

Remarks on Screen Integrable Null Hypersurfaces in Lorentzian Manifolds

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In the present paper, we show that the geometry of a screen integrable null hypersurface can be generated from an isometric immersion of a leaf of its screen distribution into the ambient space. We prove, under certain geometric conditions, that such immersions are contained in semi-Euclidean spheres or hyperbolic spaces, and the underlying null hypersurfaces are necessarily umbilic and screen totally umbilic. Where necessary, the examples are given to illustrate the main ideas.

Key words: screen integrable null hypersurfaces, normal curvature tensor of a leaf

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1. Introduction

The study of null submanifolds in semi-Riemannian manifolds was introduced by Duggal–Benjancu [5] and later updated by Duggal–Sahin [9]. In the above books, the authors laid a foundation for research on null geometry by constructing their structural equations, among other results. In fact, they introduced a non-degenerate screen distribution to construct a null transversal vector bundle, which is non-intersecting with its null tangent bundle, and developed local geometry of null curves, hypersurfaces and submanifolds. The list of other pioneers of the theory includes D.N. Kupeli [15] whose approach is purely intrinsic compared to that of [5, 9]. Since then, many researchers including but not limited to [1, 4, 6, 8, 10, 11], have studied null submanifolds and many interesting results have been obtained. Null hypersurfaces appear in general relativity as models of different types of black hole horizons (see [5, 9] for details) and their theory is quite fundamental for modern mathematical physics.

Among the most studied null hypersurfaces are those with an integrable screen distribution, and they are commonly known as *screen integrable null hypersurfaces*. They include the well-known screen conformal ones among others. It was shown in [7] that all screen integrable null hypersurfaces are locally isometric to $\mathcal{C}_\xi \times M^*$, where \mathcal{C}_ξ is a null curve tangent to the normal bundle of the hypersurface and M^* is a leaf of its screen distribution. In particular, [5] proves that a null cone of an $(n + 2)$ -dimensional Lorentzian space \mathbb{R}_1^{n+2} is screen conformal,

satisfying the above structure, with $M^* \cong \mathbb{S}^n$. Under some geometric conditions on the ambient space, Duggal and Sahin [9] also proved that a screen conformal Einstein null hypersurface is locally a triple product $\mathcal{C}_\xi \times M_\alpha^* \times M_\beta^*$, where M_α^* and M_β^* are some leaves of its screen distribution (see Theorem 2.5.17 of [9] for more details). In [5], Duggal and Bejancu tried to understand the geometry of a null hypersurface M from a leaf M^* of its screen distribution as an immersion in the ambient space. They, in fact, showed that an umbilic leaf in the ambient space implies that the underlying null hypersurface is umbilic too (see Proposition 5.1 of [5, p. 107]). A natural question then arises, *which other geometric information about the null hypersurface can be derived from the geometry of an isometric immersion of a leaf of its screen distribution into the ambient space?*

The main aim of this paper is to give some solutions to the above problem by studying null hypersurfaces of Lorentzian spaces. Consequently, we prove two main theorems in that line, Theorems 3.6 and 3.10. The paper is arranged as follows. In Section 2, we quote some basic notions needed in the rest of the paper. In Section 3, we prove several characterization results.

2. Preliminaries

Suppose M is an $(n + 1)$ -dimensional smooth manifold and $F : M \rightarrow \overline{M}$ is a smooth mapping such that each point $x \in M$ has an open neighborhood \mathcal{U} for which F restricted to \mathcal{U} is one-to-one and $F^{-1} : F(\mathcal{U}) \rightarrow M$ are smooth. Then we say that $F(M)$ is an immersed hypersurface of \overline{M} . If this condition globally holds, then $F(M)$ is called an embedded hypersurface of \overline{M} , which we assume in this paper. The embedded hypersurface has a natural manifold structure inherited from the manifold structure on \overline{M} via the embedding mapping. At each point $F(x)$ of $F(M)$, the tangent space is naturally identified with an $(n + 1)$ -dimensional subspace $T_{F(x)}M$ of the tangent space $T_{F(x)}\overline{M}$. The embedding F induces, in general, a symmetric tensor field, say g , on $F(M)$ such that $g(X, Y)|_x = \overline{g}(F_*X, F_*Y)|_{F(x)}$ for all $X, Y \in T_xM$. Here, F_* is the differential map of F defined by $F_* : T_xM \rightarrow T_{F(x)}\overline{M}$ and $(F_*X)\omega = X(\omega \circ F)$ for an arbitrary smooth function ω in a neighborhood of $F(x)$ of $F(M)$. Henceforth we write M and x instead of $F(M)$ and $F(x)$. Due to the causal character of three categories (space-like, time-like and light-like) of the vector fields of \overline{M} , there are three types of hypersurfaces M , namely, *Riemannian*, *semi-Riemannian* and *null* (or *light-like*), and g is a non-degenerate or a degenerate symmetric tensor field on M according as M is of the first two types and of the third type, respectively. The geometry of Riemannian or semi-Riemannian hypersurfaces is well known and has received a considerable attention, for example, see [13] and references cited therein. In the present paper, we focus on null hypersurfaces using the approach of Duggal–Bejancu [5].

