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On Certain Geometric Properties in Banach Spaces of Vector-Valued Functions

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We consider a certain type of geometric properties of Banach spaces, which includes, for instance, octahedrality, almost squareness, lushness and the Daugavet property. For this type of properties, we obtain a general reduction theorem, which, roughly speaking, states the following: if the property in question is stable under certain finite absolute sums (for example, finite ℓ^p-sums), then it is also stable under the formation of corresponding Köthe–Bochner spaces (for example, L^p-Bochner spaces). From this general theorem, we obtain as corollaries a number of new results as well as some alternative proofs of already known results concerning octahedral and almost square spaces and their relatives, diameter two properties, lush spaces and other classes.

Key words: absolute sums, Köthe—Bochner spaces, Lebesgue–Bochner spaces, octahedral spaces, almost square spaces, diameter two properties, lush spaces, generalised lush spaces, Daugavet property

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1. Introduction

Let $\mathbb{K}$ be the real or complex field. We consider a class $\mathcal{E}$ of Banach spaces over $\mathbb{K}$ which is closed under isometric isomorphisms, i.e., if $X \in \mathcal{E}$ and Y is isometrically isomorphic to X, then $Y \in \mathcal{E}$.

For a given Banach space X, we denote by X^* its dual space, by B_X its closed unit ball and by S_X its unit sphere. Furthermore, B_X^{fin} and S_X^{fin} will denote the sets of all finite sequences in B_X and S_X. For fixed $n \in \mathbb{N}$, B_X^n and S_X^n will stand for the sets of all sequences of length n in B_X and S_X. Given $\mathbf{x} = (x_1, \ldots, x_n) \in B_X^n$, we set $\|\mathbf{x}\|_\infty := \max_{i=1,\ldots,n}\|x_i\|$. Finally, $\mathcal{U}(X)$ denotes the set of all closed non-zero subspaces of X.

The following is our main definition.

Definition 1.1. Let X be a Banach space. A family of real-valued functions $F_{\varepsilon,U}$ on $B_U^{\text{fin}} \times B_{U^*}^{\text{fin}} \times B_U^{\text{fin}} \times B_{U^*}^{\text{fin}}$ with $U \in \mathcal{U}(X)$ and $\varepsilon > 0$ is said to be a test family for $\mathcal{E}$ in X if the following conditions are satisfied:

(i) For every $U \in \mathcal{U}(X)$, one has that $U \in \mathcal{E}$ if and only if for every $\varepsilon > 0$ and all $\mathbf{x} \in S_U^{\text{fin}}$ and $\mathbf{x}^* \in S_{U^*}^{\text{fin}}$ there exist $\mathbf{y} \in S_U^{\text{fin}}$ and $\mathbf{y}^* \in S_{U^*}^{\text{fin}}$ such that $F_{\varepsilon,U}(\mathbf{x}, \mathbf{x}^*, \mathbf{y}, \mathbf{y}^*) \leq \varepsilon$.

(ii) If $0 < \varepsilon_1 < \varepsilon_2$ and $U \in \mathcal{U}(X)$, then $F_{\varepsilon_1,U} \geq F_{\varepsilon_2,U}$.

(iii) There exists $c > 0$ such that for all $U \in \mathcal{U}(X)$, all $\varepsilon > 0$, every $\mathbf{x}, \mathbf{y} \in B_U^{\mathrm{fin}}$ and every $\mathbf{x}^*, \mathbf{y}^* \in B_{X^*}^{\mathrm{fin}}$ one has

$$F_{\varepsilon,X}(\mathbf{x}, \mathbf{x}^*, \mathbf{y}, \mathbf{y}^*) \leq c F_{\varepsilon,U}(\mathbf{x}, \mathbf{x}^*|_U, \mathbf{y}, \mathbf{y}^*|_U),$$

where for $\mathbf{x}^* = (x_1^*, \dots, x_n^*)$ we define $\mathbf{x}^*|_U = (x_1^*|_U, \dots, x_n^*|_U)$ (and analogously for $\mathbf{y}^*$).

(iv) For every $\varepsilon > 0$, all $\tau > 0$, each $\mathbf{x}^* \in B_{X^*}^{\mathrm{fin}}$, all $n \in \mathbb{N}$ and all $\mathbf{x} \in B_X^n$, there exists a $\delta > 0$ such that

$$|F_{\varepsilon,X}(\mathbf{x}, \mathbf{x}^*, \mathbf{y}, \mathbf{y}^*) - F_{\varepsilon,X}(\mathbf{z}, \mathbf{x}^*, \mathbf{y}, \mathbf{y}^*)| \leq \tau$$

holds for all $\mathbf{y} \in B_X^{\mathrm{fin}}$, all $\mathbf{y}^* \in B_{X^*}^{\mathrm{fin}}$ and every $\mathbf{z} \in B_X^n$ with $\|\mathbf{x} - \mathbf{z}\|_\infty \leq \delta$.

(v) For every $\varepsilon > 0$, all $n, m \in \mathbb{N}$ and all $\eta > 0$, there exists $\theta > 0$ such that for every $U \in \mathcal{U}(X)$ one has

$$|F_{\varepsilon,U}(\mathbf{x}, \mathbf{x}^*, \mathbf{y}, \mathbf{y}^*) - F_{\varepsilon,U}(\mathbf{x}, \mathbf{z}^*, \mathbf{y}, \mathbf{y}^*)| \leq \eta$$

for all $\mathbf{x} \in B_U^n$, all $\mathbf{y} \in B_U^{\mathrm{fin}}$, all $\mathbf{y}^* \in B_{U^*}^{\mathrm{fin}}$ and all $\mathbf{x}^*, \mathbf{z}^* \in B_{U^*}^m$ with $\|\mathbf{x}^* - \mathbf{z}^*\|_\infty \leq \theta$.

Roughly speaking, we want to show that if a Banach space property can be characterised in terms of test families and is stable under certain finite, absolute sums, then it is also stable under the formation of corresponding Köthe–Bochner function spaces.

Some examples of Banach space properties which can be described by test families will be presented in the next section (the constant c in the above definition will be 1 for all these examples). Here we continue with the necessary basics on absolute sums and Köthe–Bochner spaces.

Let I be a non-empty set, E be a subspace of $\mathbb{R}^I$ with $e_i \in E$ for all $i \in I$ and $\|\cdot\|_E$ be a complete norm on E (here e_i denotes the characteristic function of $\{i\}$).

The norm $\|\cdot\|_E$ is called *absolute* if

$$\begin{aligned} &(a_i)_{i\in I} \in E,\ (b_i)_{i\in I} \in \mathbb{R}^I \text{ and } |a_i| = |b_i|\ \forall i \in I \\ &\Rightarrow\ (b_i)_{i\in I} \in E \text{ and } \|(a_i)_{i\in I}\|_E = \|(b_i)_{i\in I}\|_E. \end{aligned}$$

The norm is called *normalised* if $\|e_i\|_E = 1$ for every $i \in I$.

Standard examples of subspaces of $\mathbb{R}^I$ with absolute normalised norm are of course the spaces $\ell^p(I)$ for $1 \leq p \leq \infty$ and the space $c_0(I)$.

We note the following lemma on absolute norms (see, e.g., [24, Remark 2.1]).

Lemma 1.2. *Let $(E, \|\,.\,\|_E)$ be a subspace of $\mathbb{R}^I$ with an absolute normalised norm. Then the following is true:*

$$\begin{aligned} &(a_i)_{i\in I} \in E,\ (b_i)_{i\in I} \in \mathbb{R}^I \text{ and } |b_i| \leq |a_i|\ \forall i \in I \\ &\Rightarrow\ (b_i)_{i\in I} \in E \text{ and } \|(b_i)_{i\in I}\|_E \leq \|(a_i)_{i\in I}\|_E. \end{aligned}$$

If $(X_i)_{i\in I}$ is a family of (real or complex) Banach spaces, we put

$$\Big[\bigoplus_{i\in I} X_i\Big]_E := \Big\{(x_i)_{i\in I} \in \prod_{i\in I} X_i : (\|x_i\|)_{i\in I} \in E\Big\}.$$

This defines a subspace of the product space $\prod_{i\in I} X_i$ which becomes a Banach space when endowed with the norm

$$\|(x_i)_{i\in I}\|_E := \|(\|x_i\|)_{i\in I}\|_E \ \forall (x_i)_{i\in I} \in \Big[\bigoplus_{i\in I} X_i\Big]_E.$$

We call this Banach space the absolute sum of the family $(X_i)_{i\in I}$ with respect to E. For $p \in [1,\infty]$ and $E = \ell^p(I)$, one obtains the usual p-sums of Banach spaces.

