Journal of Mathematical Physics, Analysis, Geometry
2019, Vol. 15, No. 4, pp. 435–447
doi: https://doi.org/10.15407/mag15.04.435

# A Study Concerning Berger Type Deformed Sasaki Metric on the Tangent Bundle

Murat Altunbas, Ramazan Simsek, and Aydın Gezer

Let $TM$ be the tangent bundle over an almost anti-paraHermitian manifold endowed with Berger type deformed Sasaki metric $g_{BS}$. In this paper, first, we obtain the Levi-Civita connection of this metric and study geodesics on $TM$. Secondly, we construct some almost anti-paraHermitian structures on $TM$ and search conditions for these structures to be anti-paraKähler and quasi-anti-paraKähler. Finally, we present certain Riemannian curvature properties of $(TM, g_{BS})$.

*Key words:* Berger type deformed Sasaki metric, paracomplex structure, geodesics, tangent bundle.

*Mathematical Subject Classification 2010:* 53C07, 53C55, 53C22.

## 1. Introduction

Let $M$ be an $n$-dimensional Riemannian manifold with a Riemannian metric $g$ and $TM$ be its tangent bundle. By means of natural lifts of the Riemannian metric $g$, from the Riemannian manifold $(M, g)$ to its tangent bundle $TM$, one can induce new (pseudo) Riemannian metrics with interesting geometric properties. The well-known example of such metrics is the Sasaki metric. This metric was constructed on the tangent bundle $TM$ of the Riemannian manifold $(M, g)$ by S. Sasaki in [18]. However, in most cases the study of some geometric properties of the tangent bundle endowed with this metric led to the flatness of the base manifold. O. Kowalski [11] has shown that it is never locally symmetric unless the base metric is locally flat. E. Musso and F. Tricerri [15] have generalized this fact: they have shown that it has constant scalar curvature if and only if the base metric is flat. The different deformations of the Sasaki metric were defined and studied by some authors (see [7–10, 14, 19]). In [19], A. Yampolsky introduced a fiber-wise deformation of the Sasaki metric on slashed and unit tangent bundles over a Kähler manifold based on the Berger deformation of metric on a unit sphere and studied geodesics of this metric.

Let $M$ be a $2k$-dimensional Riemannian manifold with a Riemannian metric $g$. An almost paracomplex manifold is an almost product manifold $(M_{2k}, \varphi)$, $\varphi^2 = id$, such that the two eigenbundles $T^+M$ and $T^-M$ associated to the two eigenvalues $+1$ and $-1$ of $\varphi$, respectively, have the same rank. The integrability of an almost paracomplex structure is equivalent to the vanishing of the Nijenhuis tensor: $N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + [X,Y]$. A paracomplex structure is an integrable almost paracomplex structure. Let $(M_{2k}, \varphi)$ be an almost paracomplex manifold. A Riemannian metric $g$ is said to be an anti-paraHermitian metric if

$$g(\varphi X, \varphi Y) = g(X,Y)$$

or, equivalently,

$$g(\varphi X, Y) = g(X, \varphi Y) \quad \text{(purity condition)}$$

for any vector fields $X, Y$ on $M_{2k}$. If $(M_{2k}, \varphi)$ is an almost paracomplex manifold with an anti-paraHermitian metric $g$, then the triple $(M_{2k}, \varphi, g)$ is said to be an almost anti-paraHermitian manifold. Moreover, $(M_{2k}, \varphi, g)$ is said to be anti-paraKähler if $\varphi$ is parallel with respect to the Levi-Civita connection $\nabla^g$ of $g$. As is well known, the anti-paraKähler condition ($\nabla^g \varphi = 0$) is equivalent to paraholomorphicity of the anti-paraHermitian metric $g$, that is, $\Phi_\varphi g = 0$, where $\Phi_\varphi$ is the Tachibana operator [17].

In this paper, we construct a new metric, will be called a Berger type deformed Sasaki metric, on the tangent bundle over an anti-paraKähler manifold. The paper can be considered as a contribution to studying a special new metrics on the tangent bundle constructed from the base metric and the almost paracomplex structure on an almost anti-paraHermitian manifold. The considered metric is far from being a subclass of the so-called $g-$natural metrics which were fully characterized by M.T.K. Abbassi and M. Sarih [1–3]. The paper aims to study the tangent bundle $TM$ with Berger type deformed Sasaki metric over an almost anti-paraHermitian manifold.

Through the paper, manifolds, tensor fields and connections are always assumed to be differentiable of class $C^\infty$ and the so-called "Einstein's summation" will be used everywhere.

## 2. Lifts to tangent bundles

Let $M$ be an $n$-dimensional Riemannian manifold with a Riemannian metric $g$ and $TM$ be its tangent bundle denoted by $\pi : TM \to M$. A system of local coordinates $(U, x^i)$ in $M$ induces on $TM$ a system of local coordinates $\left(\pi^{-1}(U), x^i, x^{\bar{i}} = u^i\right)$, $\bar{i} = n + i = n+1, \ldots, 2n$, where $(u^i)$ is the Cartesian coordinates in each tangent space $T_P M$ at $P \in M$ with respect to the natural base $\left\{ \frac{\partial}{\partial x^i}\big|_P \right\}$, $P$ being an arbitrary point in $U$ whose coordinates are $(x^i)$.

Given a vector field $X = X^i \frac{\partial}{\partial x^i}$ on $M$, the vertical lift ${}^V X$ and the horizontal lift ${}^H X$ of $X$ are given, with respect to the induced coordinates, by

$${}^V X = X^i \partial_{\bar{i}}, \quad {}^H X = X^i \partial_i - u^s \Gamma^i_{sk} X^k \partial_{\bar{i}},$$

where $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_{\bar{i}} = \frac{\partial}{\partial u^i}$ and $\Gamma^i_{sk}$ are the coefficients of the Levi-Civita connection $\nabla$ of $g$ [20].

In particular, we have the vertical spray ${}^{V}u$ and the horizontal spray ${}^{H}u$ on $TM$ defined by

$$ {}^{V}u = u^{i}\,{}^{V}(\partial_i) = u^{i}\partial_{\bar{i}}, \quad {}^{H}u = u^{i}\,{}^{H}(\partial_i) = u^{i}\partial_{i}. $$

${}^{V}u$ is also called the canonical or Liouville vector field on $TM$.