Now, let g be degenerate on M . Then there exists a *nonzero* vector field ξ on M such that $g(\xi, X) = 0$ for all $X \in \Gamma(TM)$. The *radical* or the *null space* [13, p. 53] of T_xM , at each point $x \in M$, is a subspace $\text{Rad } T_xM$ defined by $\text{Rad } T_xM = \{\xi \in T_xM : \forall X \in T_xM \ g_x(\xi, X) = 0\}$, whose dimension is

called the *nullity degree* of g and M is called a *null hypersurface* of \overline{M} . It follows that $T_x M^\perp$ is also null and satisfies $\text{Rad } T_x M = T_x M \cap T_x M^\perp$. For a hypersurface M , if $\dim(T_x M^\perp) = 1$, then $\dim(\text{Rad } T_x M) = 1$ and $\text{Rad } T_x M = T_x M^\perp$. We call $\text{Rad } TM$ a radical (null) distribution of M . Thus, for a null hypersurface M , TM and TM^\perp have a nontrivial intersection and their sum is not the whole of tangent bundle space $T\overline{M}$. In other words, a vector of $T_x \overline{M}$ cannot be decomposed uniquely into a component tangent to $T_x M$ and a component of $T_x M^\perp$. Therefore, the standard text-book definition of the second fundamental form and the Gauss–Weingarten formulas do not work in the usual way for the null case.

To overcome the above difficulty, Duggal–Bejancu [5] introduced an approach to null geometry which we follow in this paper. The approach consists of fixing, on the null hypersurface, the geometric data formed by a null section and a *screen distribution*. By the screen distribution of M , we mean a complementary bundle of TM^\perp in TM . It is then a rank n non-degenerate distribution over M . In fact, there are infinitely many possibilities of choices for such a distribution provided the hypersurface M is paracompact, but each of them is canonically isomorphic to the factor vector bundle TM/TM^\perp [15]. We denote by $S(TM)$ the screen distribution over M . Then we have the decomposition $TM = S(TM) \perp TM^\perp$, where \perp denotes the orthogonal direct sum. From [5] or [9], it is known that for a null hypersurface equipped with a screen distribution, there exists a unique rank 1 vector subbundle $\text{tr}(TM)$ of $T\overline{M}$ over M such that for any non-zero section ξ of TM^\perp on a coordinate neighborhood $\mathcal{U} \subset M$ there exists a unique section N of $\text{tr}(TM)$ on \mathcal{U} satisfying $\overline{g}(N, \xi) = 1$, $\overline{g}(N, N) = \overline{g}(N, W) = 0$ for all $W \in \Gamma(S(TM)|_{\mathcal{U}})$. It then follows that $T\overline{M}|_M = S(TM) \perp \{TM^\perp \oplus \text{tr}(TM)\} = TM \oplus \text{tr}(TM)$, where \oplus denotes the direct (non-orthogonal) sum. We call $\text{tr}(TM)$ a (null) *transversal* vector bundle along M . Throughout the paper, all manifolds are supposed to be paracompact and smooth. We denote by $\mathcal{F}(M)$ the algebra of differentiable functions on M and by $\Gamma(E)$ the $\mathcal{F}(M)$ -module of differentiable sections of a vector bundle E over M . We also assume that all associated structures are smooth.

Let ∇ and ∇^* denote the induced connections on M and $S(TM)$, respectively, and P be the projection of TM onto $S(TM)$. Then the local Gauss–Weingarten equations of M and $S(TM)$ are the following [5]:

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \quad (2.1)$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N, \quad (2.2)$$

$$\nabla_X PY = \nabla_X^* PY + C(X, PY)\xi, \quad (2.3)$$

$$\nabla_X \xi = -A_\xi^* X - \tau(X)\xi, \quad A_\xi^* \xi = 0, \quad (2.4)$$

for all $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(TM^\perp)$ and $N \in \Gamma(\text{tr}(TM))$. In the above setting, B is the local second fundamental form of M and C is the local second fundamental form on $S(TM)$. A_N and A_ξ^* are the shape operators on TM and $S(TM)$, respectively, while τ is a 1-form on TM . The above shape operators are related to their local fundamental forms by $g(A_\xi^* X, Y) = B(X, Y)$, $g(A_N X, PY) = C(X, PY)$

for any $X, Y \in \Gamma(TM)$. Moreover, $\bar{g}(A_\xi^*X, N) = 0$ and $\bar{g}(A_NX, N) = 0$ for all $X \in \Gamma(TM)$. From the above relations, we notice that A_ξ^* and A_N are both screen-valued operators.

The null hypersurface M is said to be *totally umbilic* [5] if $B = \rho \otimes g$, where ρ is a smooth function on a coordinate neighborhood $\mathcal{U} \subset TM$. In case $\rho = 0$, we say that M is *totally geodesic*. In the same line, M is called *screen totally umbilic* if $C = \varrho \otimes g$, where ϱ is a smooth function on a coordinate neighborhood $\mathcal{U} \subset TM$. When $\varrho = 0$, we say that M is *screen totally geodesic*. The *mean curvature vector* H of a null hypersurface is transversal to M and given by $H = \frac{1}{n}(\text{trace}_{S(TM)} B)N = \frac{1}{n}(\text{trace}_{S(TM)} A_\xi^*)N$. We say that M is a *minimal* null hypersurface if $H = 0$. More precisely, M is minimal if $\text{trace}_{S(TM)} A_\xi^* = 0$ (see [5, 9] for more details and examples).