The "continuous counterpart" to absolute sums are the Köthe–Bochner function spaces, whose definition we will recall now. Let $(S, \mathcal{A}, \mu)$ be a complete σ-finite measure space. For $A \in \mathcal{A}$, we denote by χ_A the characteristic function of A. A Köthe function space over $(S, \mathcal{A}, \mu)$ is a Banach space $(E, \|\cdot\|_E)$ of real-valued measurable functions on S (modulo equality μ-almost everywhere) such that

(i) $\chi_A \in E$ for every $A \in \mathcal{A}$ with $\mu(A) < \infty$,

(ii) for every $f \in E$ and every set $A \in \mathcal{A}$ with $\mu(A) < \infty$ f is μ-integrable over A,

(iii) if g is measurable and $f \in E$ such that $|g(t)| \leq |f(t)|$ μ-a.e., then $g \in E$ and $\|g\|_E \leq \|f\|_E$.

Standard examples are the spaces $L^p(\mu)$ for $1 \leq p \leq \infty$.

Further recall that, given a Banach space X, a function $f : S \to X$ is called simple if there are finitely many disjoint measurable sets $A_1, \ldots, A_n \in \mathcal{A}$ such that $\mu(A_i) < \infty$ for all $i = 1, \ldots, n$, f is constant on each A_i and $f(t) = 0$ for every $t \in S \setminus \bigcup_{i=1}^n A_i$. The function f is said to be Bochner-measurable if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions such that $\lim_{n\to\infty}\|f_n(t) - f(t)\| = 0$ μ-a.e.

For a Köthe function space E and a Banach space X, we denote by $E(X)$ the space of all Bochner-measurable functions $f : S \to X$ (modulo equality a. e.) such that $\|f(\cdot)\| \in E$. Endowed with the norm $\|f\|_{E(X)} = \|\|f(\cdot)\|\|_E$, $E(X)$ becomes a Banach space, the so-called Köthe–Bochner space induced by E and X. For $E = L^p(\mu)$, we obtain the usual Lebesgue–Bochner spaces $L^p(\mu, X)$ for $1 \leq p \leq \infty$. For more information on Köthe–Bochner spaces the reader is referred to the book [25].

2. Examples

We will now discuss a number of examples of the Banach space properties which can be described via test families. We start with the octahedral spaces and their relatives.

2.1. Octahedrality. A real Banach space X is called octahedral (OH) (see [12]) if the following holds: for every finite-dimensional subspace F of X and every $\varepsilon > 0$ there is some $y \in S_X$ such that

$$\|x + y\| \geq (1 - \varepsilon)(\|x\| + 1) \quad \forall x \in F.$$

ℓ^1 is the model example of an OH space. It is known that a Banach space has an equivalent OH norm if and only if it contains an isomorphic copy of ℓ^1 (see [11, Theorem 2.5, p. 106]).

In the paper [15], two variants of octahedrality where introduced.

X is called locally octahedral (LOH) if for every $x \in X$ and every $\varepsilon > 0$ there exists $y \in S_X$ such that

$$\|sx + y\| \geq (1 - \varepsilon)(|s|\|x\| + 1) \quad \forall s \in \mathbb{R}.$$

X is called weakly octahedral (WOH) if for every finite-dimensional subspace F of X, every $x^* \in B_{X^*}$ and each $\varepsilon > 0$ there is some $y \in S_X$ such that

$$\|x + y\| \geq (1 - \varepsilon)(|x^*(x)| + 1) \quad \forall x \in F.$$

The motivation for this definition in [15] was the study of the so-called diameter two properties. Given $x^* \in S_{X^*}$ and $\alpha > 0$, the slice of B_X induced by x^* and α is $S(x^*, \alpha) := \{z \in B_X, x^*(z) > 1 - \alpha\}$. According to [1], the space X is said to have the local diameter two property (LD2P) if every slice of B_X has diameter 2; X has the diameter two property (D2P) if every nonempty, relatively weakly open subset of B_X has diameter 2; X has the strong diameter two property (SD2P) if every convex combination of slices of B_X has diameter 2.

The following results were proved in [15]:

(a) X has LD2P $\iff$ X^* is LOH.

(b) X has D2P $\iff$ X^* is WOH.

(c) X has SD2P $\iff$ X^* is OH.

The equivalence (c) was also proved independently in [5].

It is known that the three diameter two properties are really different. For example, it follows from the results on direct sums in [15] that $c_0 \oplus_2 c_0$ has the D2P but not the SD2P (we will recall these results in Section 4).

Concerning the nonequivalence of the LD2P and the D2P, it has been shown in [6] that there is a Banach space with the LD2P whose unit ball contains relatively weakly open subsets of arbitrarily small diameter (every Banach space containing an isomorphic copy of c_0 can be renormed to become such a space [6, Theorem 2.4] (note that the abbreviation SD2P in [6] does not stand for “strong diameter two property” but for “slice diameter two property”, which coincides with the LD2P of [1]).

In [21] it was shown that Cesàro function spaces have the D2P.

It is possible to characterise all three octahedrality properties in terms of test families. To do this, we make use of the following equivalent formulations proved in [15] (other equivalent characterisations in terms of coverings of the unit ball were proved in [14]).

A Banach space X is OH if and only if for every $n \in \mathbb{N}$, all $x_1, \dots, x_n \in S_X$ and every $\varepsilon > 0$ there exists an element $y \in S_X$ such that $\|x_i + y\| \geq 2 - \varepsilon$ for all $i = 1, \dots, n$.

X is LOH if and only if for every $x \in S_X$ and all $\varepsilon > 0$ there exists $y \in S_X$ such $\|x \pm y\| \geq 2 - \varepsilon$.

Of course, the same characterisations also hold for all closed subspaces of X. Thus, if we put

$$F_{\varepsilon,U}(\mathbf{x}, \mathbf{x}^*, \mathbf{y}, \mathbf{y}^*) := \max\{2 - \|x_i + y_1\| : i = 1, \dots, n\}$$

for $U \in \mathcal{U}(X)$, $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_m) \in B_U^{\mathrm{fin}}$ and $\mathbf{x}^*, \mathbf{y}^* \in B_{U^*}^{\mathrm{fin}}$, we obtain a test family for the class of octahedral spaces in X.

If we put instead

$$F_{\varepsilon,U}(\mathbf{x}, \mathbf{x}^*, \mathbf{y}, \mathbf{y}^*) := \max\{2 - \|x_1 + y_1\|, 2 - \|x_1 - y_1\|\},$$

we obtain a test family for LOH in X.

In both cases, condition (i) in Definition 1.1 follows from the above characterisations, while conditions (ii)–(v) are easily verified.

For weak octahedrality, the following was proved in [15]: X is WOH if and only if for every $n \in \mathbb{N}$, all $x_1, \dots, x_n \in S_X$, every $x^* \in S_{X^*}$ and every $\varepsilon > 0$ there exists a $y \in S_X$ such that $\|x_i + ty\| \geq (1-\varepsilon)(|x^*(x_i)| + t)$ for all $i = 1, \dots, n$ and every $t \geq \varepsilon$.

The original formulation in [15] reads "for every $x^* \in B_{X^*}$", but it clearly suffices to take $x^* \in S_{X^*}$.

Thus, if we define

$$F_{\varepsilon,U}(\mathbf{x}, \mathbf{x}^*, \mathbf{y}, \mathbf{y}^*) := \max_{i=1,\dots,n} \sup_{t \geq \varepsilon} \left(1 - \frac{\|x_i + ty_1\|}{|x_1^*(x_i)| + t}\right)$$

for $U \in \mathcal{U}(X)$, $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_m) \in B_U^{\mathrm{fin}}$ and $\mathbf{x}^* = (x_1^*, \dots, x_k^*)$, $\mathbf{y}^* \in B_{U^*}^{\mathrm{fin}}$, then condition (i) in the definition of a test family for the class of WOH spaces is satisfied and (ii) and (iii) are clearly true as well. Conditions (iv) and (v) easily follow from the next auxiliary lemma.

