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On Subspace Convex-Cyclic Operators

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Let $\mathcal{H}$ be an infinite dimensional real or complex separable Hilbert space. We introduce a special type of a bounded linear operator T and study its important relation with the invariant subspace problem on $\mathcal{H}$: the operator T is said to be subspace convex-cyclic for a subspace $\mathcal{M}$ if there exists a vector whose orbit under T intersects the subspace $\mathcal{M}$ in a relatively dense set. We give the sufficient condition for a subspace convex-cyclic transitive operator T to be subspace convex-cyclic. We also give a special type of the Kitai criterion related to invariant subspaces which implies subspace convex-cyclicity. Finally we show a counterexample of a subspace convex-cyclic operator which is not subspace convex-cyclic transitive.

Key words: ergodic dynamical systems, convex-cyclic operators, Kitai criterion, convex-cyclic transitive operators

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1. Introduction

Ergodic dynamical systems seem to be of interest for a few decades, with an increasing number of papers appearing lately (see, for example, [1, 2, 5, 9, 14]), a large number of them concerning convex-cyclic operators.

A bounded linear operator T on an infinite dimensional separable Hilbert space is convex-cyclic (see [13]) if there exists a vector x in $\mathcal{H}$ such that $\widehat{\mathrm{Orb}(T,x)} = \{P(T)x : P \,\text{is a convex polynomial}\}$ is dense in $\mathcal{H}$ and the vector x is said to be a convex-cyclic vector for T. A bounded linear operator T is said to be cyclic if there exists a vector x in $\mathcal{H}$ such that the linear span of the orbit $[T,x] = \mathrm{span}\{T^n x : n \in \mathbb{N}\}$ is dense in $\mathcal{H}$ and x is called a cyclic vector. If the orbit $\mathrm{Orb}(T,x) = \{T^n x : n \in \mathbb{N}\}$ itself is dense in $\mathcal{H}$ without a linear span, then T is called hypercyclic and x is called a hypercyclic vector. The operator T is said to be supercyclic if the cone generated by $\mathrm{Orb}(T,x)$, i.e., $\mathbb{C}\,\mathrm{Orb}(T,x) = \{\lambda T^n x : \lambda \in \mathbb{C} \,\text{and}\, n \in \mathbb{N}\}$, is dense in $\mathcal{H}$ and x is called a supercyclic vector [11, 12]. In [4], it is mentioned that between a set and its linear span there is a convex hull, from this we get that every hypercyclic operator is convex-cyclic and every convex-cyclic operator is cyclic.

In this work, we want to modify the notions given above and introduce a new concept of a subspace convex-cyclic operator. In our case the orbit of subspace hypercyclic, subspace supercyclic and subspace convex-cyclic under hypercyclic, supercyclic and convex-cyclic operator respectively, intersected with a given selected subspace, is dense in that subspace.

The paper is organized as follows. First, we define the concept of subspace convex-cyclic operators and construct an example of a subspace convex-cyclic operator which does not need to be a convex-cyclic operator (Example 2.3). Next, we prove that being a subspace convex-cyclic transitive operator implies being a subspace convex-cyclic operator. Then we show that a "subspace convex-cyclic criterion" holds. To this end, we find an interesting relation between our new operator and invariant subspaces. We also show, by giving a proper example, that being transitive and fulfilling the above-mentioned criterion are not necessary conditions for the operator to be subspace convex-cyclic. In the end, we present some open questions concerning subspace cyclic operators.

2. Definition and examples

Let $\mathcal{H}$ be an infinite dimensional real or complex separable Hilbert space. Whenever we talk about a subspace $\mathcal{M}$ of $\mathcal{H}$, we will assume that $\mathcal{M}$ is closed topologically. Let $\mathbf{B}(\mathcal{H})$ be the algebra of all linear bounded operators on $\mathcal{H}$. We start with our main definition. Recall that we say that P is a *convex polynomial* if P has only non-negative coefficients adding up to 1, that is if $P(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$ for some $n \in \mathbb{N}$, with $a_0, a_1, \ldots, a_n \geq 0$, and $\sum_{i=0}^{n} a_i = 1$.

