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Certain Functions Defined in Terms of Cantor Series

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The present paper is devoted to certain examples of functions whose argument is represented in terms of Cantor series.

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1. Introduction

Let (a_k), where $k = 1, 2, \dots$, be a sequence of all rational numbers and every rational number be included only one time. The function

$$f(x) = \sum_{a_k < x} 2^{-k}$$

is increasing in the whole real axes, has the range in $(0, 1)$ and jumps at rational points.

The next function,

$$g(x) = \sum_{n=1}^{\infty} \left[x n^K \right] / n!,$$

is a strictly increasing function of $x > 0$ which does not take rational values (see [4, Remark to Corollary 3.4]). Here K is an arbitrary positive integer and $[y]$ is an integer part of y. By analogy, the function

$$\sum_{n=1}^{\infty} [\gamma n^{\alpha}]/n!$$

is a strictly monotonic function of $\alpha \geq 0$ and $\gamma > 0$ without rational values (see [4, Remark for Corollary 3.5]).

The present paper is devoted to certain functions defined in terms of positive Cantor series that are singular or non-differentiable.

Let $Q \equiv (q_k)$ be a fixed sequence of positive integers, $q_k > 1$, Θ_k be a sequence of the sets $\Theta_k \equiv \{0, 1, \dots, q_k - 1\}$, and $\varepsilon_k \in \Theta_k$.

The Cantor series expansion

$$\frac{\varepsilon_1}{q_1} + \frac{\varepsilon_2}{q_1 q_2} + \dots + \frac{\varepsilon_k}{q_1 q_2 \dots q_k} + \dots \tag{1.1}$$

of $x \in [0,1]$ was first studied by G. Cantor in [2]. It is easy to see that the Cantor series expansion is the q-ary expansion

$$\frac{\alpha_1}{q} + \frac{\alpha_2}{q^2} + \dots + \frac{\alpha_k}{q^k} + \dots$$

of numbers from the closed interval $[0,1]$ whenever the condition $q_k = q$ holds for all positive integers k. Here q is a fixed positive integer, $q > 1$, and $\alpha_k \in \{0, 1, \dots, q-1\}$.

By $x = \Delta^Q_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_k \dots}$, denote any number $x \in [0,1]$ represented by series (1.1). This notation is called *the representation of x by Cantor series* (1.1).

We note that certain numbers from $[0,1]$ have two different representations by Cantor series (1.1), i.e.,

$$\Delta^Q_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{m-1} \varepsilon_m 000 \cdots} = \Delta^Q_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{m-1} [\varepsilon_m - 1][q_{m+1}-1][q_{m+2}-1] \cdots} = \sum_{i=1}^{m} \frac{\varepsilon_i}{q_1 q_2 \cdots q_i}.$$

Such numbers are called *Q-rational*. The other numbers in $[0,1]$ are called *Q-irrational*.

Let $c_1, c_2, \dots, c_m$ be an ordered tuple of integers such that $c_i \in \{0, 1, \dots, q_i - 1\}$ for $i = \overline{1, m}$.

A cylinder $\Delta^Q_{c_1 c_2 \cdots c_m}$ of rank m with base $c_1 c_2 \cdots c_m$ is a set of the form

$$\Delta^Q_{c_1 c_2 \cdots c_m} \equiv \{x : x = \Delta^Q_{c_1 c_2 \cdots c_m \varepsilon_{m+1} \varepsilon_{m+2} \cdots \varepsilon_{m+k} \cdots}\},$$

i.e., any cylinder $\Delta^Q_{c_1 c_2 \cdots c_m}$ is a closed interval of the form

$$\left[\Delta^Q_{c_1 c_2 \cdots c_m 000}, \Delta^Q_{c_1 c_2 \cdots c_m [q_{m+1}-1][q_{m+2}-1][q_{m+3}-1] \cdots}\right].$$

Define *the shift operator σ of expansion* (1.1) by the rule

$$\sigma(x) = \sigma\left(\Delta^Q_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_k \cdots}\right) = \sum_{k=2}^{\infty} \frac{\varepsilon_k}{q_2 q_3 \cdots q_k} = q_1 \Delta^Q_{0 \varepsilon_2 \cdots \varepsilon_k \cdots}.$$

It is easy to see that

$$\begin{aligned} \sigma^n(x) &= \sigma^n\left(\Delta^Q_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_k \cdots}\right) \\ &= \sum_{k=n+1}^{\infty} \frac{\varepsilon_k}{q_{n+1} q_{n+2} \cdots q_k} = q_1 \cdots q_n \Delta^Q_{\underbrace{0 \dots 0}_{n} \varepsilon_{n+1} \varepsilon_{n+2} \cdots}. \end{aligned}$$

Therefore,

$$x = \sum_{i=1}^{n} \frac{\varepsilon_i}{q_1 q_2 \cdots q_i} + \frac{1}{q_1 q_2 \cdots q_n} \sigma^n(x). \tag{1.2}$$

In [13], the notion of the shift operator of the alternating Cantor series is studied in detail.

In [7], Salem modeled the function

$$s(x) = s\left(\Delta^2_{\alpha_1\alpha_2\cdots\alpha_n\cdots}\right) = \beta_{\alpha_1} + \sum_{n=2}^{\infty}\left(\beta_{\alpha_n}\prod_{i=1}^{n-1} q_i\right) = y = \Delta^{Q_2}_{\alpha_1\alpha_2\cdots\alpha_n\cdots},$$

where $q_0 > 0$, $q_1 > 0$, and $q_0 + q_1 = 1$. It is a singular function. However, generalizations of the Salem function can be non-differentiable functions or not have the derivative on a certain set. Some parers (see, for example, [9, 10, 15]) are devoted to modeling and studying generalizations of the Salem function.