Lemma 2.1. *If Y is a real Banach space and $\varepsilon > 0$, define the function $f : B_Y \times B_Y \times B_{Y^*} \to \mathbb{R}$ by*

$$f(x, y, x^*) := \sup_{t \geq \varepsilon} \left(1 - \frac{\|x + ty\|}{|x^*(x)| + t}\right) \quad \forall x, y \in B_Y, \forall x^* \in B_{Y^*}.$$

If $\delta > 0$, $x, \tilde{x}, y, \tilde{y} \in B_Y$ with $\|x - \tilde{x}\|, \|y - \tilde{y}\| \leq \delta$ and $x^*, \tilde{x}^* \in B_{Y^*}$ with $\|x^* - \tilde{x}^*\| \leq \delta$, then

$$|f(x, y, x^*) - f(\tilde{x}, \tilde{y}, \tilde{x}^*)| \leq \delta(3/\varepsilon + 2/\varepsilon^2 + 1).$$

Proof. We have $||\tilde{x}^*(\tilde{x})| - |x^*(x)|| \leq |\tilde{x}^*(\tilde{x}) - \tilde{x}^*(x)| + |\tilde{x}^*(x) - x^*(x)| \leq 2\delta$. Thus, for every $t \geq \varepsilon$, we have

$$\begin{aligned}
1 - \frac{\|\tilde{x} + t\tilde{y}\|}{|\tilde{x}^*(\tilde{x})| + t} - \left(1 - \frac{\|x + ty\|}{|x^*(x)| + t}\right) \\
&= \frac{\|x + ty\|(|\tilde{x}^*(\tilde{x})| + t) - \|\tilde{x} + t\tilde{y}\|(|x^*(x)| + t)}{(|x^*(x)| + t)(|\tilde{x}^*(\tilde{x})| + t)} \\
&\leq \frac{\|x + ty\|(|x^*(x)| + 2\delta + t) - \|\tilde{x} + t\tilde{y}\|(|x^*(x)| + t)}{(|x^*(x)| + t)(|\tilde{x}^*(\tilde{x})| + t)} \\
&\leq \frac{2\delta}{t^2}\|x + ty\| + \frac{\|x + ty\| - \|\tilde{x} + t\tilde{y}\|}{|\tilde{x}^*(\tilde{x})| + t} \\
&\leq \frac{2\delta}{t^2}(1 + t) + \frac{\|x - \tilde{x}\| + t\|y - \tilde{y}\|}{|\tilde{x}^*(\tilde{x})| + t} \\
&\leq \frac{2\delta}{t^2}(1 + t) + \frac{(1 + t)\delta}{t} = \delta(3/t + 2/t^2 + 1) \\
&\leq \delta(3/\varepsilon + 2/\varepsilon^2 + 1).
\end{aligned}$$

By symmetry, we also have

$$1 - \frac{\|x + ty\|}{|x^*(x)| + t} - \left(1 - \frac{\|\tilde{x} + t\tilde{y}\|}{|\tilde{x}^*(\tilde{x})| + t}\right) \leq \delta(3/\varepsilon + 2/\varepsilon^2 + 1)$$

for all $t \geq \varepsilon$. This implies the desired inequality. □

The above-mentioned dual characterisations from [15] allow us to write the diameter two properties in terms of test families as well.

Since a Banach space has the LD2P if and only if its dual is LOH, we obtain a test family for the LD2P in X by setting

$$F_{\varepsilon,U}(\mathbf{x}, \mathbf{x}^*, \mathbf{y}, \mathbf{y}^*) := \max\{2 - \|x_1^* + y_1^*\|, 2 - \|x_1^* - y_1^*\|\}$$

for $U \in \mathcal{U}(X)$, $\mathbf{x}, \mathbf{y} \in B_U^{\mathrm{fin}}$ and $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$, $\mathbf{y}^* = (y_1^*, \ldots, y_m^*) \in B_{U^*}^{\mathrm{fin}}$.

Likewise, since a Banach space has the SD2P if and only its dual is OH, a test family for the SD2P in X is given by

$$F_{\varepsilon,U}(\mathbf{x}, \mathbf{x}^*, \mathbf{y}, \mathbf{y}^*) := \max\{2 - \|x_i^* + y_1^*\| : i = 1, \ldots, n\}$$

for $U \in \mathcal{U}(X)$, $\mathbf{x}, \mathbf{y} \in B_U^{\mathrm{fin}}$ and $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$, $\mathbf{y}^* = (y_1^*, \ldots, y_m^*) \in B_{U^*}^{\mathrm{fin}}$. (In both cases conditions (i)–(v) are easily checked.)

We also know that a Banach space has the D2P if and only if its dual is WOH. Then we can make use of the following characterisation (see [15]) for the property WOH in dual spaces, which does not involve the bidual:

X^* is WOH if and only if for every $n \in \mathbb{N}$, all $x_1^*, \ldots, x_n^* \in S_{X^*}$, every $x \in S_X$ and every $\varepsilon > 0$ there exists $y^* \in S_{X^*}$ such that

$$\|x_i^* + ty^*\| \geq (1 - \varepsilon)(|x_i^*(x)| + t) \quad \forall i \in \{1, \ldots, n\}\ \forall t \geq \varepsilon.$$

Thus we can define a test family for the D2P in X by

$$F_{\varepsilon,U}(\mathbf{x},\mathbf{x}^*,\mathbf{y},\mathbf{y}^*) := \max_{i=1,\dots,n} \sup_{t\geq\varepsilon}\left(1-\frac{\|x_i^*+ty_1^*\|}{|x_i^*(x_1)|+t}\right)$$

for $U\in\mathcal{U}(X)$, $\mathbf{x}=(x_1,\dots,x_k), \mathbf{y}\in B_U^{\mathrm{fin}}$ and $\mathbf{x}^*=(x_1^*,\dots,x_n^*)$, $\mathbf{y}^*=(y_1^*,\dots,y_m^*)\in B_{U^*}^{\mathrm{fin}}$.

Conditions (i)–(iii) are clear, and conditions (iv) and (v) are proved by using an auxiliary lemma similar to Lemma 2.1.

We remark that it is also possible to describe the LD2P via a different test family, using directly the definition of the LD2P (and not its dual characterisation). It is easily checked that a Banach space X has the LD2P if and only if the following holds: for every $x^*\in S_{X^*}$ and every $\varepsilon>0$ there exist $y_1,y_2\in S_X$ such that $x^*(y_1),x^*(y_2)\geq 1-\varepsilon$ and $\|y_1-y_2\|\geq 2-\varepsilon$.

Thus we can define a test family for the LD2P in X as follows:

$$F_{\varepsilon,U}(\mathbf{x},\mathbf{x}^*,\mathbf{y},\mathbf{y}^*) := \max\{1-x_1^*(y_1),1-x_1^*(y_2),2-\|y_1-y_2\|\}$$

for $U\in\mathcal{U}(X)$, $\mathbf{x},\mathbf{y}=(y_1,\dots,y_m)\in B_U^{\mathrm{fin}}$ and $\mathbf{x}^*=(x_1^*,\dots,x_n^*),\mathbf{y}^*\in B_{U^*}^{\mathrm{fin}}$, where $y_2:=y_1$ if $m=1$ (once again, conditions (i)–(v) in Definition 1.1 are easily verified).

Finally, there is yet another weakening of the definition of octahedral spaces, which was introduced in [16]: X is called alternatively octahedral (AOH) if for every $n\in\mathbb{N}$, all $x_1,\dots,x_n\in S_X$ and every $\varepsilon>0$ there is some $y\in S_X$ such that

$$\max\{\|x_i+y\|,\|x_i-y\|\}\geq 2-\varepsilon\quad \forall i=1,\dots,n.$$

Every octahedral space is alternatively octahedral while, for example, c_0 is alternatively octahedral but not locally octahedral (see [16]).