Definition 2.1. Let $T \in \mathbf{B}(\mathcal{H})$ and let $\mathcal{M}$ be a non-zero subspace of $\mathcal{H}$. We say that T is a *subspace convex-cyclic operator for* $\mathcal{M}$ if there exists $x \in \mathcal{H}$ such that $\widehat{\mathrm{Orb}(T,x)} \cap \mathcal{M}$ is dense in $\mathcal{M}$, where

$$\widehat{\mathrm{Orb}(T,x)} = \{P(T)x : P \text{ is convex polynomial}\}.$$

The vector x is said to be a subspace convex-cyclic vector.

We will write $\mathcal{M}$ convex-cyclic instead of subspace convex-cyclic for $\mathcal{M}$. Moreover, let us define $\mathrm{CoC}(T,\mathcal{M}) := \{x \in \mathcal{H} : \widehat{\mathrm{Orb}(T,x)} \cap \mathcal{M} \text{ is dense in } \mathcal{M}\}$ as the set of all subspace convex-cyclic vectors for $\mathcal{M}$.

Remark 2.2. Note that $\mathcal{M}$ can be any non-empty subset, convex or not.

Example 2.3. Let T be a convex-cyclic operator on $\mathcal{H}$ and I be the identity operator on $\mathcal{H}$. Then $T \oplus I : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ is a subspace convex-cyclic operator for a subspace $\mathcal{M} = \mathcal{H} \oplus \{0\}$ with subspace convex-cyclic vector $x \oplus 0$.

In fact, since T is a convex-cyclic operator on $\mathcal{H}$, so there exists $x \in \mathcal{H}$ such that $\widehat{\mathrm{Orb}(T,x)}$ is dense in $\mathcal{H}$. Now we can consider $T \oplus I : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$. Let $\mathcal{M} := \mathcal{H} \oplus \{0\} \subseteq \mathcal{H}$ and let there exist $m := x \oplus 0$ such that

$$\begin{aligned}\widehat{\mathrm{Orb}}(T \oplus I, m) &= \{P(T \oplus I)m : P \text{ is convex polynomial}\} \\ &= \{P(T)x \oplus 0 : P \text{ is convex polynomial}\}. \\ &\subseteq \mathcal{H} \oplus \{0\} = \mathcal{M}.\end{aligned}$$

And since $\widehat{\mathrm{Orb}(T,x)}$ is dense in $\mathcal{H}$, then $\widehat{Orb}(T\oplus I,(x\oplus 0))\cap\mathcal{H}\oplus\{0\}$ is dense in $\mathcal{H}\oplus\{0\}=\mathcal{M}$, so we get that $T\oplus I$ is a subspace convex-cyclic operator.

Remark 2.4. The above example shows that if the operator T is subspace convex-cyclic, then T does not need to be convex-cyclic. To clarify it, let us recall Propositions 2.5 and 2.6 from [13] and [8, Chapter 2] respectively. According to usual practice (cf. [13]), whenever we talk about an operator on X (or Y) we will assume that it is a bounded linear operator on a separable, infinite dimensional Hilbert space X (or Y). By Λ, we also denote a linear functional in X^*.

Proposition 2.5 (see [13]). *Let $T : X \to X$ be an operator. If T is convex-cyclic, then*

1. $\|T\| > 1$,
2. $\sup\{\|T^n\| : n \geq 1\} = +\infty$,
3. $\sup\{\|T^{*n}\Lambda\| : n \in \mathbb{N}\} = +\infty$ *for every* $\Lambda \neq 0$ *in* X^*.

Proposition 2.6 (see [8]). *Let $S : X \to X$ and $T : Y \to Y$ be operators. If $S\oplus T$ is hypercyclic, then so are S and T.*

As we have mentioned before, every hypercyclic operator is convex-cyclic, so Proposition 2.6 is of great usage here.