In the present paper, two examples of certain functions with complex local structure are constructed and investigated.

Suppose that the condition $q_n \le q$ holds for all positive integers n. The first function has the form

$$f:\ x = \Delta^{Q}_{\varepsilon_1\varepsilon_2\cdots\varepsilon_n\cdots} \to \Delta^{q}_{\varepsilon_1\varepsilon_2\cdots\varepsilon_n\cdots} = y.$$

This function is interesting since the generalization of the Salem function studied in [9] can be represented as

$$F(x) = F_{\xi,Q} \circ f.$$

Here, $\circ$ denotes the composition of functions. Also, the function $F_{\xi,Q}$ is the function of the type

$$F_{\eta,Q}(y) = \beta_{\varepsilon_1(y),1} + \sum_{k=2}^{\infty}\left(\beta_{\varepsilon_k(y),k}\prod_{j=1}^{k-1} p_{\varepsilon_j(y),j}\right),$$

where $y = \Delta^{q}_{\varepsilon_1\varepsilon_2\cdots\varepsilon_n\cdots}$.

Note that the function $F_{\eta,q}$ is a distribution function of a certain random variable η whenever the elements $p_{i,n}$ of the matrix P, described in the last-mentioned examples, are non-negative.

Remark 1.1. Let η be a random variable defined by the q-ary expansion, i.e.,

$$\eta = \frac{\xi_1}{q} + \frac{\xi_2}{q^2} + \frac{\xi_3}{q^3} + \cdots + \frac{\xi_k}{q^k} + \cdots \equiv \Delta^{q}_{\xi_1\xi_2\cdots\xi_k\cdots},$$

where the digits ξ_k $(k = 1, 2, 3, \dots)$ are random and take the values $0, 1, \dots, q-1$ with probabilities $p_{0,k}, p_{1,k}, \dots, p_{q-1,k}$. That is, ξ_k are independent and $P\{\xi_k = i_k\} = p_{i_k,k}$, $i_k \in \Theta = \{0, 1, \dots, q-1\}$.

From the definition of the distribution function and the following expressions for $x = \Delta^{q}_{\alpha_1\alpha_2\cdots\alpha_k\cdots}$:

$$\begin{aligned}\{\eta < x\} = \{\xi_1 < \alpha_1(x)\} \cup \{\xi_1 = \alpha_1(x), \xi_2 < \alpha_2(x)\} \cup \cdots \\ \cup \{\xi_1 = \alpha_1(x), \xi_2 = \alpha_2(x), \dots, \xi_k < \alpha_k(x)\} \cup \cdots,\end{aligned}$$

$$P\{\xi_1 = \alpha_1(x), \xi_2 = \alpha_2(x), \dots, \xi_k < \alpha_k(x)\} = \beta_{\alpha_k(x),k} \prod_{j=1}^{k-1} p_{\alpha_j(x),j}$$

we get that the distribution function $F_{\eta,q}$ of the random variable η has the form

$$F_{\eta,q}(x) = \begin{cases} 0 & \text{for } x < 0 \\ \beta_{\alpha_1(x),1} + \sum_{k=2}^{\infty} \left[\beta_{\alpha_k(x),k} \prod_{j=1}^{k-1} p_{\alpha_j(x),j}\right] & \text{for } 0 \le x < 1 \\ 1 & \text{for } x \ge 1 \end{cases}$$

since the conditions $F_{\eta,q}(0) = 0$, $F_{\eta,q}(1) = 1$ hold and $F_{\eta,q}$ is a continuous, monotonic and non-decreasing function. Most generalizations of the Salem function were studied in [11]).

Remark 1.2. In the general case, suppose that (f_n) is a finite or infinite sequence of certain functions (the sequence may contain functions with complicated local structure). Let us consider the corresponding composition of the functions

$$\cdots \circ f_n \circ \cdots \circ f_2 \circ f_1 = f_{c,\infty}$$

or

$$f_n \circ \cdots \circ f_2 \circ f_1 = f_{c,n}.$$

Also, we can take a certain part of the composition, i.e.,

$$f_{n_0+t} \circ \cdots \circ f_{n_0+1} \circ f_{n_0} = f_{c,\overline{n_0,n_0+t}},$$

where n_0 is a fixed positive integer (a number from the set $\mathbb{N}$), $t \in \mathbb{Z}_0 = \mathbb{N} \cup \{0\}$ and $n_0 + t \le n$.

One can use this technique for modeling and studying the functions with complex local structure. Also, one can use new representations of real numbers (numeral systems) of the type

$$x' = \Delta^{f_{c,\infty}}_{i_1 i_2 \cdots i_n} = \cdots \circ f_n \circ \cdots \circ f_2 \circ f_1(x),$$
$$x' = \Delta^{f_{c,n}}_{i_1 i_2 \cdots i_n} = f_n \circ \cdots \circ f_2 \circ f_1(x)$$

or

$$z' = \Delta^{f_{c,\overline{n_0,n_0+t}}}_{i_1 i_2 \cdots i_n} = f_{n_0+t} \circ \ldots \circ f_{n_0+1} \circ f_{n_0}(z)$$

in fractal theory, applied mathematics, etc.