Now, let $r$ be the norm of a vector $u \in TM$. Then, for any smooth function $f$ of $\mathbb{R}$ to $\mathbb{R}$, we have

$$ {}^{H}X\left(f(r^2)\right) = 0, \quad {}^{V}X\left(f(r^2)\right) = 2f'(r^2)g(X,u), $$

and we get

$$ {}^{H}X(r^2) = 0, \quad {}^{V}X(r^2) = 2g(X,u). $$

Let $X$, $Y$ and $Z$ be any vector fields on $M$. Then we have [3],

$$ \begin{aligned} {}^{H}X(g(Y,u)) &= g\left((\nabla_X Y),u\right), \\ {}^{V}X(g(Y,u)) &= g\left(X,Y\right), \\ {}^{H}X\,{}^{V}(g(Y,Z)) &= X\left(g(Y,Z)\right), \\ {}^{V}X\,{}^{V}(g(Y,Z)) &= 0. \end{aligned} $$

The bracket operation of vertical and horizontal vector fields is given by the formulas [5, 20]:

$$ \begin{aligned} \left[{}^{H}X, {}^{H}Y\right] &= {}^{H}[X,Y] - {}^{V}(R(X,Y)u), \\ \left[{}^{H}X, {}^{V}Y\right] &= {}^{V}(\nabla_X Y), \\ \left[{}^{V}X, {}^{V}Y\right] &= 0 \end{aligned} \tag{2.1} $$

for all vector fields $X$ and $Y$ on $M$, where $R$ is the Riemannian curvature tensor of $g$ defined by

$$ R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}. $$

## 3. The Levi-Civita connection of the Berger type deformed Sasaki metric on the tangent bundle

In the following, let $(M_{2k}, \varphi, g)$ be an almost anti-paraHermitian manifold and $TM$ be its tangent bundle. A fiber-wise Berger type deformation of the Sasaki metric on $TM$ is defined by

$$ \begin{aligned} g_{BS}\left({}^{H}X, {}^{H}Y\right) &= g(X,Y), \\ g_{BS}\left({}^{V}X, {}^{H}Y\right) &= {}^{S}g\left({}^{H}X, {}^{V}Y\right) = 0, \\ g_{BS}\left({}^{V}X, {}^{V}Y\right) &= g(X,Y) + \delta^2 g(X,\varphi u)g(Y,\varphi u) \end{aligned} \tag{3.1} $$

for all vector fields $X, Y$ on $M_{2k}$, where $\delta$ is some constant [19]. The metric is said to be a Berger type deformed Sasaki metric. A direct consequence of usual calculations using the Koszul formula gives the following result.

**Proposition 3.1.** *Let $(M_{2k}, \varphi, g)$ be an anti-paraKähler manifold and $TM$ be its tangent bundle. The Levi-Civita connection of the Berger type deformed Sasaki metric $g_{BS}$ on $TM$ satisfies the following properties:*

(i) $\widetilde{\nabla}_{^HX}{}^HY = {}^H(\nabla_X Y) - \frac{1}{2}{}^V(R(X,Y)u)$,

(ii) $\widetilde{\nabla}_{^HX}{}^VY = \frac{1}{2}{}^H(R(u,Y)X) + {}^V(\nabla_X Y)$,

(iii) $\widetilde{\nabla}_{^VX}{}^HY = \frac{1}{2}{}^H(R(u,X)Y)$,

(iv) $\widetilde{\nabla}_{^VX}{}^VY = \frac{\delta^2}{1+\delta^2\alpha} g(X,\varphi Y)^V(\varphi u)$,

*where $\nabla$ is the Levi-Civita connection, $R$ is its Riemannian curvature tensor and $\alpha = g(u,u)$ (for complex version, see [19]).*

If we denote the horizontal and vertical projections by $\mathcal{H}$ and $\mathcal{V}$, respectively, then we can state the followings:

(i) The vertical distribution $VTM$ is totally geodesic in $TTM$ if $\mathcal{H}\widetilde{\nabla}_{^VX}{}^VY = 0$;

(ii) The horizontal distribution $HTM$ is totally geodesic in $TTM$ if $\mathcal{V}\widetilde{\nabla}_{^HX}{}^HY = 0$ for all vector fields $X, Y$ on $M$.

Hence, we can state the following result.

**Proposition 3.2.** *Let $(M_{2k}, \varphi, g)$ be an anti-paraKähler manifold and $TM$ be its tangent bundle equipped with the Berger type deformed Sasaki metric $g_{BS}$. Then*

(i) *The vertical distribution $VTM$ is totally geodesic in $TTM$;*

(ii) *The horizontal distribution $HTM$ is totally geodesic in $TTM$ if and only if $(M_{2k}, \varphi, g)$ is flat.*

*Proof.* The results come immediately from (i) and (iv) of Proposition 3.1. □

Next, we shall state some relations between the geodesics of $(TM, g_{BS})$ and those of the anti-paraKähler manifold $(M_{2k}, \varphi, g)$. The following lemma will be useful later.

**Lemma 3.3** ([21])**.** *Let $(M, g)$ be a Riemannian manifold and $x : I \to M$ be a curve on $M$. If $C : t \in I \to C(\tau) = (x(t), u(t)) \in TM$ is a curve in $TM$ such that $u(t) \in T_{x(t)}M$ (i.e., $u(t)$ is a vector field along $x(t)$), then*

$$\dot{C}(t) = {}^H\dot{x} + {}^V(\nabla_{\dot{x}} u).$$

Let $(M_{2k}, \varphi, g)$ be an anti-paraKähler manifold, $TM$ be its tangent bundle equipped with the metric $g_{BS}$, and $C(t) = (x(t), u(t))$ be a curve on $TM$ such that $u(t)$ is a vector field along $x(t)$. Direct computations with using Proposition 3.1 and Lemma 3.3 give

$$\nabla_{\dot{C}}\dot{C} = {}^{H}\{\nabla_{\dot{x}}\dot{x} + R(u, \nabla_{\dot{x}}u)\dot{x}\} + {}^{V}\left\{\nabla_{\dot{x}}\nabla_{\dot{x}}u + \frac{\delta^2}{1+\delta^2\alpha} g(\varphi(\nabla_{\dot{x}}u), \nabla_{\dot{x}}u)\varphi u\right\}.$$

From the relation above we get the following theorem.

