nu^{-1}\delta^{2d-2}. \tag{3.1}$$

Now we estimate the integral over the neighbourhood $\Sigma^{nbh}$ of $\Sigma_*$. For this end, using (2.3), for $0 \le A < B \le \infty$ we disintegrate $\left\langle J_\nu, (\Sigma^{nbh})_A^B \right\rangle$ as

$$\begin{aligned}\left\langle J_\nu, (\Sigma^{nbh})_A^B \right\rangle &= \int_{\Sigma^1} m(d\eta) \int_A^B dt\, t^{2d-1} \int_{-\theta_0}^{\theta_0} d\theta\, \frac{F(\eta,t,\theta)\mu(\eta,\theta)\cos(\lambda x\cdot y)}{t^4\theta^2+\nu^2} \\ &= \int_{\Sigma^1} m(d\eta) \int_A^B dt\, t^{2d-1} \Upsilon_\nu(\eta,t), \end{aligned} \tag{3.2}$$

where

$$\Upsilon_\nu(\eta,t) = t^{-4} \int_{-\theta_0}^{\theta_0} \frac{F(\eta,t,\theta)\mu(\eta,\theta)\cos(\lambda t^2\theta)\, d\theta}{\theta^2+\varepsilon^2}, \quad \varepsilon = \nu t^{-2}.$$

To study $\Upsilon_\nu$ we first consider the integral $\Upsilon_\nu^0$, obtained from $\Upsilon_\nu$ by freezing $F\mu$ at $\theta = 0$. Since $\mu(\eta,0) = 1$, then

$$\Upsilon_\nu^0 = 2t^{-4} F(\eta,t,0) \int_0^{\theta_0} \frac{\cos(\lambda t^2\theta)\, d\theta}{\theta^2+\varepsilon^2} = 2\nu^{-1}t^{-2}F(\eta,t,0) \int_0^{\theta_0/\varepsilon} \frac{\cos(\nu\lambda w)\, dw}{w^2+1}.$$

Consider the integral

$$2\int_0^{\theta_0/\varepsilon} \frac{\cos(\nu\lambda w)\, dw}{w^2+1} = 2\int_0^{\infty} \frac{\cos(\nu\lambda w)\, dw}{w^2+1} - 2\int_{\theta_0/\varepsilon}^{\infty} \frac{\cos(\nu\lambda w)\, dw}{w^2+1} =: I_1 - I_2.$$

Since

$$2\int_0^\infty \frac{\cos(\xi w)\, dw}{w^2+1} = \int_{-\infty}^\infty \frac{e^{i\xi w}\, dw}{w^2+1} = \pi e^{-|\xi|},$$

then $I_1 = \pi e^{-\nu\lambda}$. For $I_2$ we have an obvious bound $|I_2| \le 2\varepsilon/\theta_0 = C_1\nu t^{-2}$. So

$$\Upsilon_\nu^0(\eta,t) = \pi\nu^{-1}t^{-2}F(\eta,t,0)(e^{-\nu\lambda} + \Delta_t), \qquad |\Delta_t| \le C\nu t^{-2}. \tag{3.3}$$

Now we estimate the difference between $\Upsilon_\nu$ and $\Upsilon_\nu^0$. Writing $(F\mu)(\eta,t,\theta) - (F\mu)(\eta,t,0)$ as $A(\eta,t)\theta + B(\eta,t,\theta)\theta^2$, where $|A|,|B| \le C\langle t\rangle^{-N}$ in view of (2.5), we have

$$\Upsilon_\nu - \Upsilon_\nu^0 = t^{-4}\int_{-\theta_0}^{\theta_0} \frac{(A\theta + B\theta^2)\cos(\lambda t^2\theta)\, d\theta}{\theta^2+\varepsilon^2}.$$

Since the first integrand is odd in $\theta$, then its integral vanishes, and

$$|\Upsilon_\nu - \Upsilon_\nu^0| \le C\langle t\rangle^{-N} t^{-4} \int_{-\theta_0}^{\theta_0} \frac{\theta^2\, d\theta}{\theta^2+\varepsilon^2} \le 2C\langle t\rangle^{-N} t^{-4}\theta_0.$$

So by (3.3)

$$\begin{aligned}&\left|\Upsilon_\nu(\eta,t) - \pi\nu^{-1}t^{-2}F(\eta,t,0)e^{-\nu\lambda}\right| \\ &\qquad \le C\langle t\rangle^{-N}\left(t^{-4} + \nu^{-1}t^{-2}\,\nu t^{-2}\right) \le C'\langle t\rangle^{-N} t^{-4}. \end{aligned} \tag{3.4}$$