The second map considered in this paper can be used for modeling fractals in the space $\mathbb{R}^2$:

$$f : x = \Delta^q_{\underbrace{u \cdots u}_{\alpha_1 - 1} \alpha_1 \underbrace{u \cdots u}_{\alpha_2 - 1} \alpha_2 \cdots \underbrace{u \cdots u}_{\alpha_n - 1} \alpha_n \cdots} \to \Delta^q_{\alpha_1 \alpha_2 \cdots \alpha_n \cdots},$$

where $u \in \{0, 1, \dots, q-1\}$ is a fixed number, $\alpha_n \in \{1, 2, \dots, q-1\} \setminus \{u\}$, and $3 < q$ is a fixed positive integer. It is easy to see that one can consider this map defined in terms of other representations of real numbers (e.g., the Q_s, Q^*, Q_s^*, $\tilde{Q}$, the nega-$\widetilde{Q}$-representations and other positive and alternating representations). Actually, the functions with complex local structure defined in terms of different representations of real numbers, as well as their compositions, are useful for modeling fractals (the Moran sets) in $\mathbb{R}^2$. Regularity properties of different sets under the map generated by the functions with complex local structure and their compositions are interesting and unknown.

2. One function defined in terms of positive Cantor series

Let us consider the function

$$f(x) = f\left(\Delta^Q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots}\right) = f\left(\sum_{n=1}^{\infty} \frac{\varepsilon_n}{q_1 q_2 \cdots q_n}\right) = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{q^n} = \Delta^q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots} = y,$$

where $\varepsilon_n \in \Theta_n$ and the condition $q_n \le q$ holds for all positive integers n.

Lemma 2.1 (On the well-posedness of the definition of the function)**.** *The values of the function f for different representations of Q-rational numbers from $[0,1]$ are:*

- *identical whenever for all positive integers n the condition $q_n = q$ holds;*
- *different whenever for all positive integers n the condition $q_n < q$ holds;*
- *different for numbers from no more than a countable subset of Q-rational numbers whenever there exists a finite or infinite subsequence n_k of positive integers such that $q_{n_k} < q$ for all positive integer values of k.*

Proof. Let x be a Q-rational number. Then there exists a number n_0 such that

$$x = x_1 = \Delta^Q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{n_0-1}\varepsilon_{n_0}000\ldots} = \Delta^Q_{\varepsilon_1\varepsilon_2\cdots\varepsilon_{n_0-1}[\varepsilon_{n_0}-1][q_{n_0+1}-1][q_{n_0+2}-1][q_{n_0+3}-1]\cdots} = x_2.$$

Whence,

$$f(x_1) = \Delta^q_{\varepsilon_1\varepsilon_2\cdots\varepsilon_{n_0-1}\varepsilon_{n_0}000\ldots}, \; f(x_2) = \Delta^q_{\varepsilon_1\varepsilon_2\cdots\varepsilon_{n_0-1}[\varepsilon_{n_0}-1][q_{n_0+1}-1][q_{n_0+2}-1][q_{n_0+3}-1]\cdots}$$

and

$$f(x_2) - f(x_1) = -\frac{1}{q^{n_0}} + \sum_{n=n_0+1}^{\infty} \frac{q_n - 1}{q^n} \le 0.$$

Thus, certain Q-rational points are the points of discontinuity of the function. It is easy to see that $f(x_2) - f(x_1) = 0$ if the condition $q_n = q$ holds for all positive integers n.

From the unique representation for each Q-irrational number from $[0,1]$, it follows that the function f is well defined at any Q-irrational point. □

Remark 2.2. We should not consider the representation

$$\Delta^{Q}_{\varepsilon_1\varepsilon_2\cdots\varepsilon_{n-1}[\varepsilon_n-1][q_{n+1}-1][q_{n+2}-1][q_{n+3}-1]\cdots}$$

to get the function f to be well defined on the set of Q-rational numbers from $[0,1]$.

Lemma 2.3. *The function f has the following properties:*

1. $D(f)=[0,1]$, *where $D(f)$ is the domain of f.*
2. *Let $E(f)$ be the range of f. Then:*
   - $E(f)=[0,1]$ *whenever the condition $q_n=q$ holds for all positive integers n,*
   - $E(f)=[0,1]\setminus C_f$, *where* $C_f=C_1\cup C_2$, $C_1=\left\{y: y=\Delta^{q}_{\varepsilon_1\varepsilon_2\cdots\varepsilon_n}, \varepsilon_n\notin\{q_n,q_n+1,\dots,q-1\} \text{ for all } n \text{ such that } q_n<q\right\}$ *and* $C_2=\left\{y: y=\Delta^{q}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{n-1}[\varepsilon_n-1][q_{n+1}-1][q_{n+2}-1][q_{n+3}-1]\cdots}\right\}$.
3. $f(x)+f(1-x)=f(1)\le 1$.
4. $f\left(\sigma^k(x)\right)=\sigma^k\left(f(x)\right)$ *for any $k\in\mathbb{N}$.*

Proof. Property 1 follows from the definition of f. *Property* 2 follows from Lemma 2.1.

Let us prove *property* 3. Since

$$1-x=\sum_{n=1}^{\infty}\frac{q_n-1-\varepsilon_n}{q_1q_2\cdots q_n},$$

we have

$$f(1-x)=\sum_{n=1}^{\infty}\frac{q_n-1-\varepsilon_n}{q^n}.$$

Whence,

$$f(x)+f(1-x)=\sum_{n=1}^{\infty}\frac{\varepsilon_n}{q^n}+\sum_{n=1}^{\infty}\frac{q_n-1-\varepsilon_n}{q^n}=\sum_{n=1}^{\infty}\frac{q_n-1}{q^n}=f(1)\le 1.$$

Note that the last inequality is an equality where $y=x$, i.e., the condition $q_n=q$ holds for all positive integers n.