**Theorem 3.4.** *Let $(M_{2k}, \varphi, g)$ be an anti-paraKähler manifold, $TM$ be its tangent bundle equipped with the Berger type deformed Sasaki metric $g_{BS}$, and $C(t) = (x(t), u(t))$ be a curve on $TM$ such that $u(t)$ is a vector field along $x(t)$. Then $C$ is a geodesic on $TM$ if and only if*

$$\nabla_{\dot{x}}\dot{x} = -R(u, \nabla_{\dot{x}}u)\dot{x},$$
$$\nabla_{\dot{x}}\nabla_{\dot{x}}u = -\frac{\delta^2}{1+\delta^2\alpha} g(\varphi(\nabla_{\dot{x}}u), \nabla_{\dot{x}}u)\varphi u.$$

A curve $C(t) = (x(t), u(t))$ on $TM$ is said to be a horizontal lift of the curve $x(t)$ on $M$ if and only if $\nabla_{\dot{x}}u = 0$. Thus, we have

**Corollary 3.5.** *Let $(M_{2k}, \varphi, g)$ be an anti-paraKähler manifold, $TM$ be its tangent bundle equipped with the Berger type deformed Sasaki metric $g_{BS}$, and $C(t) = (x(t), u(t))$ be the horizontal lift of the curve $x(t)$. Then $C(t)$ is a geodesic on $(TM, g_{BS})$ if and only if $x(t)$ is a geodesic on $(M_{2k}, \varphi, g)$.*

If $x(t)$ is a curve on $M$, then the curve $C(t) = (x(t), \dot{x}(t))$ is called a natural lift of the curve $x(t)$. The following last result ends this section.

**Corollary 3.6.** *Let $(M_{2k}, \varphi, g)$ be an anti-paraKähler manifold and $TM$ be its tangent bundle equipped with the Berger type deformed Sasaki metric $g_{BS}$. The natural lift $C(t) = (x(t), \dot{x}(t))$ of any geodesic $x(t)$ is a geodesic on $(TM, g_{BS})$.*

## 4. Some almost paracomplex structures with anti-paraHermitian metrics on the tangent bundle

An almost paracomplex structure on $TM$ satisfying the purity condition: $g_{BS}(\widetilde{\varphi}\widetilde{X}, \widetilde{Y}) = g_{BS}(\widetilde{X}, \widetilde{\varphi}\widetilde{Y})$ for all vector fields $\widetilde{X}, \widetilde{Y}$ on $TM$ is defined by

$$\widetilde{\varphi}\left({}^{H}X\right) = {}^{V}X - \frac{1}{\alpha}\left(1 + \frac{1}{\sqrt{1+\alpha\delta^2}}\right) g(X, \varphi u)\, {}^{V}(\varphi u),$$
$$\widetilde{\varphi}\left({}^{V}X\right) = {}^{H}X - \frac{1}{\alpha}\left(1 + \sqrt{1+\alpha\delta^2}\right) g(X, \varphi u)\, {}^{H}(\varphi u). \tag{4.1}$$

Thus, we have the following result.

**Theorem 4.1.** *Let $(M_{2k}, \varphi, g)$ be an almost anti-paraHermitian manifold and $TM$ be its tangent bundle equipped with the Berger type deformed Sasaki metric $g_{BS}$ and the paracomplex structure $\widetilde{\varphi}$ defined by (4.1). The triple $(TM, \widetilde{\varphi}, g_{BS})$ is an almost anti-paraHermitian manifold.*

We know that the integrability of $\widetilde{\varphi}$ is equivalent to the vanishing of the Nijenhuis tensor $\widetilde{N}_{\widetilde{\varphi}}$. The Nijenhuis tensor of $\widetilde{\varphi}$ is given by

$$\widetilde{N}_{\widetilde{\varphi}}(\widetilde{X},\widetilde{Y})=[\widetilde{X},\widetilde{Y}]-\widetilde{\varphi}[\widetilde{\varphi}\widetilde{X},\widetilde{Y}]-\widetilde{\varphi}[\widetilde{X},\widetilde{\varphi}\widetilde{Y}]+[\widetilde{\varphi}\widetilde{X},\widetilde{\varphi}\widetilde{Y}],$$

where $\widetilde{X},\widetilde{Y}$ are vector fields on $TM$. It follows that

$$\begin{aligned}
\widetilde{N}_{\widetilde{\varphi}}\left({}^{V}X,{}^{V}Y\right) &= \left[{}^{V}X,{}^{V}Y\right]-\widetilde{\varphi}\left[\widetilde{\varphi}{}^{V}X,{}^{V}Y\right]-\widetilde{\varphi}\left[{}^{V}X,\widetilde{\varphi}{}^{V}Y\right]+\left[\widetilde{\varphi}{}^{V}X,\widetilde{\varphi}{}^{V}Y\right]\\
&= \left[\widetilde{\varphi}{}^{H}Z,\widetilde{\varphi}{}^{H}W\right]-\widetilde{\varphi}\left[{}^{H}Z,\widetilde{\varphi}{}^{H}W\right]-\left[\widetilde{\varphi}{}^{H}Z,{}^{H}W\right]+\left[{}^{H}Z,{}^{H}W\right]\\
&= \widetilde{N}_{\widetilde{\varphi}}\left({}^{H}Z,{}^{H}W\right),\\
\widetilde{N}_{\widetilde{\varphi}}\left({}^{V}X,{}^{H}W\right) &= \left[{}^{V}X,{}^{H}W\right]-\widetilde{\varphi}\left[\widetilde{\varphi}{}^{V}X,{}^{H}W\right]-\widetilde{\varphi}\left[{}^{V}X,\widetilde{\varphi}{}^{H}W\right]+\left[\widetilde{\varphi}{}^{V}X,\widetilde{\varphi}{}^{H}W\right]\\
&= \left[\widetilde{\varphi}{}^{H}Z,{}^{H}W\right]-\widetilde{\varphi}\left[{}^{H}Z,{}^{H}W\right]-\widetilde{\varphi}\left[\widetilde{\varphi}{}^{H}Z,\widetilde{\varphi}{}^{H}W\right]+\left[{}^{H}Z,{}^{H}W\right]\\
&= -\widetilde{\varphi}\left(\widetilde{N}_{\widetilde{\varphi}}\left({}^{H}Z,{}^{H}W\right)\right),
\end{aligned}$$

where ${}^{V}X=\widetilde{\varphi}{}^{H}Z$, ${}^{V}Y=\widetilde{\varphi}{}^{H}W$. So we can write the following lemma.