It is easily checked that

$$F_{\varepsilon,U}(\mathbf{x},\mathbf{x}^*,\mathbf{y},\mathbf{y}^*) := \max\{2-\max\{\|x_i+y_1\|,\|x_i-y_1\|\} : i=1,\dots,n\},$$

where $U\in\mathcal{U}(X)$, $\mathbf{x}=(x_1,\dots,x_n),\mathbf{y}=(y_1,\dots,y_m)\in B_U^{\mathrm{fin}}$ and $\mathbf{x}^*,\mathbf{y}^*\in B_{U^*}^{\mathrm{fin}}$, defines a test family for AOH in X.

2.2. Almost square spaces. Next we turn to the classes of almost square and locally almost square Banach spaces. These notions were introduced in [2].

A real Banach space X is said to be almost square (ASQ) if the following holds: for all $n\in\mathbb{N}$ and all $x_1,\dots,x_n\in S_X$ there exists a sequence $(y_k)_{k\in\mathbb{N}}$ in B_X such that $\|y_k\|\to 1$ and $\|x_i+y_k\|\to 1$ for all $i=1,\dots,n$.

X is called locally almost square (LASQ) if for every $x\in S_X$ there is a sequence $(y_k)_{k\in\mathbb{N}}$ in B_X such that $\|y_k\|\to 1$ and $\|x\pm y_k\|\to 1$.

According to [2], X is ASQ if and only if for every $\varepsilon>0$, every $n\in\mathbb{N}$ and all $x_1,\dots,x_n\in S_X$ there exists a $y\in S_X$ such that $\|x_i-y\|\leq 1+\varepsilon$ for all $i=1,\dots,n$, and X is LASQ if and only if for every $\varepsilon>0$ and every $x\in S_X$ there is some $y\in S_X$ such that $\|x\pm y\|\leq 1+\varepsilon$.

c_0 is the model example of an ASQ space. It was further proved in [2] that every ASQ space contains an isomorphic copy of c_0 and that every separable Banach space containing an isomorphic copy of c_0 has an equivalent ASQ norm. In [8], it was proved that the same holds also for nonseparable spaces.

In [2], it was also proved that X^* is OH (i. e., X has the SD2P) whenever X is ASQ. By [21, Proposition 2.5], every LASQ space has the LD2P.

If we define

$$F_{\varepsilon,U}(\mathbf{x},\mathbf{x}^*,\mathbf{y},\mathbf{y}^*) := \max\{\|x_i - y_1\| - 1 : i = 1,\dots,n\}$$

for $U \in \mathcal{U}(X)$, $\mathbf{x} = (x_1,\dots,x_n), \mathbf{y} = (y_1,\dots,y_m) \in B_U^{\mathrm{fin}}$ and $\mathbf{x}^*,\mathbf{y}^* \in B_{U^*}^{\mathrm{fin}}$, then we obtain a test family for ASQ in X, as is easily checked.

Likewise, a test family for LASQ in X is given by

$$F_{\varepsilon,U}(\mathbf{x},\mathbf{x}^*,\mathbf{y},\mathbf{y}^*) := \max\{\|x_1 + y_1\| - 1, \|x_1 - y_1\| - 1\}.$$

There is also an intermediate notion of weakly almost square (WASQ) spaces defined in [2] (by [21, Proposition 2.6], these spaces have the D2P) but it is not clear whether this notion can be phrased in terms of test families.

2.3. The Daugavet property. We now consider spaces with the Daugavet and the alternative Daugavet properties.

A real Banach space X is said to have the Daugavet property (DP) if the equality $\|\mathrm{id} + T\| = 1 + \|T\|$ holds for every rank-one operator $T : X \to X$ (see, for example, [18, 31]).

Examples of such spaces include $C(K)$ for compact Hausdorff spaces K without isolated points, and $L^1(\mu)$ for atomless measures μ (see the examples in [31]). In [18], the following remarkable result was proved: if X has the DP, then $\|\mathrm{id} + T\| = 1 + \|T\|$ actually holds for all weakly compact operators on X.

According to [18, Lemma 2], X has the DP if and only if for every $x \in S_X$, every $x^* \in S_{X^*}$ and all $\varepsilon > 0$ there exists $y \in S_X$ such that $x^*(y) \ge 1 - \varepsilon$ and $\|x + y\| \ge 2 - \varepsilon$.

Thus a test family for the Daugavet property in X is given by

$$F_{\varepsilon,U}(\mathbf{x},\mathbf{x}^*,\mathbf{y},\mathbf{y}^*) := \max\{1 - x_1^*(y_1), 2 - \|x_1 + y_1\|\}$$

for all $U \in \mathcal{U}(X)$, $\varepsilon > 0$, $\mathbf{x} = (x_1,\dots,x_n), \mathbf{y} = (y_1,\dots,y_m) \in B_U^{\mathrm{fin}}$ and all $\mathbf{x}^* = (x_1^*,\dots,x_k^*), \mathbf{y}^* \in B_{U^*}^{\mathrm{fin}}$ (again conditions (i)–(v) are easily verified).

The following weaker version of the DP was introduced in [29]: a real or complex Banach space X is said to have the alternative Daugavet property (ADP) if $\max_{\omega\in\mathbb{T}}\|\mathrm{id} + \omega T\| = 1 + \|T\|$ holds for every rank-one operator T on X, where $\mathbb{T} := \{\omega \in \mathbb{K} : |\omega| = 1\}$.

Again it was proved in [29] that the above equality holds for all weakly compact operators if it holds for all rank-one operators. It was also proved in [29] that X has the ADP if and only if for every $\varepsilon > 0$, every $x \in S_X$ and every $x^* \in S_{X^*}$ there is some $y \in S_X$ such that $\operatorname{Re} x^*(y) \ge 1 - \varepsilon$ and $\max_{\omega\in\mathbb{T}}\|y + \omega x\| \ge 2 - \varepsilon$.

We can thus define a test family for the ADP in X as follows:

$$F_{\varepsilon,U}(\mathbf{x},\mathbf{x}^*,\mathbf{y},\mathbf{y}^*) := \max\left\{1 - \operatorname{Re} x_1^*(y_1), 2 - \max_{\omega\in\mathbb{T}}\|y_1 + \omega x_1\|\right\}.$$

2.4. Lush spaces. Next we consider the class of lush Banach spaces which was introduced in [9] (in connection with the study of the numerical index of Banach spaces). A Banach space X is called lush provided that for every $\varepsilon > 0$ and all $x_1, x_2 \in S_X$ there exists a functional $y^* \in S_{X^*}$ such that $x_1 \in S(y^*, \varepsilon)$ and $\mathrm{d}(x_2, \mathrm{aco}S(y^*, \varepsilon)) < \varepsilon$, where aco denotes the absolutely convex hull and d is the usual inf distance.

For example, if K is a compact Hausdorff space, then $C(K)$ and, more generally, every so-called C-rich subspace of $C(K)$, is lush (see [9]).

We can define a test family for lushness in X by

$$F_{\varepsilon,U}(\mathbf{x},\mathbf{x}^*,\mathbf{y},\mathbf{y}^*) := \max\{1 - y_1^*(x_1), \mathrm{d}(x_2, \mathrm{aco}S(y_1^*, \varepsilon))\}$$

for $\varepsilon > 0$, $U \in \mathcal{U}(X)$, $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} \in B_U^{\mathrm{fin}}$ and $\mathbf{x}^*, \mathbf{y}^* = (y_1^*, \ldots, y_m^*) \in B_{U^*}^{\mathrm{fin}}$ (where we set $x_2 := x_1$ if $n = 1$ and $\mathrm{d}(x_2, \mathrm{aco}S(y_1^*, \varepsilon)) := 2$ if $\|y_1^*\| < 1$). Conditions (i)–(v) in Definition 1.1 are easily verified.

In [30], the following related notion was introduced: the space X is called generalised lush (GL) if for every $x \in S_X$ and every $\varepsilon > 0$ there is some functional $y^* \in S_{X^*}$ such that $x \in S(y^*, \varepsilon)$ and $\mathrm{d}(z, S(y^*, \varepsilon)) + \mathrm{d}(z, -S(y^*, \varepsilon)) < 2 + \varepsilon$ for every $z \in S_X$.

It was shown in [30] that every separable lush space is GL, and that $\mathbb{R}^2$ equipped with the hexagonal norm $\|(a, b)\| = \max\{|b|, |a| + 1/2|b|\}$ is GL but not lush. It is not known whether every nonseparable lush space is GL.