Let us prove *property* 4. We have

$$\begin{aligned} f\left(\sigma^k(x)\right)&=f\left(\sum_{j=k+1}^{\infty}\frac{\varepsilon_j}{q_{k+1}q_{k+2}\cdots q_j}\right)\\ &=\sum_{j=k+1}^{\infty}\frac{\varepsilon_j}{q^{j-k}}=\sigma^k\left(\sum_{n=1}^{\infty}\frac{\varepsilon_n}{q^n}\right)=\sigma^k\left(f(x)\right). \end{aligned}$$

The lemma is proved. ∎

Lemma 2.4. *The function f is continuous at Q-irrational points from $[0,1]$.*

The function f is continuous at all Q-rational points from $[0,1]$ if the condition $q_n = q$ holds for all positive integers n.

If there exist positive integers n such that $q_n < q$, then the points of the type

$$\Delta^Q_{\varepsilon_1\varepsilon_2\cdots\varepsilon_{n-1}\varepsilon_n 000\cdots} \quad \textit{and} \quad \Delta^Q_{\varepsilon_1\varepsilon_2\cdots\varepsilon_{n-1}[\varepsilon_n-1][q_{n+1}-1][q_{n+2}-1]\cdots}$$

are the points of discontinuity of the function.

Proof. Let $x = \Delta^Q_{\varepsilon_1\varepsilon_2\cdots\varepsilon_n\cdots} \in [0,1]$ be an arbitrary number.

Let x_0 be an Q-irrational number.

Then there exists $n_0 = n_0(x)$ such that

$$\begin{cases} \varepsilon_m(x) = \varepsilon_m(x_0) \quad \text{for} \quad m = \overline{1, n_0 - 1} \\ \varepsilon_{n_0}(x) \neq \varepsilon_{n_0}(x_0). \end{cases}$$

From the system, it follows that the conditions $x \to x_0$ and $n_0 \to \infty$ are equivalent and

$$|f(x) - f(x_0)| = \left| \sum_{j=n_0}^{\infty} \frac{\varepsilon_j(f(x)) - \varepsilon_j(f(x_0))}{q^k} \right| \leq \sum_{j=n_0}^{\infty} \frac{|\varepsilon_j(f(x)) - \varepsilon_j(f(x_0))|}{q^k}$$
$$\leq \sum_{j=n_0}^{\infty} \frac{q-1}{q^k} = \frac{1}{q^{n_0-1}} \to 0 \quad \text{as } n_0 \to \infty.$$

So, the function f is continuous at Q-irrational points. That is,

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Let $x_0 = \Delta^Q_{\varepsilon_1\varepsilon_2\cdots\varepsilon_n\cdots}$ be a Q-rational number.

If the condition $q_n < q$ holds for a certain $n \in \mathbb{N}$, then $q_n \leq q - 1$ and $q_n - 1 \leq q - 2$. That is,

$$\varepsilon_n \in \Theta_n = \{0, 1, \ldots, q_n - 1\} \subseteq \{0, 1, \ldots, q - 2\}.$$

Since

$$\lim_{x \to x_0 - 0} f(x) = \Delta^q_{\varepsilon_1\varepsilon_2\cdots\varepsilon_{n-1}[\varepsilon_n-1][q_{n+1}-1][q_{n+2}-1]\cdots}$$

and

$$\lim_{x \to x_0 + 0} f(x) = \Delta^q_{\varepsilon_1\varepsilon_2\cdots\varepsilon_{n-1}\varepsilon_n 000\cdots},$$

we obtain

$$\Delta_f = \lim_{x \to x_0 + 0} f(x) - \lim_{x \to x_0 - 0} f(x) = \frac{1}{q^n} - \sum_{j=n+1}^{\infty} \frac{q_j - 1}{q^j} \geq 0.$$

Notice that

$$\Delta_f \geq \frac{1}{q^n} - \sum_{j=n+1}^{\infty} \frac{q-2}{q^j} = \frac{1}{(q-1)q^n}$$

and

$$\Delta_f \leq \frac{1}{q^n} - \sum_{j=n+1}^{\infty} \frac{1}{q^j} = \frac{q-2}{(q-1)q^n}.$$

Thus, x_0 is a point of discontinuity for $q_n < q$ and

$$\frac{1}{(q-1)q^n} \leq \Delta_f \leq \frac{q-2}{(q-1)q^n}. \qquad \square$$

Lemma 2.5. *The function f is strictly increasing.*

Proof. Let $x_1 = \Delta^Q_{\alpha_1\alpha_2\cdots\alpha_n\cdots}$ and $x_2 = \Delta^Q_{\varepsilon_1\varepsilon_2\cdots\varepsilon_n\cdots}$ such that $x_1 < x_2$. Then there exists n_0 such that $\alpha_i = \varepsilon_i$ for $i = \overline{1, n_0 - 1}$ and $\alpha_{n_0} < \varepsilon_{n_0}$. So,

$$f(x_2) - f(x_1) = \frac{\varepsilon_{n_0} - \alpha_{n_0}}{q^{n_0}} + \sum_{j=n_0+1}^{\infty} \frac{\varepsilon_j - \alpha_j}{q^j}.$$

Since $\varepsilon_{n_0} > \alpha_{n_0}$ and $q_n \leq q$, we have

$$f(x_2) - f(x_1) > \frac{1}{q^{n_0}} - \sum_{j=n_0+1}^{\infty} \frac{q_j - 1}{q^j} \geq \frac{1}{q^{n_0}} + \sum_{j=n_0+1}^{\infty} \frac{1-q}{q^j} = 0. \qquad \square$$