**Lemma 4.2.** *The almost paracomplex structure $\widetilde{\varphi}$ defined by* (4.1) *is integrable if and only if $\widetilde{N}_{\widetilde{\varphi}}\left({}^{H}X,{}^{H}Y\right)=0$ for all vector fields $X,Y$ (for almost complex version, see* [10]*).*

By direct computations, we have

$$\begin{aligned}
\widetilde{N}_{\widetilde{\varphi}}\left({}^{H}X,{}^{H}Y\right) = \beta&\left\{(g(X,\varphi u)\,{}^{V}(\varphi Y)-(g(Y,\varphi u)\,{}^{V}(\varphi X)\right\}\\
&+\beta'\left\{g(X,u)g(Y,\varphi u)-g(Y,u)g(X,\varphi u)\right\}{}^{V}(\varphi u)\\
&+\beta\left\{g(\nabla_X Y,\varphi u)-g(\nabla_Y X,\varphi u)\right\}{}^{V}(\varphi u)\\
&+(\beta\beta' g(u,\varphi u)+\beta^2)\{g(X,\varphi u)g(Y,u)\\
&-g(Y,\varphi u)g(X,u)\}\,{}^{V}(\varphi u)-{}^{V}(R(X,Y)u),
\end{aligned}$$

where $\beta=\frac{1}{\alpha}(1+\frac{1}{\sqrt{1+\alpha\delta^2}})$. Here we use the following formulae:

$$\left[{}^{V}\varphi u,{}^{V}Y\right]=-{}^{V}(\varphi Y),\quad {}^{V}Xg(Y,\varphi u)=g(\nabla_X Y,\varphi u),\quad {}^{V}(\varphi u)(\beta)=2\beta' g(\varphi u,u).$$

The almost paracomplex structure $\widetilde{\varphi}$ defined by (4.1) is integrable if and only if

$$\begin{aligned}
{}^{V}(R(X,Y)u) = \beta&\left\{(g(X,\varphi u)^{V}(\varphi Y)-(g(Y,\varphi u)\,{}^{V}(\varphi X)\right\}\\
&+\beta'\{g(X,u)g(Y,\varphi u)-g(Y,u)g(X,\varphi u)\}^{V}(\varphi u)\\
&+\beta\{g(\nabla_X Y,\varphi u)-g(\nabla_Y X,\varphi u)\}\,{}^{V}(\varphi u)\\
&+(\beta\beta' g(u,\varphi u)+\beta^2)\{g(X,\varphi u)g(Y,u) \qquad (4.2)\\
&-g(Y,\varphi u)g(X,u)\}\,{}^{V}(\varphi u). \qquad (4.3)
\end{aligned}$$

It is known that if the base manifold is anti-paraKähler, then the Riemannian curvature tensor of the base manifold satisfies the equality $R(\varphi X,Y)u=R(X,\varphi Y)u$. Then, according to (4.3), this identity is never satisfied. This shows that the almost paracomplex structure $\widetilde{\varphi}$ is never integrable and the structure $(\widetilde{\varphi},g_{BS})$ on the tangent bundle $TM$ is never anti-paraKähler. Hence, we get the result below.

**Corollary 4.3.** *Let $(M_{2k}, \varphi, g)$ be an anti-paraKähler manifold and $TM$ be its tangent bundle equipped with the Berger type deformed Sasaki metric $g_{BS}$ and the paracomplex structure $\widetilde{\varphi}$ defined by (4.1). The triple $(TM, \widetilde{\varphi}, g_{BS})$ can not be an anti-paraKähler manifold.*

Let us consider the simplified almost paracomplex structure $\widetilde{\varphi}_1$ on $TM$ defined by

$$\widetilde{\varphi}_1({}^HX) = {}^VX, \quad \widetilde{\varphi}_1({}^VX) = {}^HX \tag{4.4}$$

for all vector fields $X, Y$ on $M$ [4]. For purity condition, we put

$$A(\widetilde{X}, \widetilde{Y}) = g_{BS}(\widetilde{\varphi}_1\widetilde{X}, \widetilde{Y}) - g_{BS}(\widetilde{X}, \widetilde{\varphi}_1\widetilde{Y})$$

for any vector fields $\widetilde{X}, \widetilde{Y}$ on $TM$. For all vector fields $\widetilde{X}$ and $\widetilde{Y}$, which are of the form ${}^VX, {}^VY$ or ${}^HX, {}^HY$, we have

$$\begin{aligned} A\left({}^HX, {}^HY\right) &= g_{BS}\left(\widetilde{\varphi}_1\left({}^HX\right), {}^HY\right) - g_{BS}\left({}^HX, \widetilde{\varphi}_1\left({}^HY\right)\right) = 0, \\ A\left({}^HX, {}^VY\right) &= g_{BS}\left(\widetilde{\varphi}_1\left({}^HX\right), {}^VY\right) - g_{BS}\left({}^HX, \widetilde{\varphi}_1\left({}^VY\right)\right) \\ &= g(X,Y) + \delta^2 g(X, \varphi u) g(Y, \varphi u) - g(X,Y) \\ &= \delta^2 g(X, \varphi u) g(Y, \varphi u), \\ A\left({}^VX, {}^VY\right) &= g_{BS}\left(\widetilde{\varphi}_1\left({}^VX\right), {}^VY\right) - g_{BS}\left({}^VX, \widetilde{\varphi}_1\left({}^VY\right)\right) = 0. \end{aligned}$$

From above, if $A(\widetilde{X}, \widetilde{Y}) = 0$, then $\delta = 0$. Hence we have the following theorem.