Remark 2.7. Clearly, $T\oplus I$ is not a convex-cyclic operator. In fact, assume that $T\oplus I$ is convex-cyclic on $\mathcal{H}\oplus\mathcal{H}$, then, by Proposition 2.6, the identity operator must be convex-cyclic on $\mathcal{H}$, which is impossible, because the norm of the identity operator is equal to one, and by Proposition 2.5, we get a contradiction.

3. Subspace convex-cyclic transitive operators

In this section we will define subspace convex-cyclic transitive operators and we will show that they are subspace convex-cyclic operators. First we state the classical equivalence of topological transitivity [3] and [8], also convex-cyclic operators [4].

Definition 3.1. Let $T\in\mathbf{B}(\mathcal{H})$ and let $\mathcal{M}$ be a non-zero subspace of $\mathcal{H}$. We say that T is $\mathcal{M}$ *convex-cyclic transitive with respect to* $\mathcal{M}$ if for all non-empty sets $U\subset\mathcal{M}$ and $V\subset\mathcal{M}$, both relatively open, there exists a convex polynomial P such that $U\cap P(T)(V)\neq\phi$ or $P(T)^{-1}(U)\cap V\neq\phi$ contains a relatively open non-empty subset of $\mathcal{M}$.

We use the ideas from [3,8,12] changing them to work for convex polynomial spans and generalizing them to obtain the following.

Theorem 3.2. *Let $T\in\boldsymbol{B}(\mathcal{H})$ and let $\mathcal{M}$ be a non-zero subspace of $\mathcal{H}$. Then*

$$\mathrm{CoC}(T,\mathcal{M})=\bigcap_{j=1}^{\infty}\bigcup_{P\in\mathcal{P}}P(T)^{-1}(\mathcal{B}_j),$$

where $\mathcal{P}$ is the collection of all convex polynomials and $\{\mathcal{B}_j\}$ is a countable open basis for the relative topology of $\mathcal{M}$ as a subspace of $\mathcal{H}$.

Proof. Observe that $x \in \bigcap_{j=1}^{\infty} \bigcup_{P\in\mathcal{P}} P(T)^{-1}(B_j)$ if and only if for all $j \in \mathbb{N}$ there exist convex polynomials P such that $x \in P(T)^{-1}(B_j)$ which implies $P(T)(x) \in B_j$. But, since $\{B_j\}$ is a basis for the relative topology of $\mathcal{M}$, this occurs if and only if $\widehat{\mathrm{Orb}(T,x)} \cap \mathcal{M}$ is dense in $\mathcal{M}$, that is, $x \in \mathrm{CoC}(T,\mathcal{M})$. □

Lemma 3.3. *Denote by $\mathcal{P}$ is the set of all convex polynomials. Let $T \in \boldsymbol{B}(\mathcal{H})$ and let $\mathcal{M}$ be a non-zero subspace of $\mathcal{H}$. Then the following conditions are equivalent:*

(1) *T is $\mathcal{M}$ convex-cyclic transitive with respect to $\mathcal{M}$;*

(2) *for each relatively open subsets U and V of $\mathcal{M}$, there exists $P \in \mathcal{P}$ such that $P(T)^{-1}(U) \cap V$ is a relatively open subset in $\mathcal{M}$;*

(3) *for each relatively open subsets U and V of $\mathcal{M}$, there exists $P \in \mathcal{P}$ such that $P(T)^{-1}(U) \cap V \neq \phi$ and $P(T)(\mathcal{M}) \subset \mathcal{M}$.*

Proof. (3) $\Rightarrow$ (2) Since $P(T) : \mathcal{M} \to \mathcal{M}$ is continuous and we know that U is relatively open in $\mathcal{M}$, then $P(T)^{-1}(U)$ is also relatively open in $\mathcal{M}$. Now for any relatively open V in $\mathcal{M}$ we see that $W := P(T)^{-1}(U) \cap V$ is relatively open in $\mathcal{M}$.