Let $\vartheta = \bar{g}(N, \cdot)$ be 1-form metrically equivalent to N defined on \bar{M} . Take $\eta = i^*\vartheta$ to be its restriction on M , where $i : M \rightarrow \bar{M}$ is the inclusion map. Then it is easy to show that $(\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y)$ for all $X, Y, Z \in \Gamma(TM)$. Consequently, ∇ is generally *not* a metric connection with respect to g . However, the induced connection ∇^* on $S(TM)$ is a metric connection. Denote by \bar{R} the curvature tensor of the connection $\bar{\nabla}$. Using the Gauss–Weingarten formulae (2.1)–(2.4), we obtain the following curvature relations (see details in [5, 9]):

$$\bar{g}(\bar{R}(X, Y)\xi, N) = C(Y, A_\xi^*X) - C(X, A_\xi^*Y) - 2d\tau(X, Y), \tag{2.5}$$

where $2d\tau(X, Y) = X\tau(Y) - Y\tau(X) - \tau([X, Y])$ for all $X, Y \in \Gamma(TM)|_{\mathcal{U}}$, $\xi \in \Gamma(TM^\perp)$ and $N \in \Gamma(\text{tr}(TM))$.

Suppose π is a non-degenerate plane of $T_p\bar{M}$ for $p \in \bar{M}$. Then the associated matrix G_p of \bar{g}_p , with respect to an arbitrary basis $\{u, v\}$, is of rank 2 given by (1.2.15) of [9, p. 16]. Define a real number $K(\pi) = K_p(u, v) = \bar{R}(u, v, v, u)$, where $\bar{R}(u, v, v, u)$ is the 4-linear mapping on $T_p\bar{M}$ by the curvature tensor. The smooth function K , which assigns to each non-degenerate tangent plane π the real number $K(\pi)$ is called the *sectional curvature* of \bar{M} , which is independent of the basis $\{u, v\}$. If K is a constant c at every point of $p \in \bar{M}$, then \bar{M} is of constant sectional curvature c , denoted by $\bar{M}(c)$, whose curvature tensor field \bar{R} is given by $\bar{R}(X, Y)Z = c\{\bar{g}(Y, Z)X - \bar{g}(X, Z)Y\}$, for any $X, Y, Z \in \Gamma(T\bar{M})$. In particular, if $K = 0$, then \bar{M} is called a *flat manifold* for which $\bar{R} = 0$.

3. Geometry of (M, g) from the geometry of a leaf of $S(TM)$

Assume that (M, g) is a screen integrable null hypersurface of a *Lorentzian manifold* (\bar{M}, \bar{g}) . Let M^* be a (Riemannian) leaf of its screen distribution $S(TM)$. Let $f : M^* \rightarrow \bar{M}$ be an isometric immersion of M^* in \bar{M} , as a codimension 2 nondegenerate submanifold. Then (2.1) and (2.3) give the Gauss formula of M^* (in \bar{M}) as

$$\bar{\nabla}_X Y = \nabla_X^* Y + C(X, Y)\xi + B(X, Y)N, \quad \text{for all } X, Y \in \Gamma(TM^*). \tag{3.1}$$

It is obvious from (3.1) that the second fundamental form h^* of M^* , as a submanifold of \bar{M} , is given by $h^*(X, Y) = C(X, Y)\xi + B(X, Y)N$. Next, denote by $\nabla^{*\perp}$ the normal connection on the normal bundle $TM^{*\perp}$. Then the Weingarten formula for M^* is given by

$$\bar{\nabla}_X V = -A_V X + \nabla_X^{*\perp} V, \quad \text{for all } X \in \Gamma(TM^*), V \in \Gamma(TM^{*\perp}), \quad (3.2)$$

where A_V denotes the shape operator of M^* . Since $TM^{*\perp} = TM^\perp \oplus \text{tr}(TM)$, we let $V = a\xi + bN$ such that $a, b \neq 0$. Thus, it is easy to see that $W = a\xi - bN$ is another vector field of $TM^{*\perp}$ which is orthogonal to V . From now on, we consider $TM^{*\perp}$ to be spanned by V and W . Putting all the above into account, we can express the shape operator A_V of M^* in terms of the shape operators A_ξ^* and A_N as follows.

Lemma 3.1. *Let $f : M^* \rightarrow \bar{M}$ be an isometric immersion such that (3.1) and (3.2) hold. Then the shape operator of M^* satisfies $A_V = aA_\xi^* + bA_N$, where $V = a\xi + bN$ and a, b are non-vanishing smooth functions.*

Proof. Taking the \bar{g} -product of (3.2) with $Y \in \Gamma(TM^*)$ and using the fact that $\bar{\nabla}$ is a metric connection, we get $g(A_V X, Y) = \bar{g}(V, \bar{\nabla}_X Y)$. Then, applying (3.1) to the last relation and owing to the fact that M^* is nondegenerate, we get the desired result. \square