Theorem 2.6 (On differential properties)**.**

- *If the condition $q_n = q$ holds for all positive integers n, then $f^{'}(x_0) = 1$.*
- *If for all n the condition $q_n < q$ holds or there exists only a finite number of n such that $q_n = q$, then f is a singular function.*
- *If there exists only a finite number of n such that $q_n < q$, then f is non-differentiable.*
- *If there exists an infinite subsequence (n_k) of positive integers such that $q_{n_k} < q$, then f is a singular function.*

Proof. Suppose $x_0 = \Delta^Q_{\varepsilon_1\varepsilon_2\cdots\varepsilon_{m-1}c\varepsilon_{m+1}\cdots}$, where c is a fixed number from $\{0, 1, \dots, q_m - 1\}$, and (x_m) is a sequence of numbers $x_m = \Delta^Q_{\varepsilon_1\varepsilon_2\cdots\varepsilon_{m-1}\varepsilon_m\varepsilon_{m+1}\cdots}$. Then

$$x_m - x_0 = \frac{\varepsilon_m - c}{q_1 q_2 \cdots q_m} \quad \text{and} \quad f(x_m) - f(x_0) = \frac{\varepsilon_m - c}{q^m}.$$

As the conditions $x_m \to x_0$ and $m \to \infty$ are equivalent, we have

$$\lim_{m\to\infty} \frac{f(x_m) - f(x_0)}{x_m - x_0} = \lim_{m\to\infty} \frac{q_1 q_2 \cdots q_m}{q^m}. \tag{2.1}$$

Let us consider the cylinders $\Delta^Q_{c_1c_2\cdots c_n}$. The value $\mu_f\left(\Delta^Q_{c_1c_2\cdots c_n}\right)$, defined by the equality

$$\begin{aligned}\mu_f\left(\Delta^Q_{c_1c_2\cdots c_n}\right) &= f\left(\sup \Delta^Q_{c_1c_2\cdots c_n}\right) - f\left(\inf \Delta^Q_{c_1c_2\cdots c_n}\right)\\ &= f\left(\Delta^Q_{\varepsilon_1\varepsilon_2\cdots\varepsilon_n[q_{n+1}-1][q_{n+2}-1]\cdots}\right) - f\left(\Delta^Q_{\varepsilon_1\varepsilon_2\cdots\varepsilon_n 000\cdots}\right),\end{aligned}$$

is called *the change* μ_f *of the function* f *on a cylinder* $\Delta^Q_{c_1c_2\cdots c_n}$.

Thus, for $x_0 \in \Delta^Q_{c_1c_2\cdots c_n}$, we obtain

$$f'(x_0) = \lim_{n\to\infty} \frac{\mu_f\left(\Delta^Q_{c_1c_2\cdots c_n}\right)}{\left|\Delta^Q_{c_1c_2\cdots c_n}\right|} = \lim_{n\to\infty}\left(\frac{q_1q_2\cdots q_n}{q^n}\sum_{j=n+1}^{\infty}\frac{q_j-1}{q^j}\right). \tag{2.2}$$

Since $2 \le q_n \le q$, we have

$$\frac{1}{q-1}\lim_{n\to\infty}\left(\frac{q_1q_2\cdots q_n}{q^n}\right) \le \lim_{n\to\infty}\left(\frac{q_1q_2\cdots q_n}{q^n}\sum_{j=n+1}^{\infty}\frac{q_j-1}{q^j}\right)$$
$$\le \lim_{n\to\infty}\left(\frac{q_1q_2\cdots q_n}{q^n}\right).$$

Therefore,

- $f'(x_0) = 1$ if the condition $q_n = q$ holds for all positive integers n;
- $f'(x_0) = 0$, i.e., f is a singular function if for all n the condition $q_n < q$ holds or there exists only a finite number of n such that $q_n = q$;
- f is non-differentiable if there exists only a finite number of n such that $q_n < q$ (since limits (2.1) and (2.2) are different);
- f is a singular function if there exists an infinite subsequence (n_k) of positive integers such that $q_{n_k} < q$.

□

Theorem 2.7. *The Lebesgue integral of the function* f *can be calculated by the formula*

$$\int_{[0,1]} f(x)dx = \frac{1}{2}\sum_{n=1}^{\infty}\frac{q_n-1}{q^n}.$$

Proof. We have

$$0 \le f(x) \le \sum_{n=1}^{\infty}\frac{q_n-1}{q^n}.$$

Suppose that

$$T = \{0, \Delta^q_{1000\ldots}, \Delta^q_{2000\ldots}, \ldots, \Delta^q_{[q_1-1]000\ldots}, \ldots, \Delta^q_{[q_1-1][q_2-1]\cdots[q_n-1]\ldots}\},$$
$$E_n = \{x : y_{n-1} \le f(x) < y_n\} = \Delta^Q_{c_1c_2\ldots c_n}, \quad c_n \in \Theta_n.$$

We get

$$\lambda(E_n) = \frac{1}{q_1q_2\cdots q_n},$$

where $\lambda(\cdot)$ is the Lebesgue measure of a set.

Also, $\overline{y} \in [y_{n-1}, y_n)$. Suppose that $\overline{y} = y_{n-1}$. It is easy to see that the conditions $\lambda(E_n) \to 0$ and $n \to \infty$ are equivalent.