(2) $\Rightarrow$ (1) Since for each relatively open subsets U and V, $P(T)^{-1}(U) \cap V$ is a relatively open subset in $\mathcal{M}$, so $P(T)^{-1}(U) \cap V \neq \phi$; now $W := P(T)^{-1}(U) \cap V$ is relatively open in $\mathcal{M}$, and T is $\mathcal{M}$ convex-cyclic transitive with respect to $\mathcal{M}$.

(1) $\Rightarrow$ (3) By the definition of $\mathcal{M}$ convex-cyclic transitive operators, there are U and V that are relatively open subsets in $\mathcal{M}$ such that $W := P(T)^{-1}(U) \cap V \neq \phi$, and the set W is relatively open in $\mathcal{M}$, and $W \subset P(T)^{-1}(U)$. Then $P(T)(W) \subset U$ and $U \subset \mathcal{M}$, so we get that

$$P(T)(W) \subset \mathcal{M}.$$

Let $x \in \mathcal{M}$. We must show that $P(T)(\mathcal{M}) \subset \mathcal{M}$. Take $w_0 \in W$. Since W is relatively open in $\mathcal{M}$ and $x \in \mathcal{M}$, so there exists $r > 0$ such that $w_0 + rx \in W$.

But $P(T)(W) \subseteq \mathcal{M}$, that is,

$$P(T)(w_0 + rx) = P(T)(w_0) + rP(T)x \in \mathcal{M},$$

so $P(T)(w_0) \in \mathcal{M}$ and $\mathcal{M}$ is a subspace, thus

$$r^{-1}\left(-P(T)(w_0) + P(T)(w_0) + rP(T)(x)\right) \in \mathcal{M},$$

that is, $P(T)(x) \in \mathcal{M}$. This is true for any $x \in \mathcal{M}$, and hence for $P(T)(x) \in \mathcal{M}$, that is, $P(T)(\mathcal{M}) \subseteq \mathcal{M}$. □

Theorem 3.4. *Let $T \in \boldsymbol{B}(\mathcal{H})$ and let $\mathcal{M}$ be a non-zero subspace of $\mathcal{H}$. If T is $\mathcal{M}$ convex-cyclic transitive, then T is $\mathcal{M}$ convex-cyclic.*

Proof. It is a direct consequence of the proofs of Lemma 3.3 and Theorem 3.2. □

Remark 3.5. It is natural to ask whether the converse of Theorem 3.4 is true or not. We will answer this question later in Proposition 5.7.

4. Subspace convex-cyclic criterion

In this section, we introduce a type of the Kitai criterion [10], which is a sufficient criterion for an operator to be $\mathcal{M}$ convex-cyclic. Also, we will relate it with invariant subspaces and we will see that the converse of Theorem 3.4 in general is not true.

Theorem 4.1. *Let $T \in \boldsymbol{B}(\mathcal{H})$ and let $\mathcal{M}$ be a non-zero subspace of $\mathcal{H}$. Assume that there exists X and Y, dense subsets of $\mathcal{M}$, such that for every $x \in X$ and $y \in Y$, there exists a sequence $\{P_k\}_{k\geq 1}$ of convex polynomials such that*

1. *$P_k(T)x \to 0$ for all $x \in X$,*
2. *for each $y \in Y$, there exists a sequence $\{x_k\}$ in $\mathcal{M}$ such that $x_k \to 0$ and $P_k(T)x_k \to y$,*
3. *$\mathcal{M}$ is an invariant subspace for $P_k(T)$ for all $k \geq 0$.*

Then T is an $\mathcal{M}$ convex-cyclic operator.