The main result in [30] is that every GL-space X has the Mazur-Ulam property (MUP), i.e., if Y is any Banach space and $T : S_X \to S_Y$ is a surjective isometry, then T can be extended to an isometric isomorphism between X and Y.

It is not obvious whether the property GL can be described via test families. However, there is the following (at least formally) weaker version of GL-spaces: X is said to have the property $(**)$ if for all $x_1, x_2 \in S_X$ and each $\varepsilon > 0$ one can find $y^* \in S_{X^*}$ such that $x_1 \in S(y^*, \varepsilon)$ and $\mathrm{d}(x_2, S(y^*, \varepsilon)) + \mathrm{d}(x_2, -S(y^*, \varepsilon)) < 2 + \varepsilon$.

This notion was introduced in the author's paper [17] (with the help of an anonymous referee) and the following observations were made:

(a) Every lush space has property $(**)$.

(b) For separable spaces, $(**)$ is equivalent to GL.

(c) Every space with property $(**)$ has the MUP.

A test family for $(**)$ in X can be defined by

$$F_{\varepsilon,U}(\mathbf{x},\mathbf{x}^*,\mathbf{y},\mathbf{y}^*) := \max\{1 - y_1^*(x_1), \mathrm{d}(x_2, S(y_1^*, \varepsilon)) + \mathrm{d}(x_2, -S(y_1^*, \varepsilon)) - 2\}$$

for $\varepsilon > 0$, $U \in \mathcal{U}(X)$, $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} \in B_U^{\mathrm{fin}}$ and $\mathbf{x}^*, \mathbf{y}^* = (y_1^*, \ldots, y_m^*) \in B_{U^*}^{\mathrm{fin}}$, where $x_2 := x_1$ if $n = 1$ and $\mathrm{d}(x_2, S(y_1^*, \varepsilon)) := \mathrm{d}(x_2, -S(y_1^*, \varepsilon)) := 2$ if $\|y_1^*\| < 1$.

3. Main result

Given a complete σ-finite measure space $(S,\mathcal{A},\mu)$, a Köthe function space E over $(S,\mathcal{A},\mu)$ and pairwise disjoint sets $A_1,\dots,A_N\in\mathcal{A}$ with $0<\mu(A_i)<\infty$ for $i=1,\dots,N$, we define

$$\|(a_1,\dots,a_N)\|_{E(A_1,\dots,A_N)}:=\left\|\sum_{i=1}^N \frac{a_i}{\|\chi_{A_i}\|_E}\chi_{A_i}\right\|_E \quad \forall (a_1,\dots,a_N)\in\mathbb{R}^N.$$

Then $\|\cdot\|_{E(A_1,\dots,A_N)}$ is an absolute normalized norm on $\mathbb{R}^N$. If $p\in[1,\infty]$ and $E=L^p(\mu)$, then this norm coincides with the usual p-norm on $\mathbb{R}^N$, regardless of the choice of $A_1,\dots,A_N$.

For a Banach space X, we denote by $E(A_1,\dots,A_N,X)$ the N-fold absolute sum of X with respect to $\|\cdot\|_{E(A_1,\dots,A_N)}$.

The following theorem is the main result of this paper.

Theorem 3.1. *Let $(S,\mathcal{A},\mu)$ be a complete σ-finite measure space and E be a Köthe function space over $(S,\mathcal{A},\mu)$. Suppose that X is a Banach space such that the simple functions are dense in $E(X)$ and $E(A_1,\dots,A_N,X)\in\mathcal{E}$ for every $N\in\mathbb{N}$ and all pairwise disjoint sets $A_1,\dots,A_N\in\mathcal{A}$ with $0<\mu(A_i)<\infty$ for each i. Suppose further that there exists a test family for $\mathcal{E}$ in $E(X)$. Then $E(X)\in\mathcal{E}$.*

Proof. Let $(F_{\varepsilon,U})_{\varepsilon>0,U\in\mathcal{U}(E(X))}$ be a test family for $\mathcal{E}$ in $E(X)$. Let $\mathbf{f}=(f_1,\dots,f_n)\in S^n_{E(X)}$, $\Phi=(\varphi_1,\dots,\varphi_m)\in S^m_{E(X)^*}$ and $\varepsilon>0$. Choose $\delta>0$ such that:

(a) For all $\mathbf{y}\in B^{\mathrm{fin}}_{E(X)}$, all $\mathbf{y}^*\in B^{\mathrm{fin}}_{E(X)^*}$ and all $\mathbf{z}\in B^n_{E(X)}$ with $\|\mathbf{f}-\mathbf{z}\|_\infty\le\delta$, we have

$$|F_{\varepsilon,E(X)}(\mathbf{f},\Phi,\mathbf{y},\mathbf{y}^*)-F_{\varepsilon,E(X)}(\mathbf{z},\Phi,\mathbf{y},\mathbf{y}^*)|\le\frac{\varepsilon}{2}.$$

(b) For every $U\in\mathcal{U}(E(X))$, for all $\mathbf{x}\in B^n_U$, all $\mathbf{y}\in B^{\mathrm{fin}}_U$, every $\mathbf{y}^*\in B^{\mathrm{fin}}_{U^*}$ and all $\mathbf{x}^*,\mathbf{z}^*\in B^m_{U^*}$ with $\|\mathbf{x}^*-\mathbf{z}^*\|_\infty\le\delta$, we have

$$|F_{\varepsilon,U}(\mathbf{x},\mathbf{x}^*,\mathbf{y},\mathbf{y}^*)-F_{\varepsilon,U}(\mathbf{x},\mathbf{z}^*,\mathbf{y},\mathbf{y}^*)|\le\frac{\varepsilon}{4c},$$

where c is the constant from Definition 1.1 (iii).

This is possible because of (iv) and (v) in Definition 1.1.

Put $\tilde{\varepsilon}:=\min\{\varepsilon,\varepsilon/4c\}$.

We can find simple functions $h_1,\dots,h_n\in E(X)$ such that $\|h_i\|_{E(X)}=1$ and $\|f_i-h_i\|_{E(X)}\le\delta$ for all $i=1,\dots,n$.

Also, there are simple functions $g_1,\dots,g_m\in E(X)$ with $\|g_j\|_{E(X)}=1$ and $|\varphi_j(g_j)|\ge 1-\delta$ for all $j=1,\dots,m$.

Fix pairwise disjoint sets $A_1,\dots,A_N\in\mathcal{A}$ with $0<\mu(A_i)<\infty$ such that each h_i and each g_j belong to the subspace

$$U:=\left\{\sum_{k=1}^N x_k\chi_{A_k}:x_1,\dots,x_N\in X\right\}\subseteq E(X).$$

By considering the map $T : E(A_1, \dots, A_N, X) \to U$ defined by

$$T(x_1, \dots, x_N) := \sum_{k=1}^{N} \frac{x_k}{\|\chi_{A_k}\|_E} \chi_{A_k},$$

we see that U is isometrically isomorphic to $E(A_1, \dots, A_N, X)$.

By assumption, we have $E(A_1, \dots, A_N, X) \in \mathcal{E}$, thus $U \in \mathcal{E}$.

Since $g_j \in S_U$, we have $1 \geq \|\varphi_j|_U\| \geq 1 - \delta$ for each j. Hence $\psi_j := \varphi_j|_U / \|\varphi_j|_U\| \in S_{U^*}$ with

$$\|\psi_j - \varphi_j|_U\| = |1 - \|\varphi_j|_U\|| \leq \delta \quad \forall j = 1, \dots, m. \tag{3.1}$$

Put $\Psi := (\psi_1, \dots, \psi_m) \in S_{U^*}^m$ and $\mathbf{h} = (h_1, \dots, h_n) \in S_U^n$. Since $U \in \mathcal{E}$, we can find $\mathbf{u} = (u_1, \dots, u_l) \in S_U^{\text{fin}}$ and $\mathbf{u}^* = (u_1^*, \dots, u_s^*) \in S_{U^*}^{\text{fin}}$ such that $F_{\tilde{\varepsilon},U}(\mathbf{h}, \Psi, \mathbf{u}, \mathbf{u}^*) \leq \tilde{\varepsilon}$.