Let $\{V, W\}$ be an orthonormal basis of $T_x M^{*\perp}$ at $x \in M^*$. Then the mean curvature vector of a leaf M^* in \bar{M} is the vector $H^* = \frac{1}{2}[(\text{trace } A_V)V + (\text{trace } A_W)W]$. We say that M^* is minimal in \bar{M} if H^* vanishes. It then follows that M^* is minimal if and only if $\text{trace } A_V = 0$ and $\text{trace } A_W = 0$. In view of Lemma 3.1, one can easily see that minimality of a leaf M^* implies minimality of the underlying null hypersurface (M, g) . Let us consider the curvature tensor of the normal bundle $TM^{*\perp}$ as $R^{*\perp} : T_x M^* \times T_x M^* \times T_x M^{*\perp} \rightarrow T_x M^{*\perp}$, given by

$$R^{*\perp}(X, Y)V = \nabla_X^{*\perp} \nabla_Y^{*\perp} V - \nabla_Y^{*\perp} \nabla_X^{*\perp} V - \nabla_{[X, Y]}^{*\perp} V, \quad (3.3)$$

for any $X, Y \in \Gamma(TM^*)$ and $V \in \Gamma(TM^{*\perp})$. The importance of the 1-form τ in the study of null geometry has been shown in [5] and [9]. In fact, it has been shown that the Ricci tensor of a null submanifold is symmetric if and only if τ is closed, that is, $d\tau = 0$. In what follows, we show that the normal curvature $R^{*\perp}$ of a leaf M^* is directly linked to the 1-form τ of (2.2).

Proposition 3.2. *Let (M, g) be a screen integrable null hypersurface of a Lorentzian manifold $(\bar{M}(c), \bar{g})$. Then the normal curvature $R^{*\perp}$ of any leaf M^* of $S(TM)$ satisfies*

$$R^{*\perp}(X, Y)V = \{C(X, A_V Y) - C(Y, A_V X)\}\xi + \{B(X, A_V Y) - B(Y, A_V X)\}N = 2d\tau(X, Y)W, \quad (3.4)$$

for any vector fields $X, Y \in \Gamma(TM^*)$ and $V, W \in \Gamma(TM^{*\perp})$.

Proof. A direct calculation with using (3.1) and (3.2) leads to

$$\begin{aligned} \overline{R}(X, Y)V &= -\nabla_X^* A_V Y - C(X, A_V Y)\xi - B(X, A_V Y)N - A_{\nabla_Y^* \perp V} X \\ &\quad + \nabla_X^{*\perp} \nabla_Y^* V + \nabla_Y^* A_V X + C(Y, A_V X)\xi + B(Y, A_V X)N \\ &\quad + A_{\nabla_X^* \perp V} Y - \nabla_Y^{*\perp} \nabla_X^* V + A_V[X, Y] - \nabla_{[X, Y]}^{*\perp} V \\ &= -\nabla_X^* A_V Y + \nabla_Y^* A_V X - A_{\nabla_Y^* \perp V} X + A_{\nabla_X^* \perp V} Y + A_V[X, Y] \\ &\quad + R^{*\perp}(X, Y)V + \{C(Y, A_V X) - C(X, A_V Y)\}\xi \\ &\quad + \{B(Y, A_V X) - B(X, A_V Y)\}N, \quad \text{for all } X, Y \in \Gamma(TM^*). \end{aligned} \tag{3.5}$$

Since \overline{M} is a space of constant curvature c , we have $\overline{R}(X, Y)V = 0$ for any $X, Y \in \Gamma(TM^*)$ and $V \in \Gamma(TM^{*\perp})$. Thus, (3.5) gives

$$\begin{aligned} R^{*\perp}(X, Y)V + \{C(Y, A_V X) - C(X, A_V Y)\}\xi \\ + \{B(Y, A_V X) - B(X, A_V Y)\}N = 0, \end{aligned} \tag{3.6}$$

which proves (3.4). Next, applying Lemma 3.1 to (3.6) and using the fact that $B(A_\xi^* X, Y) = B(X, A_\xi^* Y)$ and $C(A_N X, Y) = C(X, A_N Y)$, for all $X, Y \in \Gamma(S(TM))$, we get

$$\begin{aligned} R^{*\perp}(X, Y)V + \{C(Y, A_\xi^* Y) - C(X, A_\xi^* X)\}a\xi \\ + \{B(Y, A_N X) - B(X, A_N Y)\}bN = 0. \end{aligned} \tag{3.7}$$

As $B(Y, A_N X) = g(A_N X, A_\xi^* Y) = C(X, A_\xi^* Y)$, (3.7) reduces to

$$R^{*\perp}(X, Y)V = \{C(Y, A_\xi^* Y) - C(X, A_\xi^* X)\}(a\xi - bN). \tag{3.8}$$

Next, as \overline{M} is a space of constant curvature c , we have $\overline{R}(X, Y)\xi = 0$ for all $X, Y \in \Gamma(TM)$. Thus, in view of (2.5) and (3.8), we conclude that

$$R^{*\perp}(X, Y)V = 2d\tau(X, Y)W,$$

where $W = a\xi - bN$, which proves (3.4) and thus the proof is completed. □

From Proposition 3.2 we have

Corollary 3.3. *In view of Proposition 3.2, the following are equivalent:*

1. $d\tau$ vanishes on $S(TM)$.
2. $A_\xi^* \circ A_N = A_N \circ A_\xi^*$.
3. $R^{*\perp}(x) = 0$.
4. The normal bundle of M^* is parallel.