**Theorem 4.4.** *If the Berger type deformed Sasaki metric $g_{BS}$ on the tangent bundle over the anti-paraKähler manifold $(M_{2k}, \varphi, g)$ is anti-paraHermitian with respect to the paracomplex structure $\widetilde{\varphi}_1$ defined by (4.4), then it reduces to the Sasaki metric.*

Now, consider another almost paracomplex structure $\widetilde{\varphi}_2$ on $TM$ defined by

$$\widetilde{\varphi}_2\left({}^HX\right) = {}^HX, \quad \widetilde{\varphi}_2\left({}^VX\right) = -{}^VX \tag{4.5}$$

for all vector fields $X, Y$ on $M$ [4]. From (3.1) and (4.5), it is easy to see that the Berger type deformed Sasaki metric $g_{BS}$ is pure with respect to the almost paracomplex structure $\widetilde{\varphi}_2$.

We now analyze the paraholomorphy property of the Berger type deformed Sasaki metric $g_{BS}$ with respect to the almost paracomplex structure $\widetilde{\varphi}_2$. We calculate

$$\begin{aligned} (\Phi_{\widetilde{\varphi}_2} g_{BS})(\widetilde{X}, \widetilde{Y}, \widetilde{Z}) &= (\widetilde{\varphi}_2\widetilde{X})\Big(g_{BS}(\widetilde{Y}, \widetilde{Z})\Big) - \widetilde{X}\Big(g_{BS}(\widetilde{\varphi}_2\widetilde{Y}, \widetilde{Z})\Big) \\ &\quad + g_{BS}\Big((L_{\widetilde{Y}}\widetilde{\varphi}_2)\widetilde{X}, \widetilde{Z}\Big) + g_{BS}\Big(\widetilde{Y}, (L_{\widetilde{Z}}\widetilde{\varphi}_2)\widetilde{X}\Big) \end{aligned}$$

for all vector fields $\widetilde{X}, \widetilde{Y}, \widetilde{Z}$ on $TM$. Then we yield

$$(\Phi_{\widetilde{\varphi}_2} g_{BS})\left({}^VX, {}^VY, {}^HZ\right) = 0,$$

$$
\begin{aligned}
&(\Phi_{\widetilde{\varphi}_2} g_{BS}) \left({}^VX, {}^VY, {}^VZ\right) = 0, \\
&(\Phi_{\widetilde{\varphi}_2} g_{BS}) \left({}^VX, {}^HY, {}^VZ\right) = 0, \\
&(\Phi_{\widetilde{\varphi}_2} g_{BS}) \left({}^VX, {}^HY, {}^HZ\right) = 0, \\
&(\Phi_{\widetilde{\varphi}_2} g_{BS}) \left({}^HX, {}^VY, {}^HZ\right) = 2g_{BS}({}^V(R(X,Z)u), {}^VY), \\
&(\Phi_{\widetilde{\varphi}_2} g_{BS}) \left({}^HX, {}^VY, {}^VZ\right) = 0, \\
&(\Phi_{\widetilde{\varphi}_2} g_{BS}) \left({}^HX, {}^HY, {}^HZ\right) = 0, \\
&(\Phi_{\widetilde{\varphi}_2} g_{BS}) \left({}^HX, {}^HY, {}^VZ\right) = 2g_{BS}\left({}^V(R(X,Y)u), {}^VZ\right)
\end{aligned} \tag{4.6}
$$

which give the following result.

**Theorem 4.5.** *Let $(M_{2k}, \varphi, g)$ be an anti-paraKähler manifold and $TM$ be its tangent bundle equipped with the Berger type deformed Sasaki metric $g_{BS}$ and the paracomplex structure $\widetilde{\varphi}_2$ defined by (4.5). The triple $(TM, \widetilde{\varphi}_2, g_{BS})$ is an anti-paraKähler manifold if and only if $M_{2k}$ is flat.*

Let $(M_{2k}, \varphi, g)$ be a non-integrable almost paracomplex manifold with an anti-paraHermitian metric. If $\underset{X,Y,Z}{\sigma} g((\nabla_X \varphi)Y, Z) = 0$, where $\sigma$ is the cyclic sum by three arguments, then the triple $(M_{2k}, \varphi, g)$ is a quasi-anti-paraKähler manifold (for complex version, see [13]). It is known that $\underset{X,Y,Z}{\sigma} g((\nabla_X \varphi)Y, Z) = 0$ is equivalent to $\underset{X,Y,Z}{\sigma} (\Phi_\varphi g)(X, Y, Z) = 0$ [16]. By means of (4.6), we can easily find

$$\underset{\widetilde{X},\widetilde{Y},\widetilde{Z}}{\sigma} (\Phi_{\widetilde{\varphi}_2} g_{BS})(\widetilde{X}, \widetilde{Y}, \widetilde{Z}) = 0$$

for all vector fields $\widetilde{X}, \widetilde{Y}, \widetilde{Z}$ on $TM$. Hence, we have the following theorem.

**Theorem 4.6.** *Let $(M_{2k}, \varphi, g)$ be an anti-paraKähler manifold and $TM$ be its tangent bundle equipped with the Berger type deformed Sasaki metric $g_{BS}$ and the paracomplex structure $\widetilde{\varphi}_2$ defined by (4.5). The triple $(TM, \widetilde{\varphi}_2, g_{BS})$ is a quasi-anti-paraKähler manifold.*

Let $\nabla$ be an arbitrary linear connection on $M$ and $S$ be the $(1,2)$-tensor field defined by

$$S(X,Y) = \frac{1}{2}\{(\nabla_{FY} F)X + F((\nabla_Y F)X) - F((\nabla_X F)Y)\},$$

where $F$ is an almost paracomplex (product) structure. By a straightforward computation, one can easily prove that

$$\overline{\nabla} = \nabla - S$$

is an almost paracomplex connection on $M$, i.e., $\overline{\nabla} F = 0$. Consider the almost paracomplex structure $\widetilde{\varphi}_2$ defined by (4.5) and the Levi-Civita connection $\widetilde{\nabla}$ given by Proposition 3.1. Then we can construct any almost paracomplex connection on $TM$ by

$$\overline{\nabla}_{\widetilde{X}} \widetilde{Y} = \widetilde{\nabla}_{\widetilde{X}} \widetilde{Y} - \widetilde{S}(\widetilde{X}, \widetilde{Y}), \tag{4.7}$$

where

$$\widetilde{S}(\widetilde{X},\widetilde{Y})=\frac{1}{2}\Big\{(\widetilde{\nabla}_{\widetilde{\varphi}_2\widetilde{Y}}\widetilde{\varphi}_2)\widetilde{X}+\widetilde{\varphi}_2\big((\nabla_{\widetilde{Y}}\widetilde{\varphi}_2)\widetilde{X}\big)-\widetilde{\varphi}_2\big((\nabla_{\widetilde{X}}\widetilde{\varphi}_2)\widetilde{Y}\big)\Big\}.$$