Proof. To prove that T is an $\mathcal{M}$ convex-cyclic operator, we will use Lemma 3.3 and Theorem 3.4. Let U and V be non-empty relatively open subsets of $\mathcal{M}$. We will show that there exists $k \geq 0$ such that $P_k(T)(U) \cap V \neq \phi$. Since X and Y are dense in $\mathcal{M}$, there exists $v \in X \cap V$ and $u \in Y \cap U$. Furthermore, since U and V are relatively open, there exists $\epsilon > 0$ such that the $\mathcal{M}$-ball centered at v of radius ϵ is contained in V and the $\mathcal{M}$-ball centered at u of radius ϵ is contained in U. By hypothesis, given these $v \in X$ and $u \in Y$, one can choose k large enough such that there exists $x_k \in \mathcal{M}$ with $\|P_k(T)v\| < \frac{\epsilon}{2}$, $\|x_k\| < \epsilon$ and $\|P_k(T)x_k - u\| < \frac{\epsilon}{2}$. We have:

(1) Since $v \in \mathcal{M}$ and $x_k \in \mathcal{M}$, it follows that $v + x_k \in \mathcal{M}$. Also, since

$$\|(v + x_k) - v\| = \|x_k\| < \epsilon,$$

it follows that $v + x_k$ is in the $\mathcal{M} - ball$ centered at v of radius ϵ, and hence $v + x_k \in V$.

(2) Since v and x_k are in $\mathcal{M}$ and $\mathcal{M}$ is invariant under $P_k(T)$, it follows that $P_k(T)(v + x_k) \in \mathcal{M}$. Also,

$$\|P_k(T)(v + x_k) - u\| \leq \|P_k(T)(v)\| + \|P_k(T)(x_k) - u\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and hence $P_k(T)(v + x_k)$ is in the $\mathcal{M} - ball$ centered at u of radius ϵ, and thus $P_k(T)(v + x_k) \in U$.

So, by steps (1) and (2), T is $\mathcal{M}$ convex-cyclic transitive and by Theorem 3.4 we get that $v + x_k \in P_k(T)^{-1}(U) \cap V$, that is, $P_k(T)^{-1}(U) \cap V \neq \phi$, which means that T is an $\mathcal{M}$ convex-cyclic operator. $\square$

As we have mentioned before, most of the papers on convex-cyclic operators use the Kitai criterion [10], which does not contain an analogue of our condition 3. We clarify the need of this additional condition in details in Example 5.2.

Theorem 4.2. *Let $T \in \boldsymbol{B}(\mathcal{H})$ and let $\mathcal{M}$ be a non-zero subspace of $\mathcal{H}$. Assume that there exist subsets of $\mathcal{M}$, X and Y, of which only Y is dense in $\mathcal{M}$, such that for every $x \in X$ and $y \in Y$ there exists a sequence $\{P_k\}_{k\geq 0}$ of convex polynomials such that*

(1) *$P_k(T)x \to 0$ for all $x \in X$;*

(2) *for each $y \in Y$, there exists a sequence $\{x_k\}$ in $\mathcal{M}$ such that $x_k \to 0$ and $P_k(T)x_k \to y$;*

(3) *$X \subset \bigcap\limits_{k=1}^{\infty} P_k(T)^{-1}(\mathcal{M})$.*

Then T is an $\mathcal{M}$ convex-cyclic operator.

Proof. We will use the idea from [11]. Let $\{\xi_j\}_{j=1}^{\infty}$ be a sequence of positive numbers such that

$$\lim_{j\to\infty}\left(j\xi_j + \sum_{i=j+1}^{\infty}\xi_i\right) = 0.$$

In fact, from condition (1), for all $\lambda_j > 0$, $\|P_{k_j}(T)(x)\| < \lambda_j$, and from condition (2), for all $\epsilon_j > 0$, $\|P_{k_j}(T)(x_j) - y_j\| < \epsilon_j$.