Corollary 3.4. *If (M, g) is totally umbilic or screen totally umbilic in \overline{M} , then $R^{*\perp}(x) = 0$.*

Remark 3.5. Condition (2) of Corollary 3.3 implies that $A_V \circ A_W = A_W \circ A_V$. In fact, by a simple calculation, while considering Lemma 3.1, we get $A_V \circ A_W = a^2(A_\xi^* \circ A_\xi^*) - ab(A_\xi^* \circ A_N - A_N \circ A_\xi^*) - b^2(A_N \circ A_N)$. In view of (2) of Corollary 3.3, we have $A_V \circ A_W = a^2(A_\xi^* \circ A_\xi^*) - b^2(A_N \circ A_N)$. On the other hand, $A_W \circ A_V = a^2(A_\xi^* \circ A_\xi^*) - b^2(A_N \circ A_N)$, which, if compared with the previous relation, proves the assertion. It then follows that the vanishing of $d\tau$ on a leaf M^* implies simultaneous diagonalisation of A_V for all $V \in \Gamma(TM^{*\perp})$.

Next, we prove the following result.

Theorem 3.6. *Let (M, g) be a screen integrable null hypersurface of \mathbb{R}_1^{n+2} , and let $f : M^* \rightarrow \mathbb{R}_1^{n+2}$ be an isometric immersion of a leaf M^* of $S(TM)$ as a codimension 2 submanifold of \mathbb{R}_1^{n+2} . Suppose there exists a nonzero normal vector field V to M^* in \mathbb{R}_1^{n+2} such that $d\tau = 0$ on $S(TM)$ and $A_V^* = \lambda I$, $\lambda \neq 0$. Then $f(M^*)$ is contained inside*

1. $\mathbb{S}_1^{n+1} \left(\frac{\sqrt{\epsilon}}{\lambda} \right)$ if $\epsilon > 0$,
2. $\mathbb{H}_0^{n+1} \left(\frac{-\sqrt{-\epsilon}}{\lambda} \right)$ if $\epsilon < 0$,

where $\epsilon = \bar{g}(V, V)$. Furthermore, the null hypersurface (M, g) is proper quasi-screen conformal in \mathbb{R}_1^{n+2} . Moreover, if $A_W = 0$, where V, W span the normal bundle $TM^{*\perp}$, then (M, g) is proper totally umbilic and screen totally umbilic in \mathbb{R}_1^{n+2} .

Proof. Observe that the vector $f(x) + \frac{1}{\lambda}V$ is constant for all $x \in M^*$. Let us denote it by \tilde{c} , then we have $\bar{g}(f(x) - \tilde{c}, f(x) - \tilde{c}) = \frac{1}{\lambda^2}\bar{g}(V, V) = \frac{\epsilon}{\lambda^2}$. As $d\tau = 0$ on $S(TM)$, then $TM^{*\perp}$ is parallel by Theorem 3.3. Consequently, V is parallel, and therefore $f(M^*)$ is contained in a sphere or a hyperbolic space with center \tilde{c} by [12]. This proves parts (1) and (2). Furthermore, the condition $A_V = \lambda I$ together with Lemma 3.1 implies that $A_V = aA_\xi^* + bA_N = \lambda I$, where $V = a\xi + bN$. Then, in view of [14], the null hypersurface (M, g) is quasi-screen conformal in \mathbb{R}_1^{n+2} . On the other hand, if $A_W = 0$, we have $aA_\xi^* - bA_N = 0$. Combining this relation with the previous one gives $A_\xi^* = \frac{\lambda}{2a}I$ and $A_N = \frac{\lambda}{2b}I$, which shows that (M, g) is totally umbilic and screen totally umbilic in \mathbb{R}_1^{n+2} , and the theorem is proved. \square

Corollary 3.7. *In case the vector field V , in Theorem 3.6, is the mean curvature vector of M^* in \mathbb{R}_1^{n+2} , then M^* is immersed minimally in \mathbb{S}_1^{n+1} or \mathbb{H}_0^{n+1} . (Such an immersion is called pseudo umbilic by Chen and Yano [16]). Moreover, the underlying null hypersurface (M, g) is proper totally umbilic, screen totally umbilic and screen conformal in \mathbb{R}_1^{n+1} .*

Proof. If V is the mean curvature vector of M^* in \mathbb{R}_1^{n+2} , then V is parallel to the position vector $f(x) - \tilde{c}$ and therefore, by [12], M^* is minimal in either \mathbb{S}_1^{n+1} or \mathbb{H}_0^{n+1} . In view of (3.1) and the fact that $f(M^*)$ is pseudo umbilic, we have $C(X, Y)\xi + B(X, Y)N = g(X, Y)V = ag(X, Y)\xi + bg(X, Y)N$ for all $X, Y \in$

$\Gamma(TM^*)$. Taking the \bar{g} -product of the previous relation with ξ and N in turns, we get $B(X, Y) = bg(X, Y)$ and $C(X, Y) = ag(X, Y)$, respectively. Thus, (M, g) is totally umbilic and screen totally umbilic in \mathbb{R}_1^{n+1} . As $a, b \neq 0$, we deduce that $C(X, Y) = \psi B(X, Y)$, with $\psi = \frac{a}{b}$, showing that M is screen conformal, which completes the proof. \square

As an example, we have the following.