Next, we compute

$$\begin{aligned}
\widetilde{S}({}^HX,{}^HY) &= \frac{1}{2}\Big\{(\widetilde{\nabla}_{\widetilde{\varphi}_2{}^HY}\widetilde{\varphi}_2)^HX+\widetilde{\varphi}_2((\nabla_{{}^HY}\widetilde{\varphi}_2)^HX)-\widetilde{\varphi}_2((\nabla_{{}^HX}\widetilde{\varphi}_2)^HY)\Big\}\\
&= \frac{1}{2}\Big\{(\widetilde{\nabla}_{\widetilde{\varphi}_2{}^HY}\widetilde{\varphi}_2{}^HX)-\widetilde{\varphi}_2(\widetilde{\nabla}_{\widetilde{\varphi}_2{}^HY}{}^HX)+\widetilde{\varphi}_2((\nabla_{{}^HY}\widetilde{\varphi}_2{}^HX)\\
&\quad -\widetilde{\varphi}_2\widetilde{\nabla}_{{}^HY}{}^HX)-\widetilde{\varphi}_2(\nabla_{{}^HX}\widetilde{\varphi}_2{}^HY-\widetilde{\varphi}_2\widetilde{\nabla}_{{}^HX}{}^HY))\Big\}\\
&= \frac{1}{2}\Big\{(\widetilde{\nabla}_{{}^HY}{}^HX)-\widetilde{\varphi}_2(\widetilde{\nabla}_{{}^HY}{}^HX)+\widetilde{\varphi}_2((\nabla_{{}^HY}{}^HX)\\
&\quad -\widetilde{\varphi}_2(\nabla_{{}^HY}{}^HX))-\widetilde{\varphi}_2\Big((\nabla_{{}^HX}{}^HY)-\widetilde{\varphi}_2(\nabla_{{}^HX}{}^HY)\Big)\Big\}\\
&= \frac{1}{2}\Big\{{}^H(\nabla_YX)-\frac{1}{2}{}^V(R(Y,X)u)-{}^H(\nabla_YX)-\frac{1}{2}{}^V(R(Y,X)u)\\
&\quad +{}^H(\nabla_YX)+\frac{1}{2}{}^V(R(Y,X)u)-{}^H(\nabla_YX)+\frac{1}{2}{}^V(R(Y,X)u)\\
&\quad -{}^H(\nabla_XY)-\frac{1}{2}{}^V(R(X,Y)u)+{}^H(\nabla_XY)-\frac{1}{2}{}^V(R(X,Y)u)\Big\}\\
&= -\frac{1}{2}{}^V(R(X,Y)u),
\end{aligned}$$

similarly,

$$\widetilde{S}({}^VX,{}^VY)=0,\quad \widetilde{S}({}^VX,{}^HY)=-{}^H(R(u,X)Y),\quad \widetilde{S}({}^HX,{}^VY)=\frac{1}{2}{}^H(R(u,Y)X).$$

Using the above formulae, from (4.7), we get the following theorem.

**Theorem 4.7.** *Let $(M_{2k},\varphi,g)$ be an anti-paraKähler manifold and $TM$ be its tangent bundle equipped with the Berger type deformed Sasaki metric $g_{BS}$ and the paracomplex structure $\widetilde{\varphi}_2$ defined by (4.5). Then the almost paracomplex connection $\overline{\nabla}$ constructed by the Levi-Civita connection $\widetilde{\nabla}$ is as follows:*

(i) $\overline{\nabla}_{{}^HX}{}^HY={}^H(\nabla_XY)$,

(ii) $\overline{\nabla}_{{}^HX}{}^VY={}^V(\nabla_XY)$,

(iii) $\overline{\nabla}_{{}^VX}{}^HY=\frac{3}{2}{}^H(R(u,X)Y)$,

(iv) $\widetilde{\nabla}_{{}^VX}{}^VY=\dfrac{\delta^2}{1+\delta^2\alpha}g(X,\varphi Y)\,{}^V(\varphi u)$

*for all vector fields $X,Y$ on $M$.*

The torsion tensor $\overline{T}$ of the almost paracomplex connection $\overline{\nabla}$ has the form:

$$\overline{T}(^{H}X, ^{H}Y) = {}^{V}(R(X,Y)u),$$
$$\overline{T}(^{V}X, ^{H}Y) = \frac{3}{2}{}^{H}(R(u,X)Y),$$
$$\overline{T}(^{V}X, ^{V}Y) = 0,$$

i.e., the almost paracomplex connection $\overline{\nabla}$ is symmetric if and only if the base manifold $M_{2k}$ is flat. As is well-known, an almost paracomplex manifold has a symmetric almost paracomplex connection if and only if the almost paracomplex structure is integrable [6, 12]. Hence, from Theorem 4.5 we have

**Corollary 4.8.** *Let $(M_{2k}, \varphi, g)$ be an anti-paraKähler manifold and $TM$ be its tangent bundle equipped with the Berger type deformed Sasaki metric $g_{BS}$ and the paracomplex structure $\widetilde{\varphi}_2$ defined by (4.5). The manifold $(TM, \widetilde{\varphi}_2, g_{BS})$ has a symmetric almost paracomplex connection if and only if $M_{2k}$ is flat. In this case, the Levi-Civita connection $\widetilde{\nabla}$ and the almost paracomplex connection $\overline{\nabla}$ coincide with each other.*