Hence,

$$\int_{[0,1]} f(x)dx = \lim_{n\to\infty} \sum_n \frac{\Delta^q_{c_1c_2\cdots c_n000\cdots}}{q_1q_2\cdots q_n}$$
$$= \lim_{n\to\infty} \left(\frac{(q_n-1)q_1q_2\cdots q_n}{2q^n q_1q_2\cdots q_n} \right) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{q_n-1}{q^n}.$$

We have that

$$\frac{1}{2(q-1)} \le \int_{[0,1]} f(x)dx \le \frac{1}{2},$$

and the integral is equal to $\frac{1}{2}$ if $f(x)=x$. □

3. Fractal in $\mathbb{R}^2$ defined in terms of a certain map

Let us consider the function

$$g : x = \Delta^q_{\underbrace{u\ldots u}_{\alpha_1-1}\alpha_1\underbrace{u\ldots u}_{\alpha_2-1}\alpha_2\ldots\underbrace{u\ldots u}_{\alpha_n-1}\alpha_n\ldots} \to \Delta^q_{\alpha_1\alpha_2\cdots\alpha_n\cdots},$$

where $u \in \{0,1,\ldots,q-1\}$ is a fixed number, $\alpha_n \in \Theta = \{1,2,\ldots,q-1\} \setminus \{u\}$, and $3 < q$ is a fixed positive integer. This function can be represented as

$$g : x = \frac{u}{s-1} + \sum_{n=1}^{\infty} \frac{\alpha_n - u}{q^{\alpha_1+\alpha_2+\cdots+\alpha_n}} \to \sum_{n=1}^{\infty} \frac{\alpha_n}{q^n} = g(x) = y.$$

Theorem 3.1. *The function g has the following properties:*

1. *The domain $D(g)$ of the function g is an uncountable, perfect, and nowhere dense set of zero Lebesgue measure as well as a self-similar fractal whose Hausdorff dimension α_0 satisfies the equation*

$$\sum_{\substack{p\in\{1,2,\ldots,q-1\}\\ p\neq u}} \left(\frac{1}{s}\right)^{p\alpha_0} = 1.$$

2. *The range of g is a self-similar fractal*

$$E(g) = \{y : y = \Delta^q_{\alpha_1\alpha_2\cdots\alpha_n\cdots}, \alpha_n \in \Theta\}$$

whose Hausdorff dimension α_0 can be calculated by the formula

$$\alpha_0(E(g)) = \log_q |\Theta|,$$

where $|\cdot|$ is the number of elements of a set.

3. *The function g is well-defined and bijective on its domain.*

4. *On the domain, the function g is:*

- *decreasing whenever $u \in \{0,1\}$ for all $q > 3$;*
- *increasing whenever $u \in \{s-2, s-1\}$ for all $q > 3$;*
- *non-monotonic whenever $u \in \{2,3,\dots,s-3\}$ and $q > 4$.*

5. *The function g is continuous at any point of the domain.*

6. *The function g is non-differentiable on the domain.*

7. *The following relations are true:*

$$g\left(\sigma^{\alpha_1}(x)\right) = \sigma(g(x)),$$
$$g\left(\sigma^{\alpha_1+\alpha_2+\dots+\alpha_n}(x)\right) = \sigma^n(g(x)),$$

where σ is the shift operator.

8. *The function does not preserve the Hausdorff dimension.*

Proof. For any fixed $u \in \{0,1,\dots,q-1\}$, the domain $D(g)$ of the function g is an uncountable, perfect, and nowhere dense set of zero Lebesgue measure. It is also a self-similar fractal whose Hausdorff dimension α_0 satisfies the equation (see [16, 17]):

$$\sum_{\substack{p\in\{1,2,\dots,q-1\}\\ p\neq u}} \left(\frac{1}{s}\right)^{p\alpha_0} = 1.$$

This set does not contain q-rational numbers, i.e., the numbers of the form

$$\Delta^q_{\alpha_1\alpha_2\cdots\alpha_n000\cdots} = \Delta^q_{\alpha_1\alpha_2\cdots\alpha_n[q-1][q-1][q-1]\cdots}.$$

Thus, any element of the domain $D(g)$ of the function g has the unique q-representation. Therefore the condition $g(x_1) \neq g(x_2)$ holds for $x_1 \neq x_2$. The value $g(x) \in E(g)$ is assigned to an arbitrary $x \in D(g)$ and vice versa.

Let us consider the difference

$$|g(x) - g(x_0)| = \left|\sum_{n=1}^{\infty} \frac{\beta_n - \alpha_n}{q^n}\right|,$$

where $x_0 = \Delta^q_{\underbrace{u\cdots u}_{\alpha_1-1}\alpha_1\underbrace{u\cdots u}_{\alpha_2-1}\alpha_2\cdots\underbrace{u\cdots u}_{\alpha_n-1}\alpha_n\cdots}$ is a fixed number from $D(g)$ and $x = \Delta^q_{\underbrace{u\cdots u}_{\beta_1-1}\beta_1\underbrace{u\cdots u}_{\beta_2-1}\beta_2\cdots\underbrace{u\cdots u}_{\beta_n-1}\beta_n\cdots}$. It is easy to see that the conditions $x \to x_0$ and $\beta_n \to \alpha_n$ are equivalent, $n = 1, 2, \dots$. Hence,

$$\lim_{x\to x_0} |g(x) - g(x_0)| = \lim_{\beta_n\to\alpha_n} \left|\sum_{n=1}^{\infty} \frac{\beta_n - \alpha_n}{q^n}\right| = 0.$$

From the definition of g it follows that the set

$$E(g) = \{y : y = \Delta^q_{\alpha_1\alpha_2\cdots\alpha_n\cdots}, \alpha_n \in \Theta\}$$

is the range of g. It follows from Theorem 2 in [18] that $E(g)$ is a self-similar fractal whose Hausdorff dimension α_0 can be calculated by the formula

$$\alpha_0(E(g)) = \log_q |\Theta|,$$

where $|\cdot|$ is the number of elements of a set.

Thus *properties* 1–3 and 5 are proved.