Example 3.8 (The null cone Λ_0^{n+1} of \mathbb{R}_1^{n+2}). Let \mathbb{R}_1^{n+2} be the space \mathbb{R}^{n+2} endowed with the semi-Euclidean metric

$$\bar{g}(x, y) = -x^0y^0 + \sum_{a=1}^{n+1} x^ay^a, \quad \text{where } x = \sum_{A=0}^{n+1} x^A \frac{\partial}{\partial x^A}.$$

The null cone Λ_0^{n+1} is given by the equation $-(x^0)^2 + \sum_{a=1}^{n+1} (x^a)^2 = 0, x^0 \neq 0$. It is known that Λ_0^{n+1} is a null hypersurface of \mathbb{R}_1^{n+2} and the radical distribution is spanned by a global vector field

$$\xi = \sum_{A=0}^{n+1} x^A \frac{\partial}{\partial x^A} \tag{3.9}$$

on Λ_0^{n+1} . The unique section N , spanning the transversal bundle $\text{tr}(T\Lambda_0^{n+1})$, is given by

$$N = \frac{1}{2(x^0)^2} \left\{ -x^0 \frac{\partial}{\partial x^0} + \sum_{a=1}^{n+1} x^a \frac{\partial}{\partial x^a} \right\}, \tag{3.10}$$

and it is also globally defined. As ξ is the position vector field, we get

$$\bar{\nabla}_X \xi = \nabla_X \xi = X, \quad \forall X \in \Gamma(T\Lambda_0^{n+1}). \tag{3.11}$$

Then, $A_\xi^* X + \tau(X)\xi + X = 0$. Since A_ξ^* is $\Gamma(S(T\Lambda_0^{n+1}))$ -valued, we have

$$A_\xi^* X = -PX \quad \text{and} \quad \tau(X) = -\eta(X), \quad \forall X \in \Gamma(T\Lambda_0^{n+1}). \tag{3.12}$$

Note that any $X \in \Gamma(S(T\Lambda_0^{n+1}))$ is expressed as $X = \sum_{a=1}^{n+1} X^a \frac{\partial}{\partial x^a}$, where (X^1, \dots, X^{n+1}) satisfy $\sum_{a=1}^{n+1} x^a X^a = 0$. Then,

$$\nabla_\xi X = \bar{\nabla}_\xi X = \sum_{A=0, a=1}^{n+1} x^A \frac{\partial X^a}{\partial x^A} \frac{\partial}{\partial x^a}, \tag{3.13}$$

from which we obtain

$$g(\nabla_\xi X, \xi) = \sum_{A=0, a=1}^{n+1} x^a X^A \frac{\partial X^a}{\partial x^A} = -\sum_{a=1}^{n+1} x^a X^a = 0. \tag{3.14}$$

From (3.13) and (3.14), we have $\nabla_\xi X \in \Gamma(S(T\Lambda_0^{n+1}))$, that is, $A_N \xi = 0$. Moreover, by simple calculations, we have

$$C(X, Y) = \bar{g}(\nabla_X Y, N) = \bar{g}(\bar{\nabla}_X Y, N) = -\frac{1}{2(x^0)^2}g(X, Y). \tag{3.15}$$

Clearly, $S(T\Lambda_0^{n+1})$ is integrable. Denote by M^* its leaf. Then

$$\bar{\nabla}_X Y = \nabla_X^* Y + \frac{g(X, Y)}{x^0}(-\frac{1}{2x^0}\xi - x^0 N), \quad \forall X, Y \in \Gamma(TM^*). \tag{3.16}$$

It is obvious that M^* is a totally umbilic Riemannian submanifold of codimension 2 of \mathbb{R}_1^{n+2} . Moreover, using (3.12) and (3.15), we have

$$d\tau(X, Y) = \frac{1}{2}\{C(X, PY) - C(Y, PX)\} = 0, \quad \forall X, Y \in \Gamma(T\Lambda_0^{n+1}). \tag{3.17}$$

As $x^0 \neq 0$, we may suppose $x^0 > 0$ (for $x^0 < 0$ we proceed analogously), and in the normal bundle $TM^{*\perp}$, consider the vector fields

$$V_1 = -\frac{1}{2x^0}\xi - x^0 N \quad \text{and} \quad V_2 = -\frac{1}{2x^0}\xi + x^0 N. \tag{3.18}$$

Note that $\{V_1, V_2\}$ is an orthonormal basis, where V_1 and V_2 are space-like and time-like, respectively. Using the expressions of A_ξ^* and A_N , we get $A_{V_1} = \frac{1}{x^0}I$, from which $\lambda = \frac{1}{x^0}$. From the expressions of ξ and N , we have $\bar{\nabla}_X V_1 = -\frac{1}{x^0}X$ and $\bar{\nabla}_X V_2 = 0$ for all $X \in \Gamma(TM^*)$. Therefore, from the Weingarten formula (3.2) for M^* , we get $\nabla_X^{*\perp} V_1 = 0$ and $\nabla_X^{*\perp} V_2 = 0$. Clearly, $\{V_1, V_2\}$ is a parallel basis with respect to the normal connection $\nabla^{*\perp}$ of M^* . As the vector field V_1 is space-like, that is, $\bar{g}(V_1, V_1) = 1$ and also parallel to the mean curvature of M^* in \mathbb{R}_1^{n+2} (see (3.16)), by Corollary 3.7, we conclude that M^* is minimally immersed in the sphere $\mathbb{S}_1^{n+1}(x^0)$. Note also that (M, g) is totally umbilic, screen totally umbilic and screen conformal in \mathbb{R}_1^{n+2} .