## 5. The Riemannian curvature tensors

The Riemannian curvature tensor $\widetilde{R}$ is characterized by

$$\widetilde{R}(\widetilde{X}, \widetilde{Y})\widetilde{Z} = [\widetilde{\nabla}_{\widetilde{X}}, \widetilde{\nabla}_{\widetilde{Y}}]\widetilde{Z} - \widetilde{\nabla}_{[\widetilde{X}, \widetilde{Y}]}\widetilde{Z}$$

for all vector fields $X, \widetilde{Y}$ and $\widetilde{Z}$ on $TM$. One can check the Riemannian curvature tensor formula for the pairs $\widetilde{X} = {}^{H}X, {}^{V}X$, $\widetilde{Y} = {}^{H}Y, {}^{V}Y$ and $\widetilde{Z} = {}^{H}Z, {}^{V}Z$. Using the Proposition 3.1 and the Bianchi identities for $R$, standard calculations give

**Theorem 5.1.** *Let $(M_{2k}, \varphi, g)$ be an anti-paraKähler manifold and $TM$ be its tangent bundle equipped with the Berger type deformed Sasaki metric $g_{BS}$. Then the corresponding Riemannian curvature tensor $\widetilde{R}$ is given by*

$$\begin{aligned}
\widetilde{R}(^{H}X, ^{H}Y)^{H}Z &= \frac{1}{2}{}^{V}((\nabla_Z R)(X,Y)u) \\
&\quad + {}^{H}(R(X,Y)Z + \frac{1}{4}R(u, R(Z,Y)u)X \\
&\quad + \frac{1}{4}R(u, R(X,Z)u)Y + \frac{1}{2}R(u, R(X,Y)u)Z), \\
\widetilde{R}(^{H}X, ^{H}Y)^{V}Z &= {}^{V}(R(X,Y)Z + \frac{1}{4}R(R(u,Z)Y, X)u - \frac{1}{4}R(R(u,Z)X, Y)u) \\
&\quad + \frac{1}{2}{}^{H}((\nabla_X R)(u,Z)Y - (\nabla_Y R)(u,Z)X) \\
&\quad + \frac{\delta}{1+\delta^2\alpha} g(\varphi Z, R(X,Y)u)^{V}(\varphi u), \\
\widetilde{R}(^{H}X, ^{V}Y)^{H}Z &= {}^{V}\left(\frac{1}{4}R(R(u,Y)Z, X)u + \frac{1}{2}R(X,Z)Y\right)
\end{aligned}$$

$$+\frac{1}{2}{}^{H}((\nabla_X R)(u,Y)Z)+\frac{\delta^2}{1+\delta^2\alpha}g(\varphi Y,R(X,Z)u)\,{}^{V}(\varphi u),$$
$$\widetilde{R}({}^{H}X,{}^{V}Y)^{V}Z={}^{H}\!\left(-\frac{1}{2}R(Y,Z)X-\frac{1}{4}R(u,Y)R(u,Z)X\right),$$
$$\widetilde{R}({}^{V}X,{}^{V}Y)^{H}Z={}^{H}(R(X,Y)Z+\frac{1}{4}R(u,X)(R(u,Y)Z)-\frac{1}{4}R(u,Y)R(u,X)Z),$$
$$\widetilde{R}({}^{V}X,{}^{V}Y)^{V}Z=\frac{\delta^4}{(1+\delta^2\alpha)^2}(g(Y,u)g(X,\varphi Z)-g(X,u)g(Y,\varphi Z))\,{}^{V}(\varphi u).$$

Now, we consider the sectional curvature $\widetilde{K}$ of $(TM, g_{BS})$, namely,

$$\widetilde{K}(\widetilde{X},\widetilde{Y})=\frac{\widetilde{g}(\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{Y},\widetilde{X})}{\widetilde{g}(\widetilde{X},\widetilde{X})\widetilde{g}(\widetilde{Y},\widetilde{Y})-\widetilde{g}(\widetilde{X},\widetilde{Y})^2}$$

for vector fields $\widetilde{X},\widetilde{Y}$ on $TM$. With help of (3.1) and Theorem 5.1, standard calculations give the following result.

**Proposition 5.2.** *Let $(M_{2k},\varphi,g)$ be an anti-paraKähler manifold and $TM$ be its tangent bundle equipped with the Berger type deformed Sasaki metric $g_{BS}$. Then the corresponding sectional curvature $\widetilde{K}$ is given by*

$$\widetilde{K}({}^{H}X,{}^{H}Y)=K(X,Y)-\frac{3}{4}\left\|R(X,Y)u\right\|^2,$$
$$\widetilde{K}({}^{H}X,{}^{V}Y)=\frac{\frac{1}{4}\left\|R(u,Y)X\right\|^2}{1+\delta^2(g(Y,\varphi u)^2+g(X,\varphi u)^2)},$$
$$\widetilde{K}({}^{V}X,{}^{V}Y)=\frac{\frac{\delta^4}{1+\delta^2\alpha}\left(g(Y,u)g(X,\varphi Y)-g(X,u)g(Y,\varphi Y)\right)g(X,\varphi u)}{1+\delta^2(g(Y,\varphi u)^2+g(X,\varphi u)^2)},$$

*where $K(X,Y)$ is the sectional curvature of the plane spanned by $X$ and $Y$, and $\|\cdot\|$ denotes the norm of the vector with respect to the Riemannian metric $g$ in a point.*

To compute the scalar curvature $\widetilde{r}$, we rewrite the sectional curvature $\widetilde{K}$ in terms of orthonormal frame. Let the set $\{e_i\}_{1\leq i\leq n}$ be an orthonormal frame of $T_pM$, where $e_1=\frac{u}{\|u\|}$. In this case, the set $\{E_1,\dots,E_{2n}\}(n=2k)$, which is defined as below, is an orthonomal frame of $T_{(p,u)}TM$,

$$E_i={}^{H}(e_i),\,E_{n+1}=\frac{1}{\sqrt{1+\alpha\delta^2}}{}^{V}(\varphi e_1),\,E_{n+k}={}^{V}(\varphi e_k),\quad i=1,\dots,n,\ k=2,\dots n.$$

With respect to the orthonormal frame $\{E_1,\dots,E_{2n}\}$, the sectional curvature is expressed as follows:

$$\widetilde{K}(E_i,E_j)=K(e_i,e_j)-\frac{3}{4}\left|R(e_i,e_j)u\right|^2,$$
$$\widetilde{K}(E_i,E_{n+1})=0,$$
$$\widetilde{K}(E_i,E_{n+k})=\frac{\left|R(u,\varphi e_k)e_i\right|^2}{4},$$

$$\widetilde{K}(E_{m+1}, E_{m+k}) = 0,$$
$$\widetilde{K}(E_{m+k}, E_{m+l}) = 0.$$