So, we can define a sequence of positive numbers $\{\xi_j\}_{j=1}^{\infty}$ as follows: $\xi_i = \lambda_i$ for $i = 1, 2, \ldots, j$, and $\xi_i = \epsilon_i$ for $i = j+1, \ldots$ such that

$$\lim_{j\to\infty}\left(j\xi_j + \sum_{i=j+1}^{\infty}\xi_i\right) = 0.$$

Since $\mathcal{H}$ is separable, we can assume that $Y = \{y_j\}_{j=1}^{\infty}$ for some sequence $\{y_j\}_{j=1}^{\infty}$. We can construct the sequences $\{x_j\}_{j=1}^{\infty} \subset X$ and $\{k_j\}_{j=1}^{\infty}$ by induction. Let $x_1 \in X$ and k_1 be such that

$$\|x_1\| + \|P_{k_1}(T)(x_1) - y_1\| < \xi_1.$$

For each j, choose k_j and $x_j \in X$ such that

$$\|x_j\| + \|P_{k_j}(T)(x_i)\| + \|P_{k_i}(T)(x_j)\| + \|P_{k_j}(T)(x_j) - y_j\| < \xi_j$$

for all $i < j$. Since $\sum\limits_{i=1}^{\infty}\|x_i\| < \sum\limits_{i=1}^{\infty}\xi_i$, we can let $x = \sum\limits_{i=1}^{\infty}x_i$, and then x is well defined.

From condition (3), for every j, we have $P_{k_j}(T)(x) \in \mathcal{M}$ and

$$\begin{aligned}\|P_{k_j}(T)(x) - y_j\| &= \|P_{k_j}(T)(x_j) - y_j + \sum_{i=1}^{j-1} P_{k_j}(T)(x_i) + \sum_{i=j+1}^{\infty} P_{k_j}(T)(x_i)\| \\ &\leq \|P_{k_j}(T)(x_j) - y_j\| + \sum_{i=1}^{j-1} \|P_{k_j}(T)(x_i)\| + \sum_{i=j+1}^{\infty} \|P_{k_j}(T)(x_i)\|. \\ &\leq j\xi_j + \sum_{i=j+1}^{\infty} \xi_i.\end{aligned}$$

Thus,

$$\lim_{j\to\infty} \|P_{k_j}(T)(x) - y_j\| = 0,$$

that is, there exists $x \in X \subseteq \mathcal{M}$ such that $\widehat{\mathrm{Orb}(T,x)} \cap \mathcal{M}$ is dense in $\mathcal{M}$. Then T is an $\mathcal{M}$ convex-cyclic operator. □

Remark 4.3. Notice that in Theorem 4.2 the set X is not dense in $\mathcal{M}$.

Similar to Theorem 4.1, in the previous theorem conditions (1) and (2) are not sufficient for T to be an $\mathcal{M}$ convex-cyclic operator, which we will show in Example 5.2 in the next section.

5. Examples

Before we start this section, we need some notions. Let $(n_k)_{k=1}^{\infty}$ and $(m_k)_{k=1}^{\infty}$ be increasing sequences of positive integers such that $n_k < m_k < n_{k+1}$ for all k. In ℓ^p, $p \geq 1$, denote by $\{e_n\}_{n=0}^{\infty}$ the canonical basis for ℓ^p, and let B, $Be_n = e_{n-1}$, be a backward shift operator. Consider the closed linear subspace $\mathcal{M}$ generated by the set $\{e_j : n_k \leq j \leq m_k$ for some $k \geq 1\}$. If we set $n_k = \{0, 3, 9, 27, 30, \cdots\}$ and $m_k = \{1, 4, 13, 28, 37, \cdots\}$, then we obtain

$$\mathcal{M} := \overline{\mathrm{Lin}\{e_0, e_1, e_3, e_4, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{27}, e_{28}, e_{30}, e_{31}, \ldots, e_{36}, e_{37}, \ldots\}}.$$