Let us prove *property* 4. Let $x_1 = \Delta^q_{\underbrace{u\ldots u}_{\alpha_1-1}\alpha_1\underbrace{u\cdots u}_{\alpha_2-1}\alpha_2\cdots\underbrace{u\ldots u}_{\alpha_n-1}\alpha_n\cdots}$ and $x_2 = \Delta^q_{\underbrace{u\cdots u}_{\beta_1-1}\beta_1\underbrace{u\cdots u}_{\beta_2-1}\beta_2\cdots\underbrace{u\cdots u}_{\beta_n-1}\beta_n\cdots}$ such that $x_1 \neq x_2$. Then there exists n_0 such that $\alpha_i = \beta_i$ for $i = \overline{1, n_0 - 1}$ and $\alpha_{n_0} \neq \beta_{n_0}$. Suppose that $\alpha_{n_0} < \beta_{n_0}$. Consider the following numbers:

$$x_1 = \Delta^q_{\underbrace{u\cdots u}_{\alpha_1-1}\alpha_1\underbrace{u\cdots u}_{\alpha_2-1}\alpha_2\cdots\underbrace{u\cdots u}_{\alpha_{n_0-1}-1}\alpha_{n_0-1}\underbrace{u\cdots u}_{\alpha_{n_0}-1}\alpha_{n_0}\cdots}$$

and

$$x_2 = \Delta^q_{\underbrace{u\cdots u}_{\beta_1-1}\beta_1\underbrace{u\cdots u}_{\beta_2-1}\beta_2\cdots\underbrace{u\cdots u}_{\beta_{n_0-1}-1}\beta_{n_0-1}\underbrace{u\cdots u}_{\beta_{n_0}-1}\beta_{n_0}\cdots}$$

when $\alpha_{n_0} < \beta_{n_0}$. It is sufficient to consider the numbers

$$\Delta^q_{\underbrace{u\cdots u}_{\alpha_{n_0}-1}\alpha_{n_0}\cdots} \quad \text{and} \quad \Delta^q_{\underbrace{u\cdots u}_{\beta_{n_0}-1}\beta_{n_0}\cdots}.$$

Then we obtain the following cases:

- $g(x_1) < g(x_2)$ for $x_1 > x_2$. The last condition is true for the case where $u = 0$ or $u = 1$. Thus g is decreasing.
- $g(x_1) < g(x_2)$ for $x_1 < x_2$. The last condition is true for the case where $u = q - 1$ or $u = q - 2$.
- If $g(x_1) < g(x_2)$ for $x_1 > x_2$ and $x_1 < x_2$, then this condition is true for the case where $u \in \{2, 3, \ldots, q - 3\}$ and $q > 4$. Thus g is non-monotonic.

For $q = 4$, g is increasing if $u \in \{2, 3\}$ and decreasing if $u \in \{0, 1\}$.

Let us prove *property* 6. Consider a sequence (x_n) of numbers

$$x_n = \Delta^q_{\underbrace{u\ldots u}_{\alpha_1-1}\alpha_1\underbrace{u\ldots u}_{\alpha_2-1}\alpha_2\ldots\underbrace{u\ldots u}_{\alpha_{n-1}-1}\alpha_{n-1}\underbrace{u\ldots u}_{\alpha_n-1}\alpha_n\underbrace{u\ldots u}_{\alpha_{n+1}-1}\alpha_{n+1}\ldots}$$

and a fixed number

$$x_0 = \Delta^q_{\underbrace{u\ldots u}_{\alpha_1-1}\alpha_1\underbrace{u\ldots u}_{\alpha_2-1}\alpha_2\ldots\underbrace{u\ldots u}_{\alpha_{n_0-1}-1}\alpha_{n-1}\underbrace{u\ldots u}_{c-1}c\underbrace{u\ldots u}_{\alpha_{n+1}-1}\alpha_{n+1}\ldots},$$

where c is a fixed number. Then

$$\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0} = \lim_{x\to x_0}\frac{\frac{\alpha_n-c}{q^n}}{\frac{\alpha_n}{q^{\alpha_1+\alpha_2+\cdots+\alpha_{n-1}+\alpha_n}}-\frac{c}{q^{\alpha_1+\alpha_2+\cdots+\alpha_{n-1}+c}}}$$

$$= \lim_{\alpha_n \to c} \frac{(\alpha_n - c)q^{\alpha_1+\alpha_2+\cdots+\alpha_{n-1}+\alpha_n+c}}{q^n(\alpha_n q^c - cq^{\alpha_n})}$$
$$= \lim_{\alpha_n \to c} \frac{q^{\alpha_1+\alpha_2+\cdots+\alpha_{n-1}+\alpha_n+c}}{q^{c+n}}.$$

Thus the function is non-differentiable.

Property 7. It is easy to see that

$$\begin{aligned} g\left(\sigma^{\alpha_1}(x)\right) &= g(\Delta^q_{\underbrace{u\cdots u}_{\alpha_2-1}\alpha_2\cdots\underbrace{u\cdots u}_{\alpha_{n-1}-1}\alpha_{n-1}\underbrace{u\cdots u}_{\alpha_n-1}\alpha_n\underbrace{u\cdots u}_{\alpha_{n+1}-1}\alpha_{n+1}\cdots}) \\ &= \Delta^q_{\alpha_2\alpha_3\cdots\alpha_n\cdots} = \sigma(g(x)), \\ g\left(\sigma^{\alpha_1+\alpha_2+\cdots+\alpha_n}(x)\right) &= g(\Delta^q_{\underbrace{u\cdots u}_{\alpha_{n+1}-1}\alpha_{n+1}\cdots\underbrace{u\cdots u}_{\alpha_{n+2}-1}\alpha_{n+2}\cdots}) \\ &= \Delta^q_{\alpha_{n+1}\alpha_{n+3}\cdots} = \sigma^n(g(x)). \end{aligned}$$