Because of $\tilde{\varepsilon} \leq \varepsilon$ and (ii) in Definition 1.1, it follows that $F_{\varepsilon,U}(\mathbf{h}, \Psi, \mathbf{u}, \mathbf{u}^*) \leq \tilde{\varepsilon}$.

Then (b) and (3.1) imply $F_{\varepsilon,U}(\mathbf{h}, \Phi|_U, \mathbf{u}, \mathbf{u}^*) \leq \tilde{\varepsilon} + \varepsilon/4c \leq \varepsilon/2c$.

By the Hahn–Banach theorem, there are functionals $\omega_1, \dots, \omega_s \in S_{E(X)^*}$ such that $\omega_i|_U = u_i^*$ for $i = 1, \dots, s$. Let $\Omega := (\omega_1, \dots, \omega_s)$.

Now it follows from (iii) in Definition 1.1 that $F_{\varepsilon,E(X)}(\mathbf{h}, \Phi, \mathbf{u}, \Omega) \leq \varepsilon/2$.

Since $\|\mathbf{f} - \mathbf{h}\|_\infty \leq \delta$, (a) implies $F_{\varepsilon,E(X)}(\mathbf{f}, \Phi, \mathbf{u}, \Omega) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$ and the proof is finished. □

Every Köthe function space E is a Banach lattice in its natural ordering ($f \leq g$ if and only if $f(s) \leq g(s)$ for a.e. $s \in S$). It is well known that if $(E, \leq)$ is order continuous, then for every Banach space X the simple functions lie dense in $E(X)$. This includes in particular the case of L^p-spaces for $1 \leq p < \infty$. So, from the above theorem, we obtain the following corollary ($\ell_N^p(X)$ denotes the N-fold p-sum of X).

Corollary 3.2. *Let $(S, \mathcal{A}, \mu)$ be a complete σ-finite measure space and $1 \leq p < \infty$. If X is a Banach space such that $\ell_N^p(X) \in \mathcal{E}$ for every $N \in \mathbb{N}$ and there exists a test family for $\mathcal{E}$ in $L^p(\mu, X)$, then $L^p(\mu, X) \in \mathcal{E}$.*

In the case $p = \infty$, it is well known that one still has the density of $\{f \in L^\infty(\mu, X) : \operatorname{ran}(f) \text{ is countable}\}$ in $L^\infty(\mu, X)$, where $\operatorname{ran}(f)$ denotes the range of f. Thus one can prove the following Theorem in an analogous way to the proof of Theorem 3.1 (we omit the details).

Theorem 3.3. *Let $(S, \mathcal{A}, \mu)$ be a complete σ-finite measure space. If X is a Banach space such that $\ell_N^\infty(X) \in \mathcal{E}$ for every $N \in \mathbb{N}$ and $\ell^\infty(X) \in \mathcal{E}$ and there exists a test family for $\mathcal{E}$ in $L^\infty(\mu, X)$, then $L^\infty(\mu, X) \in \mathcal{E}$.*

Here $\ell^\infty(X)$ stands for $\left[\bigoplus_{n \in \mathbb{N}} X\right]_{\ell^\infty}$.

We also have a reduction result for the case of infinite absolute sums to finite sums, which reads as follows.

Proposition 3.4. *Let I be an index set and E be a subspace of $\mathbb{R}^I$ endowed with an absolute normalised norm such that* $\operatorname{span}\{e_i : i \in I\}$ *is dense in E. Let $(X_i)_{i\in I}$ be a family of Banach spaces such that $\left[\bigoplus_{i\in J} X_i\right]_E \in \mathcal{E}$ for every nonempty, finite subset $J \subseteq I$. If there is a test family for $\mathcal{E}$ in $\left[\bigoplus_{i\in I} X_i\right]_E$, then $\left[\bigoplus_{i\in I} X_i\right]_E \in \mathcal{E}$.*

The notation $\left[\bigoplus_{i\in J} X_i\right]_E$ means that all summands with indices in $I \setminus J$ are $\{0\}$. The proof is similar to that of Theorem 3.1 and will therefore be omitted.

As an immediate consequence of Proposition 3.4, we get the following results for p-sums and c_0-sums.

Corollary 3.5. *If I is any index set, $1 \leq p < \infty$, $(X_i)_{i\in I}$ is a family of Banach spaces such that $\left[\bigoplus_{i\in J} X_i\right]_p \in \mathcal{E}$ for every nonempty finite subset $J \subseteq I$, and there exists a test family for $\mathcal{E}$ in $\left[\bigoplus_{i\in I} X_i\right]_p$, then $\left[\bigoplus_{i\in I} X_i\right]_p \in \mathcal{E}$.*

Corollary 3.6. *If I is any index set, $(X_i)_{i\in I}$ is a family of Banach spaces such that $\left[\bigoplus_{i\in J} X_i\right]_\infty \in \mathcal{E}$ for every nonempty finite subset $J \subseteq I$, and there exists a test family for $\mathcal{E}$ in $\left[\bigoplus_{i\in I} X_i\right]_{c_0}$, then $\left[\bigoplus_{i\in I} X_i\right]_{c_0} \in \mathcal{E}$.*

4. Applications

In this section, we will apply the abstract results to the examples discussed earlier. This will yield some new results as well as some alternative proofs of already known results.

We first collect what is known about sums of octahedral spaces and their relatives. The following results were proved in [15]: if X and Y are real Banach spaces, then

(a) X or Y is LOH/WOH/OH $\Rightarrow X \oplus_1 Y$ is LOH/WOH/OH.

(b) X and Y are LOH/WOH $\Rightarrow X \oplus_p Y$ is LOH/WOH for every $p \in (1, \infty]$.

(c) X and Y are OH $\Rightarrow X \oplus_\infty Y$ is OH.

(d) For $p \in (1, \infty)$ $X \oplus_p Y$ is never OH.

In [2], the following generalization was obtained: if I is an index set and E is a subspace of $\mathbb{R}^I$ with an absolute normalised norm, and $(X_i)_{i\in I}$ is a family of LOH spaces, then $\left[\bigoplus_{i\in I} X_i\right]_E$ is also LOH. If each X_i is WOH and, moreover, $\operatorname{span}\{e_i : i \in I\}$ is dense in E, then $\left[\bigoplus_{i\in I} X_i\right]_E$ is also WOH.

It is also easily checked that $\ell^\infty(X)$ is OH whenever X is OH (the proof is analogous to the proof of (c) above that was given in [15]).

Combining all this with our Theorems 3.1, 3.3 and the fact that OH, WOH and LOH can be described by test families (see Section 2), we obtain the following results.

Theorem 4.1. *If $(S, \mathcal{A}, \mu)$ is a complete σ-finite measure space, E is a Köthe function space over $(S, \mathcal{A}, \mu)$ and X is an LOH/WOH space such that the simple* *functions are dense in $E(X)$ (for instance, if E is order continuous), then $E(X)$ is also LOH/WOH.*

In particular, if $p \in [1, \infty)$ and X is LOH/WOH, then so is $L^p(\mu, X)$.

Also, $L^\infty(\mu, X)$ is LOH if X is LOH.

Proposition 4.2. *If $(S, \mathcal{A}, \mu)$ is a complete σ-finite measure space and X is an OH space, then $L^1(\mu, X)$ and $L^\infty(\mu, X)$ are also OH.*

This result is not optimal. In fact, it is not difficult to see that $L^1(\mu, X)$ is OH for *any* Banach space X (provided that $L^1(\mu)$ is infinite-dimensional), see the examples at the end of [22].

Now we turn to the diameter two properties. In [15], the following results were derived via duality from the corresponding results on octahedrality in direct sums.

(a) X or Y has the LD2P/D2P/SD2P $\Rightarrow$ $X \oplus_\infty Y$ has the LD2P/D2P/SD2P,

(b) X and Y have the LD2P/D2P $\Rightarrow$ $X \oplus_p Y$ has the LD2P/D2P for every $p \in [1, \infty)$,

(c) X and Y have the SD2P $\Rightarrow$ $X \oplus_1 Y$ has the SD2P,

(d) For $p \in (1, \infty)$, $X \oplus_p Y$ never has the SD2P.