Example 5.1. Let S be a linear operator on ℓ^p, $p \geq 1$, defined as $Se_n = \frac{1}{2}e_{n+1}$. Let $\{n_k\}_{k=1}^{\infty}$ be an increasing sequence of positive integer numbers such that $n_0 = 0$ and $n_{k+1} > 2\sum_{i=1}^{k} n_i$. Let $L_0 = \mathcal{M}_0 = \mathrm{Lin}\{e_0\}$ be a linear space generated by e_0, $L_1 = S^{n_1}\mathcal{M}_0$, $\mathcal{M}_1 = \mathcal{M}_0 \oplus L_1$. In general, let $L_{k+1} = S^{n_{k+1}}\mathcal{M}_k$, $\mathcal{M}_{k+1} = \mathcal{M}_k \oplus L_{k+1}$. Define

$$\mathcal{M} = \overline{\bigcup_{k\geq 0} \mathcal{M}_k}.$$

By induction, it is easy to show that $\mathcal{M}_k \subset \mathrm{Lin}\{e_j : j \leq \sum_{i=0}^{k} n_i\}$, and

$$L_{k+1} \subset \mathrm{Lin}\left\{e_j : n_{k+1} \leq j \leq \sum_{i=0}^{k+1} n_i\right\}.$$

Thus, if $x \in \mathcal{M}$, then $x(j) = 0$ for all $\sum\limits_{i=0}^{k} n_i < j < n_{k+1}$.
As a special case, for more clarity let $n_0 = 0$ and since

$$n_{k+1} > 2\sum_{i=0}^{k} n_i,$$

we can suppose that

$$\begin{aligned}
n_1 &= n_{0+1} = 1 > 2\sum_{i=0}^{0} n_i = n_0 = 0,\\
n_2 &= n_{1+1} = 3 > 2\sum_{i=0}^{1} n_i = 2(n_0 + n_1) = 2(0+1) = 2,\\
n_3 &= n_{2+1} = 9 > 2\sum_{i=0}^{2} n_i = 2(n_0 + n_1 + n_2) = 2(0+1+3) = 8,\\
n_4 &= n_{3+1} = 27 > 2\sum_{i=0}^{3} n_i = 26.\\
&\dots
\end{aligned}$$

So, $\{n_k\}_{k=0}^{\infty} = \{0, 1, 3, 9, 27, \dots\}$. Then,

$$\begin{aligned}
L_0 &= \mathcal{M}_0 = \mathrm{Lin}\{e_0\},\\
L_1 &= S^{n_1}(\mathcal{M}_0) = S^1(\mathcal{M}_0) = S(\alpha e_0) = \frac{\alpha}{2} e_1 \in \mathrm{Lin}\{e_1\},\\
\mathcal{M}_1 &= \mathcal{M}_0 \oplus L_1 = \mathrm{Lin}\{e_0\} + \mathrm{Lin}\{e_1\} = \mathrm{Lin}\{\{e_0\} \cup \{e_1\}\} = \mathrm{Lin}\{e_0, e_1\},\\
L_2 &= S^{n_2}(\mathcal{M}_1) = S^3(\mathcal{M}_1) = S^3\left(\alpha_1 e_0 + \alpha_2 e_1)\right) = \frac{\alpha_1}{2^3} e_3 + \frac{\alpha_2}{2^4} e_4 \in \mathrm{Lin}\{e_3, e_4\},\\
\mathcal{M}_2 &= \mathcal{M}_1 \oplus L_2 = \mathrm{Lin}\{e_0, e_1\} + \mathrm{Lin}\{e_3, e_4\} = \mathrm{Lin}\{e_0, e_1, e_3, e_4\},\\
L_3 &= S^{n_3}(\mathcal{M}_2) = S^9(\mathcal{M}_2) = S^9\left(\alpha_1 e_0 + \alpha_2 e_1 + \alpha_3 e_3 + \alpha_4 e_4\right)\\
&= \frac{\alpha_1}{2^9} e_9 + \frac{\alpha_2}{2^{10}} e_{10} + \frac{\alpha_3}{2^{12}} e_{12} + \frac{\alpha_4}{2^{13}} e_{13} \in \mathrm{Lin}\{e_9, e_{10}, e_{12}, e_{13}\},\\
\mathcal{M}_3 &= \mathcal{M}_2 \oplus L_3 = \mathrm{Lin}\{e_0, e_1, e_3, e_4\} \oplus \mathrm{Lin}\{e_9, e_{10}, e_{12}, e_{13}\}\\
&= \mathrm{Lin}\{e_0, e_1, e_3, e_4, e_9, e_{10}, e_{12}, e_{13}\}\\
&\dots
\end{aligned}$$