A subbundle \mathcal{D} of the normal bundle $TM^{*\perp}$ is said to be *parallel* in the normal bundle if it is invariant by parallel translation with respect to the normal connection $\nabla^{*\perp}$; that is, if $V \in \Gamma(\mathcal{D})$, then $\nabla_X^{*\perp} V \in \Gamma(\mathcal{D})$ for any $X \in \Gamma(TM^*)$. We also say that the curvature tensor $R^{*\perp}$ of the normal connection $\nabla^{*\perp}$ is *parallel* in the normal bundle if $\nabla^{*\perp} R^{*\perp} = 0$; that is, for any $X, Y, Z \in \Gamma(TM^*)$ and $V \in \Gamma(TM^{*\perp})$, we have

$$(\nabla_Z^{*\perp} R^{*\perp}(X, Y))V = \nabla_Z^{*\perp} R^{*\perp}(X, Y)V - R^{*\perp}(X, Y)\nabla_Z^{*\perp} V = 0. \tag{3.19}$$

As an example, the curvature tensor $R^{*\perp}$ of the normal bundle $TM^{*\perp}$ of Example 3.8 is parallel in the normal bundle. This is due to the fact that the normal bundle is parallel; that is, $\nabla^{*\perp} V = 0$ for any $V \in \Gamma(TM^{*\perp})$.

Next, we define the *first normal space* $\mathcal{Q}(x)$ at $x \in M^*$ as the orthogonal complement in $T_x M^{*\perp}$ of $\{V(x) \in T_x M^{*\perp} : A_{V(x)} = 0\}$.

Lemma 3.9. *Let (M, g) be a screen integrable null hypersurface of $\overline{M}(c)$. Assume that $f : M^* \rightarrow \overline{M}$ is an immersion of a leaf M^* of $S(TM)$ as a codimension 2 submanifold of \overline{M} . Suppose that the curvature tensor $R^{*\perp}$ of the normal bundle to M^* is parallel in the normal bundle. For each $x \in M^*$, let $\mathcal{D}(x) = \{V(x) \in T_x M^{*\perp} : \forall X, Y \ R^{*\perp}(X, Y)V = 0\}$. Then \mathcal{D} is parallel in the normal bundle $TM^{*\perp}$. Moreover, $d\tau$ vanishes on TM^* .*

Proof. The proof of the first assertion follows the similar steps as in [2]. The vanishing of $d\tau$ on TM^* follows from Proposition 3.2 and the definition of $\mathcal{D}(x)$. □

Next, we prove the following result.

Theorem 3.10. *Let (M, g) be an $(n + 1)$ -dimensional screen integrable null hypersurface of a Lorentzian manifold $\overline{M}(c)$. Assume that $f : M^* \rightarrow \overline{M}$ is a minimal, non totally geodesic immersion of a leaf M^* of $S(TM)$ as a codimension 2 submanifold of \overline{M} . Suppose that the curvature tensor $R^{*\perp}$ of the normal bundle to M^* is parallel in the normal bundle. Then there exists an $(n + 1)$ -dimensional totally geodesic submanifold M' of \overline{M} such that f is a minimal immersion of M^* in M' . Furthermore, the underlying null hypersurface (M, g) is minimal in \overline{M} and its shape operator A_N satisfies $\text{trace}_{S(TM)} A_N = 0$.*

Proof. First note that if f is non totally geodesic, then the first normal space of f has constant dimension 1. We first prove the case where the normal bundle $TM^{*\perp}$ is parallel. To that end, let $\mathcal{Q}(x)$ be the first normal space at x . As $\dim \mathcal{Q}$ is constant, $\mathcal{P} = \mathcal{Q}^\perp$, where \perp is the orthogonal complement in the normal bundle $TM^{*\perp}$, is a subbundle of the normal bundle. We want to show that \mathcal{P} is parallel in the normal bundle and then use a result of [12] to draw conclusions. Given $x \in M^*$, choose a unit vector field V_1 , spanning \mathcal{Q} at each point in a neighborhood \mathcal{U} of $x \in M^*$. Let us extend the above field to $\{V_1, V_2\}$ so that the latter spans the normal space at a point of \mathcal{U} . Consider the vector field V_2 which generates the subbundle \mathcal{P} over \mathcal{U} . It then suffices to show that \mathcal{P} is parallel in the normal bundle $TM^{*\perp}$. Given $y \in \mathcal{U}$, let X_1, \dots, X_n be coordinate vector fields in a neighborhood $\mathcal{U}' \subset \mathcal{U}$ of y which diagonalize, at y , all the shape operators of M^* . This is possible because the normal bundle is flat, and it is a result of Cartan that the normal bundle is flat if and only if at each point all the second fundamental forms are simultaneously diagonalizable. As \overline{M} is a space of constant curvature c , we have $\overline{R}(X_j, X_i)V_\alpha = 0$ for $1 \leq i, j \leq n$ and $\alpha = 1, 2$. Then, by (3.5), we have

$$\begin{aligned} & -\nabla_{X_j}^* A_{V_\alpha} X_i + \nabla_{X_i}^* A_{V_\alpha} X_j - A_{\nabla_{X_i}^{*\perp} V_\alpha} X_j + A_{\nabla_{X_j}^{*\perp} V_\alpha} X_i + A_{V_\alpha} [X_j, X_i] \\ & + R^{*\perp}(X_j, X_i)V_\alpha + \{C(X_i, A_{V_\alpha} X_j) - C(X_j, A_{V_\alpha} X_i)\}\xi \\ & + \{B(X_i, A_{V_\alpha} X_j) - B(X_j, A_{V_\alpha} X_i)\}N = 0 \quad \text{for all } X_i, X_j \in T_x M^*. \end{aligned} \tag{3.20}$$