## 4. End of the proof of Theorem 1.1

1) In view of (3.2), (3.4) and since $N > 2d - 2$, for $\delta \in (0,1]$ we have

$$\Big|\Big\langle J_\nu, \Big(\Sigma^{nbh}\Big)_\delta^\infty\Big\rangle - \pi\nu^{-1}e^{-\nu\lambda}\int_{\Sigma^1} m\, d\eta \int_\delta^\infty dt\, t^{2d-3}F(\eta,t,0)\Big| \\ \le C\int_\delta^\infty t^{2d-5}\langle t\rangle^{-N}\, dt \le C_1\chi_d(\delta).$$

2) Since $d \ge 2$ and $N > 2d-2$, then by estimate (2.5) the integral

$$\int_{\Sigma^1} m\, d\eta \int_0^\infty dt\, t^{2d-3}F(\eta,t,0)$$

converges absolutely, and by (2.2) it equals

$$\int_{\Sigma^1} m\, d\eta \int_0^\infty dt\, t^{2d-3}F(\eta,t,0) = \int_{\Sigma_*} |z|^{-1}F(z)\, d_{\Sigma_*}z.$$

3) Applying 1) and 2) to $F$ replaced by $F_0 = C'\langle z\rangle^{-N}$ and using that $|F| \le |F_0|$ by (1.1) we find that the integral $\langle J_\nu, (\Sigma^{nbh})_\delta^\infty\rangle$ also converges absolutely.

4) As $|\pi(\xi,\theta)| \le \sqrt{2}\,|\xi|$, then $(\Sigma^{nbh})_0^\delta \subset S_0^{\sqrt{2}\delta} \subset K_{\sqrt{2}\delta}$. Therefore by (3.1)

$$\Big|\Big\langle J_\nu, \Big(\Sigma^{nbh}\Big)_0^\delta\Big\rangle - \pi\nu^{-1}e^{-\nu\lambda}\int_{\Sigma^1} m\, d\eta \int_0^\delta dt\, t^{2d-3}F(\eta,t,0)\Big| \\ \le \Big\langle |J_\nu|, K_{\sqrt{2}\delta}\Big\rangle + \pi\nu^{-1}e^{-\nu\lambda}\int_{\Sigma^1} m\, d\eta \int_0^\delta dt\, t^{2d-3}|F(\eta,t,0)| \\ \le C_1\nu^{-1}\delta^{2d-2} + C_2\nu^{-1}\delta^2,$$

for any $0 < \delta \le 1$. Choosing $\delta = \sqrt{\nu}$, from here and 1)–3) we find that

$$\Big|\Big\langle J_\nu, \Sigma^{nbh}\Big\rangle - \pi\nu^{-1}e^{-\nu\lambda}\int_{\Sigma^1} m\, d\eta \int_0^\infty dt\, t^{2d-3}F(\eta,t,0)\Big| \le C\chi_d(\nu),$$

and that the integral $\langle J_\nu, \Sigma^{nbh}\rangle$ converges absolutely.

5) Finally, let us estimate the integral over $\mathbb{R}^{2d} \setminus \Sigma^{nbh}$:

$$\Big\langle |J_\nu|, \mathbb{R}^{2d}\setminus\Sigma^{nbh}\Big\rangle \le \int_{\{|z|\le\sqrt{\nu}\}} \frac{|F|\, dz}{\omega^2+\nu^2} + C_d\int_{\sqrt{\nu}}^\infty dr\, r^{2d-1}\int_{S^r\setminus\Sigma^{nbh}} \frac{|F(z)|\, d_{S^r}}{\omega^2+\nu^2}.$$

By item 3) of Lemma 2.1, $|\omega| \ge Cr^2$ in $S^r \setminus \Sigma^{nbh}$. Jointly with (3.1) this implies that

$$\Big\langle |J_\nu|, \mathbb{R}^{2d}\setminus\Sigma^{nbh}\Big\rangle| \le C + C\int_{\sqrt{\nu}}^\infty r^{2d-1}r^{-4}\langle r\rangle^{-N}\, dr \le C_1\chi_d(\nu).$$

So the integral $J_\nu$ converges absolutely and, in view of 2) and 4),

$$\left| J_\nu - \pi\nu^{-1}e^{-\nu\lambda}\int_{\Sigma^1} m\, d\eta \int_0^\infty dt\, t^{2d-3}F(\eta,t,0)\right| \\ = \left| J_\nu - \pi\nu^{-1}e^{-\nu\lambda}\int_{\Sigma_*} |z|^{-1}F(z)\, d_{\Sigma_*}z \right| \le C\chi_d(\nu). \quad (4.1)$$

This proves Theorem 1.1.