For the relationship between the scalar curvature $\widetilde{r}$ of $(TM, g_{BS})$ and the scalar curvature $r$ of $(M, g)$, we have

**Proposition 5.3.** *Let $(M_{2k}, \varphi, g)$ be an anti-paraKähler manifold and $TM$ be its tangent bundle equipped with the Berger type deformed Sasaki metric $g_{BS}$. Then the corresponding scalar curvature $\widetilde{r}$ is given by*

$$\widetilde{r} = r - \sum_{i,j=1}^{n} \left( \frac{3}{4} |R(e_i, e_j)u|^2 + \frac{1}{2} |R(u, \varphi e_j)e_i|^2 \right).$$

## References


[1] M.T.K. Abbassi, *Note on the classification theorems of g-natural metrics on the tangent bundle of a Riemannian manifold $(M, g)$,* Comment. Math. Univ. Carolin. **45** (2004), No. 4, 591–596.

[2] M.T.K. Abbassi and M. Sarih, *On some hereditary properties of Riemannian g-natural metrics on tangent bundles of Riemannian manifolds,* Differential Geom. Appl. **22** (2005), No. 1, 19–47.

[3] M.T.K. Abbassi and M. Sarih, *On natural metrics on tangent bundles of Riemannian manifolds,* Arch. Math. **41** (2005), 71–92.

[4] V. Cruceanu, P. Fortuny, and P.M. Gadea, *A survey on paracomplex geometry,* Rocky Mountain J. Math. **26** (1996), No. 1, 83–115.

[5] P. Dombrowski, *On the geometry of the tangent bundles,* J. Reine and Angew. Math. **210** (1962), 73–88.

[6] A. Fujimoto, *Theory of G-structures,* Publ. Study Group of Geometry, **1**, Tokyo Univ., Tokyo, 1972.

[7] A. Gezer, *On the tangent bundle with deformed Sasaki metric,* Int. Electron. J. Geom. **6** (2013), No. 2, 19–31.

[8] A. Gezer and M. Altunbas, *Some notes concerning Riemannian metrics of Cheeger Gromoll type,* J. Math. Anal. Appl. **396** (2012), No. 1, 119–132.

[9] A. Gezer and M. Altunbas, *On the geometry of the rescaled Riemannian metric on tensor bundles of arbitrary type,* Kodai Math. J. **38** (2015), No. 1, 37–64.

[10] H.Z. Hou and L. Sun, *Geometry of tangent bundle with Cheeger–Gromoll type metric,* J. Math. Anal. App. **402** (2013), 493–504.

[11] O. Kowalski, *Curvature of the induced Riemannian metric of the tangent bundle of Riemannian manifold,* J. Reine Angew. Math. **250** (1971), 124–129.

[12] M. de Leon and P.R. Rodrigues, *Methods of Differential Geometry in Analytical Mechanics,* North-Holland Mathematics Studies, 1989.

[13] M. Manev and D. Mekerov, *On Lie groups as quasi-Kähler manifolds with Killing Norden metric,* Adv. Geom. **8** (2008), No. 3, 343–352.

[14] M.I. Munteanu, *Some aspects on the geometry of the tangent bundles and tangent sphere bundles of a Riemannian manifold*, Mediterr. J. Math. **5** (2008), No. 1, 43–59.

[15] E. Musso and F. Tricerri, *Riemannian Metrics on Tangent Bundles,* Ann. Mat. Pura. Appl. **150** (1988), No. 4, 1–19.

[16] A.A. Salimov, M. Iscan, and K. Akbulut, *Notes on para-Norden-Walker 4-manifolds,* Int. J. Geom. Methods Mod. Phys. **7** (2010), No. 8, 1331–1347.

[17] A.A. Salimov, M. Iscan, and F. Etayo, *Paraholomorphic B-manifold and its properties,* Topology Appl. **154** (2007), No. 4, 925–933.

[18] S. Sasaki, *On the differential geometry of tangent bundles of Riemannian manifolds,* Tohoku Math. J. **10** (1958), 338–358.

[19] A. Yampolsky, *On geodesics of tangent bundle with fiberwise deformed Sasaki metric over Kahlerian manifolds*, Zh. Mat. Fiz. Anal., Geom. **8** (2012), No. 2, 177–189.

[20] K. Yano and S. Ishihara, *Tangent and Cotangent Bundles*, Marcel Dekker, Inc., New York 1973.

[21] A. Zagane and M. Djaa, *On geodesics of warped Sasaki metric,* Mathematical Sci. and App. E-Notes **5** (2017), No. 1, 85–92.

Received September 17, 2018, revised December 10, 2018.

Murat Altunbas,
*Erzincan Binali Yıldırım University, Department of Mathematics, Erzincan, Turkey,*
E-mail: maltunbas@erzincan.edu.tr

Ramazan Simsek,
*Bayburt University, Technical Sciences Vocational School, Bayburt, Turkey,*
E-mail: rsimsek@bayburt.edu.tr

Aydın Gezer,
*Ataturk University, Department of Mathematics, Erzurum, 25240, Turkey,*
E-mail: aydingzr@gmail.com

## Дослідження щодо метрики Сасакі, деформованої за типом Берже, в дотичному розшаруванні

Murat Altunbas, Ramazan Simsek, and Aydın Gezer

Нехай $TM$ буде дотичним розшаруванням майже анти-пара-ермітового многовиду, наділеного метрикою Сасакі, деформованою за типом Берже $g_{BS}$. У цій роботі ми спочатку одержуємо зв'язність Леві-Чивіти цієї метрики та досліджуємо геодезичні лінії на $TM$. Потім ми будуємо майже анти-пара-ермітові стуктури на $TM$ і знаходимо умови, за яких ці структури є анти-пара-келеровими та квазі-анти-пара-келеровими. Нарешті ми описуємо деякі властивості ріманової кривини $(TM, g_{BS})$.

*Ключові слова:* метрика Сасакі, деформована за типом Берже, паракомплексна структура, геодезія, дотичне розшарування.