All these results have been known before (they are scattered in [1, 3, 4, 13, 26], see [15] for a detailed account), but the previous proofs were based on different methods. In [3], it was shown that the LD2P and the D2P are stable under sums with respect to an arbitrary absolute norm.

Since LD2P, D2P and SD2P can be described by test families (see Section 2), we obtain the following stability result from Theorem 3.1.

Theorem 4.3. *Let $(S, \mathcal{A}, \mu)$ be a complete σ-finite measure space, E be a Köthe function space over $(S, \mathcal{A}, \mu)$ and X be a Banach space such that the simple functions are dense in $E(X)$ (for instance, if E is order continuous). If X has the LD2P/D2P, then $E(X)$ also has the LD2P/D2P.*

In particular, if $p \in [1, \infty)$ and X has the LD2P/D2P, then so does $L^p(\mu, X)$.

Further, if X has the SD2P, then $L^1(\mu, X)$ also has the SD2P.

In [3] it was already proved that $L^p(\mu, X)$ has the D2P whenever $1 \leq p < \infty$, μ is a finite measure and X has the D2P (this proof also uses simple functions). Also, for the special case $p = 1$, better results are already known, for instance, it has been proved in [4, Theorem 2.13] that for a finite measure μ the space $L^1(\mu, X)$ has the D2P if and only if X has the D2P or μ has no atoms (and $L^\infty(\mu, X)$ has the D2P if and only if $L^\infty(\mu)$ is infinite-dimensional or X has the D2P).

Even more, it is known that the Daugavet property implies the SD2P (see [1, Theorem 4.4]) and that $L^1(\mu, X)$ and $L^\infty(\mu, X)$ have the Daugavet property for any atomless measure μ and any Banach space X ([31], see the discussion for the DP below).

Also, if X or Y has the LD2P, then so does $X\hat{\otimes}_\pi Y$ (see [1, Theorem 2.7]), and if X and Y have the SD2P, then so does $X\hat{\otimes}_\pi Y$ (see [7]), where $\hat{\otimes}_\pi$ denotes the projective tensor product, and it is well known that $L^1(\mu, X) = L^1(\mu)\hat{\otimes}_\pi X$. For more information on octahedrality and related properties in tensor products see also [22, 23].

For AOH spaces, the following equivalent characterisation can be proved: X is AOH if and only if for every $n \in \mathbb{N}$, all $x_1, \dots, x_n \in S_X$ and each $\varepsilon > 0$ there is some $y \in S_X$ such that

$$\max\{\|x_i + ty\|, \|x_i - ty\|\} \geq (1 - \varepsilon)(1 + t) \quad \forall t > 0, \forall i \in \{1, \dots, n\}.$$

The proof is analogous to the proof of the corresponding characterisation for octahedral spaces in [15] and will therefore be skipped.

Using this characterisation, one can show that $X \oplus_1 Y$ is AOH if X or Y is AOH and that $X \oplus_\infty Y$ is AOH if X and Y are AOH. The latter result also extends to $\ell^\infty(X)$. Again the proofs are analogous to those for the corresponding results on OH spaces in [15] and thus we will skip them.

Using our Theorems 3.1 and 3.3, we can now obtain the following result.

Proposition 4.4. *If $(S, \mathcal{A}, \mu)$ is a complete σ-finite measure space and X is an AOH space, then $L^1(\mu, X)$ and $L^\infty(\mu, X)$ are also AOH.*

Again, if $L^1(\mu)$ is infinite-dimensional, then $L^1(\mu, X)$ is even OH for *any* Banach space X ([22]).

Concerning the sums of ASQ and LASQ spaces, the following was proved in [2]: if I is any index set, E is a subspace of $\mathbb{R}^I$ with an absolute normalised norm, and $(X_i)_{i\in I}$ is a family of LASQ spaces, then $\left[\bigoplus_{i\in I} X_i\right]_E$ is also LASQ. Further, $X \oplus_\infty Y$ is ASQ/LASQ if and only if X or Y is ASQ/LASQ. Analogously to the proof of the "if" part in [2], one can show that $\ell^\infty(X)$ is ASQ/LASQ whenever X is ASQ/LASQ (it has also been proved in [2] that for $p \in [1, \infty)$ the sum $X \oplus_p Y$ is never ASQ).

If we combine these facts with Theorems 3.1, 3.3 and the fact that ASQ and LASQ can be expressed in terms of test families (Section 2), we obtain the following stability result.

Theorem 4.5. *If $(S, \mathcal{A}, \mu)$ is a complete σ-finite measure space, E is a Köthe function space over $(S, \mathcal{A}, \mu)$ and X is an LASQ space such that the simple functions are dense in $E(X)$ (for instance, if E is order continuous), then $E(X)$ is also LASQ.*

In particular, if $p \in [1, \infty)$ and X is LASQ, then so is $L^p(\mu, X)$.

Moreover, $L^\infty(\mu, X)$ is ASQ/LASQ whenever X is ASQ/LASQ.

Now we consider spaces with the Daugavet property. It has been shown in [20] that $L^1([0, 1], X)$ and $L^\infty([0, 1], X)$ have the DP if X has it. More generally, $L^1(\mu, X)$ has the DP for every *atomless* measure μ and *every* Banach space X, see [31, p.81].

In [32], it was already proved that the ℓ^1- or ℓ^∞-sum of any (finite or infinite) sequence of Banach spaces with the Daugavet property has again the Daugavet property (the Daugavet property for weakly compact operators was considered in [32], but this is equivalent to considering just rank-one operators by [18, Theorem 2.3]). In [18], a different proof for the stability of the DP by finite or infinite ℓ^1- and c_0-sums has been given (the cases of infinite sums are reduced to the corresponding finite sums by a density argument, similarly to the general reduction results for sums that we have stated in Section 3).

Putting everything together, the following characterisation was obtained in [27, Remark 9]: $L^1(\mu, X)$ has the DP if and only if X has the DP or μ has no atoms. Likewise, $L^\infty(\mu, X)$ has the DP if and only if X has the DP or μ has no atoms (see [28]).

Analogous results also hold for the alternative Daugavet property: the space $L^1(\mu, X)$ has the ADP if and only if X has the ADP or μ has no atoms if and only if $L^\infty(\mu, X)$ has the ADP (see [29]). Also, the ADP is stable under arbitrary ℓ^1-, c_0- and ℓ^∞-sums (see again [29]).

Using the stability results for sums and our Theorems 3.1 and 3.3, we obtain an alternative proof of the following known result.

Theorem 4.6. *If $(S, \mathcal{A}, \mu)$ is a complete σ-finite measure space and X is a Banach space with the DP/ADP, then $L^1(\mu, X)$ and $L^\infty(\mu, X)$ also have the DP/ADP.*

Concerning lush spaces, the following has been proved in [10]: if $\|\cdot\|_E$ is an absolute norm on $\mathbb{R}^n$, then the sum of every collection $X_1, \dots, X_n$ of lush spaces with respect to $\|\cdot\|_E$ is again lush if and only if $(\mathbb{R}^n, \|\cdot\|_E)$ is lush.

It was also proved in [10] that the ℓ^1-, c_0- and ℓ^∞-sums of any family $(X_i)_{i\in I}$ of lush spaces are again lush (also here the cases of ℓ^1- and c_0-sums are reduced to the corresponding finite sums).

Recently the following stability result has been proved in [19, Corollaries 8.10 and 8.13].

Theorem 4.7 ([19]). *Let $(S, \mathcal{A}, \mu)$ be a σ-finite measure space and X be a Banach space. Then $L^\infty(\mu, X)$ is lush if and only if X is lush if and only if $L^1(\mu, X)$ is lush.*

In fact, even more general results are proved in [19] for the so-called lush operators.

If we use instead the above-mentioned results on sums of lush spaces in combination with our Theorems 3.1 and 3.3, we obtain an alternative proof for the fact that lushness of X is sufficient for lushness of $L^1(\mu, X)$ and $L^\infty(\mu, X)$ (the proofs in [19] did not use a reduction to sums, but they also used the density of the simple functions (respectively, functions with countable range) in $L^1(\mu, X)$ (respectively, $L^\infty(\mu, X)$).

Let us now turn to generalised lushness. It was proved in [30] that the property GL is stable under arbitrary ℓ^1-, c_0- and ℓ^∞-sums. The same results also hold for the property $(**)$, with completely analogous proofs.