For $V_\alpha \in \mathcal{P}$, that is $\alpha = 2$, (3.20) gives

$$R^{*\perp}(X_j, X_i)V_2 - A_{\nabla_{X_i}^{*\perp} V_2} X_j + A_{\nabla_{X_j}^{*\perp} V_2} X_i = 0,$$

from which we get

$$-A_{\nabla_{X_i}^* V_2} X_j + A_{\nabla_{X_j}^* V_2} X_i = 0 \quad (3.21)$$

after considering the assumption $R^{*\perp} = 0$. Since X_i, \dots, X_n diagonalize all the shape operators at y , we have, at y , $A_{\nabla_{X_j}^* V_2} X_i = a_j^i X_i$ and $A_{\nabla_{X_i}^* V_2} X_j = b_i^j X_j$ for some numbers a_j^i, b_i^j . Therefore, (3.21) implies that $A_{\nabla_{X_j(y)}^* V_2} X_i(y) = 0$ for all $i \neq j$. As the immersion is minimal, we have $A_{\nabla_{X_j(y)}^* V_2} X_i(y) = 0$ for all i, j , and therefore, $A_{\nabla_{X_j(y)}^* V_2} = 0$ for all j . This in turn implies that $\nabla_{X_j(y)}^* V_2 \in \mathcal{P}(y)$, following the definition of \mathcal{P} . That is, \mathcal{P} over \mathcal{U} is parallel in the normal bundle. This implies that \mathcal{Q} is also parallel in the normal bundle. In fact, take $V_2 \in \mathcal{P}$ and $V_1 \in \mathcal{Q}$. As $\nabla^{*\perp}$ is a metric connection, we have $X\bar{g}(V_2, V_1) = \bar{g}(\nabla_X^{*\perp} V_2, V_1) + \bar{g}(V_2, \nabla_X^{*\perp} V_1) = 0$, from which we see that \mathcal{Q} is also parallel in the normal bundle. By Theorem 0.2 of [12, p. 33], there exists a totally geodesic submanifold M' of \bar{M} such that $\dim M' = n + 1$ and $f(M^*) \subset M'$.

Next, since $\dim \mathcal{Q} = 1$, we have $\dim \mathcal{P} = 1$. Let $V_2 \in \mathcal{P}$, then $A_{V_2} = 0$ by the definition of \mathcal{P} . Thus, Lemma 3.1 gives $aA_\xi^* + bA_N = 0$. Taking the trace of this relation along TM^* , we get $a \operatorname{trace} A_\xi^* + b \operatorname{trace} A_N = 0$. On the other hand, since M^* is minimal in \bar{M} , we have $(\operatorname{trace} A_{V_1})V_1 + (\operatorname{trace} A_{V_2})V_2 = 0$, where $V_1 \in \mathcal{Q}$. In view of Lemma 3.1, we have $a \operatorname{trace} A_\xi^* - b \operatorname{trace} A_N = 0$. Therefore, solving gives $\operatorname{trace} A_\xi^* = 0$ and $\operatorname{trace} A_N = 0$, showing that (M, g) is minimal in \bar{M} . This completes the proof of the first case.

Turn to the second case, where the normal bundle is not flat. Set $\mathcal{D}(x) = \{V(x) \in T_x M^{*\perp} : \forall X, Y \ R^{*\perp}(X, Y)V = 0\}$. By Lemma 3.9, \mathcal{D} is parallel in the normal bundle. Let \mathcal{P} be the orthogonal complement of \mathcal{Q} in the normal bundle $TM^{*\perp}$. It is obvious that $\mathcal{P} \subset \mathcal{D}$. Observe that, by Lemma 3.9, \mathcal{D} is parallel and, by (3.6), all the shape operators $A_V, V \in \mathcal{D}$, can be simultaneously diagonalized. Therefore, we can apply the same arguments as in the first case above, with \mathcal{D} in place of $TM^{*\perp}$, to conclude that \mathcal{P} , hence \mathcal{Q} , is parallel in the normal bundle. Again, as in the first case, by Megid's Theorem 0.2 in [12], $f(M^*) \subset M'$, where M' is a totally geodesic submanifold of \bar{M} with dimension $n + 1$. The minimality of (M, g) , as a null hypersurface of \bar{M} , also follows as in the previous case. This completes the proof of the second case, and thus the theorem is proved. \square

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Нотатки щодо скрін-інтегровних ізотропних гіперповерхонь у лоренцових многовидах

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У статті показано, що геометрія скрін-інтегровної ізотропної гіперповерхні може породжуватися ізометричним зануренням шару її скрін-розшарування в обхопний простір. Ми доводимо, за певних геометричних умов, що такі занурення містяться в псевдо-евклідових сферах або гіперболічних просторах, при цьому вихідні ізотропні гіперповерхні обов'язково є омбілічними та скрін-цілком омбілічними. За необхідності наведено приклади, що ілюструють основні ідеї.

Ключові слова: скрін-інтегровані ізотропні гіперповерхні, тензор нормальної кривини шару