Property 8. It is obvious that there exists a set S such that $\alpha_0(S) \neq \alpha_0(g(S))$, where $\alpha_0(\cdot)$ is the Hausdorff dimension of a set. □

Theorem 3.2. *The Hausdorff dimension of the graph of the function g is equal to* 1*.*

Proof. Suppose that

$$X = [0,1] \times [0,1] = \left\{(x,y) : x = \sum_{m=1}^{\infty} \frac{\alpha_m}{q^m}, \quad \alpha_m \in \Theta_q = \{0,1,\dots,q-1\}, \right.$$
$$\left. y = \sum_{m=1}^{\infty} \frac{\beta_m}{q^m}, \quad \beta_m \in \Theta_q \right\}.$$

Then the set

$$\sqcap_{(\alpha_1\beta_1)(\alpha_2\beta_2)\cdots(\alpha_m\beta_m)} = \Delta^q_{\alpha_1\alpha_2\cdots\alpha_m} \times \Delta^q_{\beta_1\beta_2\cdots\beta_m}$$

is a square with the length of a side q^{-m}. This square is called *a square of rank m with base $(\alpha_1\beta_1)(\alpha_2\beta_2)\dots(\alpha_m\beta_m)$*.

If $E \subset X$, then the number

$$\alpha^K(E) = \inf\{\alpha : \widehat{H}_\alpha(E) = 0\} = \sup\{\alpha : \widehat{H}_\alpha(E) = \infty\},$$

where

$$\widehat{H}_\alpha(E) = \lim_{\varepsilon \to 0}\left[\inf_{d \leq \varepsilon} K(E,d)d^\alpha\right]$$

and $K(E,d)$ is the minimal number of squares of diameter d required to cover the set E. The value $K(E,d)$ is called *the fractal cell entropy dimension of the set E*. It is easily seen that $\alpha^K(E) \geq \alpha_0(E)$.

From the definition and properties of the function g it follows that the graph of the function belongs to $\tau = |\Theta|$ squares from q^2 first-rank squares (here τ is equal to $(s-1)$ for $u=0$ and equal to $(s-2)$ for $u \neq 0$):

$$\sqcap_{(i_1 i_1)} = \Big[\Delta^q_{\underbrace{u\ldots u}_{i_1-1} i_1}, \Delta^q_{i_1}\Big], \quad i_1 \in \Theta_q.$$

The graph of the function f belongs to τ^2 squares from q^4 second-rank squares:

$$\sqcap_{(i_1 i_2)(i_1 i_2)} = \Big[\Delta^q_{\underbrace{u\ldots u}_{i_1-1} i_1 \underbrace{u\ldots u}_{i_2-1} i_2}, \Delta^q_{i_1 i_2}\Big], \quad i_1, i_2 \in \Theta_q.$$

The graph Γ_g of the function g belongs to τ^m squares of rank m with sides $q^{\alpha_1+\alpha_2+\cdots+\alpha_m}$ and q^{-m}. Then

$$\widehat{H}_\alpha(\Gamma_g) = \lim_{m\to\infty} \tau^m \left(\sqrt{q^{-2(\alpha_1+\alpha_2+\cdots+\alpha_m)} + q^{-2m}}\right)^\alpha.$$

Since $q^{-m(q-1)} \leq q^{-(\alpha_1+\alpha_2+\cdots+\alpha_m)} \leq q^{-m}$, we get

$$\begin{aligned}\widehat{H}_\alpha(\Gamma_g) &= \lim_{m\to\infty} \tau^m \left(2q^{-2m}\right)^{\frac{\alpha}{2}} = \lim_{m\to\infty} \tau^m \left(2q^{-2m}\right)^{\frac{\alpha}{2}} \\ &= \lim_{m\to\infty} \left(2^{\frac{\alpha}{2}} \tau^m q^{-m\alpha}\right) = \lim_{m\to\infty} \left(2^{\frac{\alpha}{2}} \left(\frac{\tau}{q^\alpha}\right)^m\right)\end{aligned}$$

for $\alpha_1 + \alpha_2 + \cdots + \alpha_m = m$ and

$$\widehat{H}_\alpha(\Gamma_g) = \lim_{m\to\infty} \tau^m \left(q^{-2m(q-1)} + q^{-2m}\right)^{\frac{\alpha}{2}} = \lim_{m\to\infty} \left(\left(\frac{\tau^{\frac{1}{\alpha}}}{q}\right)^{2m} + \left(q^{1-q}\tau^{\frac{1}{\alpha}}\right)^{2m}\right)^{\frac{\alpha}{2}}$$

for $\alpha_1 + \alpha_2 + \cdots + \alpha_m = m(q-1)$.

It is obvious that if $\left(\frac{\tau}{q^\alpha}\right)^m \to 0$, $\left(\frac{\tau^{\frac{1}{\alpha}}}{q}\right)^{2m} \to 0$, and $\left(q^{1-q}\tau^{\frac{1}{\alpha}}\right)^{2m} \to 0$ for $\alpha > 1$, and the graph of the function has self-similar properties, then $\alpha^K(\Gamma_g) = \alpha_0(\Gamma_g) = 1$. □

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Деякі функції, визначені в термінах рядів Кантора

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Цю статтю присвячено деяким прикладам функцій, аргумент яких подано в термінах рядів Кантора.

Ключові слова: ніде недиференційовна функція, сингулярна функція, розвинення дійсного числа, немонотонна функція, розмірність Гаусдорфа