## 5. Comments

i) The only part of the proof, where we use that $N > 2d-2$ is Step 2) in Section 4: there this relation is evoked to establish the absolute convergence of the integral $J_0$; everywhere else it suffices to assume that $N > 2d-4$. Accordingly, if $F$ satisfies (1.1) with $N > 2d-4$ and $\langle |F|, \Sigma_1^\infty\rangle < \infty$, then (1.5) holds, since $\langle |F|, \Sigma_0^1\rangle < \infty$, see Step 4) Section 4.

ii) Our approach does not apply to study integrals (1.4), where the divisor $(x\cdot y)^2+\nu^2$ is replaced by $(x\cdot y)^2 + (\nu\Gamma(x,y))^2$ and $\Gamma \ne \mathrm{Const}$. But it applies to integrals

$$J_\nu^s = \int_{\mathbb{R}^{2d}} dz\, \frac{F(z)\sin(\lambda x\cdot y)}{(x\cdot y)^2+\nu^2},$$

under certain restrictions on $\lambda$. E.g., if $1 \le \lambda \le \nu^{-1}$ and $d \ge 3$, then $J_\nu^s = O(1)$ as $\nu \to 0$, and the leading term again is given by an integral over $\Sigma_*$. The case $d = 2$ is a bit more complicated.

iii) The approach allows to study integrals (1.4), where the quadratic form $z \mapsto x\cdot y$ is replaced by any non-degenerate indefinite quadratic form of $z \in \mathbb{R}^M$, $M \ge 4$.

iv) The restriction $M \ge 4$ in iii) (and $d \ge 2$ in the main text, where $\dim z = 2d$) was imposed since near the origin the disparity (4.1) is controlled by the integral $\int_0 t^{M-5}\, dt$, which strongly diverges if $M < 4$. The difficulty disappears if $F$ vanishes near zero. This may be illustrated by the following easy example:

*Example* 5.1. Consider

$$J'_\nu = \int_{\mathbb{R}^2} \frac{F(x,y)\cos(\lambda xy)}{x^2y^2+\nu^2}\, dxdy,$$

where $F \in C_0^2(\mathbb{R}^2)$ vanishes near the origin. Now $2d = 2$, the quadric $\Sigma' = \{xy = 0\}$ is one dimensional, has a singularity at the origin and its smooth part $\Sigma'^* = \Sigma' \setminus 0$ has four connected components. Consider one of them: $\mathcal{C}_1 = \{(x,y) : y = 0, x > 0\}$. Now the coordinate $\xi$ is a point in $\mathbb{R}_+$ with $(x_\xi, y_\xi) = (\xi, 0)$ and with the normal $N(\xi) = (0,\xi)$, the set $\Sigma_1 \cap \mathcal{C}_1$ is the single point $(1,0)$ and the coordinate $(\eta, t, \theta)$ in the vicinity of $\mathcal{C}_1$ degenerates to $(t,\theta)$, $t > 0$, $|\theta| < \theta_0$, with the coordinate-map $(t,\theta) \mapsto (t, t\theta)$. The relations (2.2) and (2.3) are now obvious, and the integral (3.1) vanishes if $\delta > 0$ is sufficiently small. Interpreting $z = (x, y)$ as a complex number, we write the assertion of Theorem 1.1 as

$$\left| J'_\nu - \pi\nu^{-1} e^{-\nu\lambda} \int_{\Sigma'} \frac{F(z)}{|z|}\, dz \right| \le C,$$

where the integral is a contour integral in the complex plane.

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## Асимптотичні властивості інтегралів від часток, коли чисельник осцілює, а знаменник вироджується

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Ми вивчаємо асимптотичне поводження при $\nu \to 0$ інтегралів в $\mathbb{R}^{2d} = \{(x, y)\}$ від виразів вигляду $F(x, y)\cos(\lambda x \cdot y)/\big((x \cdot y)^2 + \nu^2\big)$, де $\lambda \ge 0$ і $F$ досить швидко спадає на нескінченності. Подібні інтеграли виникають в теорії хвильової турбулентності.

*Ключові слова:* асимптотичні інтеграли, інтеграли, що осцілюють, чотирихвильові взаємодії.