Now we can apply our Theorems 3.1 and 3.3 to obtain the following result.

Theorem 4.8. *If $(S, \mathcal{A}, \mu)$ is a complete σ-finite measure space and X is a Banach space with property $(**)$, then $L^1(\mu, X)$ and $L^\infty(\mu, X)$ also have the property $(**)$.*

We recall (see Subsection 2.4) that $(**)$ implies the MUP and $(**)$ is equivalent to GL for separable spaces, but it is not known whether this equivalence is true in general nor if it is in general possible to describe the property GL by test families. Thus we cannot apply our general reduction theorems directly to GL-spaces. However, it is still possible to show that GL is stable with respect to L^1-Bochner spaces by a similar proof technique. This is carried out in the next section.

5. GL-spaces

Here we show directly that $L^1(\mu, X)$ is GL whenever X is GL. The argument is similar to the proof for ℓ^1-sums given in [30] in combination with an approximation by simple functions.

Theorem 5.1. *Let $(S, \mathcal{A}, \mu)$ be a complete σ-finite measure space. If X is a GL-space, then so is $L^1(\mu, X)$.*

Proof. Let $f \in L^1(\mu, X)$ with $\|f\|_1 = 1$ and let $\varepsilon \in (0,1)$. Choose $\eta \in (0,1)$ such that

$$\begin{aligned}
&(2+\varepsilon/2)(1+\eta) + 4\eta < 2 + \varepsilon, \\
&(1-\varepsilon/2)(1-\eta) - \eta > 1 - \varepsilon, \\
&(1-\varepsilon/2)\frac{1-\eta}{1+\eta} > 1 - \varepsilon.
\end{aligned}$$

We can find a simple function g on S such that $\|f - g\|_1 \leq \eta$. Write $g = \sum_{i=1}^N x_i \chi_{A_i}$ with pairwise disjoint sets $A_1, \ldots, A_N \in \mathcal{A}$ and $x_1, \ldots, x_N \in X$.

Since X is GL, we can find functionals $x_1^*, \ldots, x_N^* \in S_{X^*}$ such that $x_i^*(x_i) \geq (1-\varepsilon/2)\|x_i\|$ and

$$\mathrm{d}(y, S(x_i^*, \varepsilon/2)) + \mathrm{d}(y, -S(x_i^*, \varepsilon/2)) < 2 + \frac{\varepsilon}{2} \quad \forall y \in S_X. \tag{5.1}$$

Let $h = \sum_{i=1}^N x_i^* \chi_{A_i}$ and $\varphi(v) = \int_S h(s)(v(s))\,\mathrm{d}\mu(s)$ for $v \in L^1(\mu, X)$. Then $\varphi \in L^1(\mu, X)^*$ with $\|\varphi\| = 1$.

We further have

$$\varphi(g) = \sum_{i=1}^N \int_{A_i} x_i^*(x_i)\,\mathrm{d}\mu(s) \geq (1-\varepsilon/2) \sum_{i=1}^N \int_{A_i} \|x_i\|\,\mathrm{d}\mu(s) = (1-\varepsilon/2)\|g\|_1.$$

Since $\|f-g\|_1 \leq \eta$ and $\|f\|_1 = 1$, it follows that $\varphi(f) \geq \varphi(g) - \eta \geq (1-\varepsilon/2)(1-\eta) - \eta$. Thus the choice of η implies $f \in S(\varphi, \varepsilon)$.

Now take any function $w \in L^1(\mu, X)$ with $\|w\|_1 = 1$. There exists a simple function $\tilde{w}$ on S such that $\|w - \tilde{w}\|_1 \leq \eta$. Write $\tilde{w} = \sum_{j=1}^M y_j \chi_{B_j}$ with pairwise disjoint sets $B_1, \ldots, B_M \in \mathcal{A}$ and $y_1, \ldots, y_M \in X$.

We put $C_{ij} := A_i \cap B_j$ for $(i,j) \in I := \{1,\dots,N\} \times \{1,\dots,M\}$. By (5.1), we can find, for each pair $(i,j) \in I$, vectors $u_{ij}, v_{ij} \in B_X$ such that $x_i^*(u_{ij}) > 1 - \varepsilon/2$, $-x_i^*(v_{ij}) > 1 - \varepsilon/2$ and

$$\|y_j - \|y_j\|u_{ij}\| + \|y_j - \|y_j\|v_{ij}\| \le (2+\varepsilon/2)\|y_j\|. \tag{5.2}$$

Let $u = \sum_{(i,j)\in I} \|y_j\| u_{ij} \chi_{C_{ij}}$ and $v = \sum_{(i,j)\in I} \|y_j\| v_{ij} \chi_{C_{ij}}$.

Since $\|u_{ij}\| \le 1$, we have $\|u(s)\| \le \|y_j\| = \|\tilde{w}(s)\|$ for all $s \in C_{ij}$ and all $(i,j) \in I$. Hence $\|u\|_1 \le \|\tilde{w}\|_1 \le \|w - \tilde{w}\|_1 + \|w\|_1 \le 1 + \eta$. Analogously, one can see that $\|v\|_1 \le 1 + \eta$.

Thus we have $\tilde{u} := u/(1+\eta) \in B_{L^1(\mu,X)}$ and $\tilde{v} := v/(1+\eta) \in B_{L^1(\mu,X)}$.

We further have

$$\begin{aligned}\varphi(\tilde{u}) &= \frac{1}{1+\eta} \sum_{(i,j)\in I} \int_{C_{ij}} x_i^*(u_{ij}) \|y_j\| \, \mathrm{d}\mu(s) \ge \frac{1-\varepsilon/2}{1+\eta} \sum_{(i,j)\in I} \int_{C_{ij}} \|y_j\| \, \mathrm{d}\mu(s) \\ &= \|\tilde{w}\|_1 \frac{1-\varepsilon/2}{1+\eta} \ge (1-\eta)\frac{1-\varepsilon/2}{1+\eta} > 1 - \varepsilon.\end{aligned}$$

Thus $\tilde{u} \in S(\varphi,\varepsilon)$, and analogously one can show that $\tilde{v} \in -S(\varphi,\varepsilon)$.

It further follows from (5.2) that

$$\|\tilde{w}(s) - u(s)\| + \|\tilde{w}(s) - v(s)\| \le (2+\varepsilon/2)\|\tilde{w}(s)\| \quad \forall s \in S.$$

Hence, $\|\tilde{w} - u\|_1 + \|\tilde{w} - v\|_1 \le (2+\varepsilon/2)\|\tilde{w}\|_1 \le (2+\varepsilon/2)(1+\eta)$.

Since $\|w - \tilde{w}\|_1 \le \eta$, we get $\|w - u\|_1 + \|w - v\|_1 \le (2+\varepsilon/2)(1+\eta) + 2\eta$.

We also have $\|u - \tilde{u}\|_1 \le \eta$ and $\|v - \tilde{v}\|_1 \le \eta$. Thus, $\|w - \tilde{u}\|_1 + \|w - \tilde{v}\|_1 \le (2+\varepsilon/2)(1+\eta) + 4\eta < 2 + \varepsilon$, and we complete the proof. □

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Про певні геометричні властивості в банахових просторах вектор-значних функцій

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Розглянуто певний тип геометричних властивостей банахових просторів, куди входять, наприклад, октаедричність, майже квадратність, пухлість та властивість Даугавета. Для цього типу властивостей отримано загальну теорему редукції, яка стверджує приблизно таке: якщо властивість, про яку йдеться, стабільна за певних скінчених абсолютних сум (наприклад, скінчених ℓ^p-сум), то вона також стабільна при утворенні відповідних просторів Кете–Бохнера (наприклад, L^p-бохнерових просторів). З цієї загальної теореми отримано в якості наслідків декілька нових результатів, а також деякі альтернативні доведення вже відомих результатів, що стосуються восьмигранного та майже квадратного просторів та їхніх похідних, властивостей діаметру 2, пухлих просторів та інших класів.

Ключові слова: абсолютні суми, простори Кете–Бохнера, простори Лебега–Бохнера, восьмигранні простори, майже квадратні простори, властивості діаметру 2, пухлі простори, узагальнені пухлі простори, властивість Даугавета