and so on for $k = 4, 5, \dots$, we see that

$$\begin{aligned}
L_1 &= L_{0+1} \subseteq \mathrm{Lin}\{e_j : n_{0+1} \le j \le \sum_{i=0}^{0+1} n_i\} = \mathrm{Lin}\{e_1\},\\
L_2 &= L_{1+1} \subseteq \mathrm{Lin}\{e_j : n_{1+1} \le j \le \sum_{i=0}^{1+1} n_i\} = \mathrm{Lin}\{e_j; 3 \le j \le 4\},
\end{aligned}$$

$$
\begin{aligned}
L_3 = L_{2+1} &\subseteq \operatorname{Lin}\{e_j : n_{2+1} = 9 \le j \le \sum_{i=0}^{2+1} n_i = 0+1+3+9\}, \\
&= \operatorname{Lin}\{e_j; 9 \le j \le 13\}
\end{aligned}
$$

Also

$$
\begin{aligned}
\mathcal{M}_1 &\subset \{e_j : j \le \sum_{i=0}^{1} n_i = 0+1\}, \\
\mathcal{M}_2 &\subset \{e_j : j \le \sum_{i=0}^{2} n_i = 0+1+3 = 4\}, \\
\mathcal{M}_3 &\subset \{e_j : j \le \sum_{i=0}^{3} n_i = 0+1+3+9 = 13\}.
\end{aligned}
$$

Also notice that $e_2 \notin \mathcal{M}_2$, and $e_2, e_5, e_6, e_7, e_8, e_{11} \notin \mathcal{M}_3$ which can be represented by the following conditions: if $x \in \mathcal{M}$ and $\mathcal{M} = \overline{\bigcup_{k\ge 0} \mathcal{M}_k}$, then for $k = 1$, we have $x(j) = 0$ for $\sum_{i=0}^{k} n_i = 1 < j < 3 = n_{k+1}$; and for $k = 2$, we have $x(j) = 0$ for $\sum_{i=0}^{k} n_i = 4 < j < 9 = n_{k+1}$. Thus we can define $\mathcal{M}_k, \mathcal{M}$, and L_k as above, which satisfy all conditions of Theorem 4.2.

Example 5.2. Let B be a backward shift operator, and $\mathcal{M}$ be a closed subspace defined as in Example 5.1. Then the operator $T = 2B$ satisfies the first two conditions of Theorem 4.2, but T is not an $\mathcal{M}$ convex-cyclic operator.

Proof. Let $Y = (y_j)_{j=1}^{\infty} \subset c_{00} \bigcap \mathcal{M}$ be a dense sequence in $\mathcal{M}$. Then there exists an increasing sequence $(k_j)_{j=1}^{\infty}$ such that $y_i \in \mathcal{M}_{k_j}, i \le j$.
Let $X := \bigcup_{j\ge 1} \{S^{n_{k_i}} y_j \ : \ i \ge j \}$.
Now we will verify that T with X, Y and $\left(P_{n_{k_i}}\right)_{i=1}^{\infty}$, a sequence of convex polynomials, satisfy the first two conditions of Theorem 4.2.

(1) Let $x \in X$ be an arbitrary element. Then we will show that $P_{n_{k_r}}(T)x \to 0$. Since $x \in X$, then there exists $j \ge 1$ such that $x := S^{n_k} y_j$. For $i \ge j$, we choose n_{k_i} large enough such that $\dfrac{1}{2^{n_{k_i}}} \to 0$. Since B is a backward shift operator, we have

$$
\begin{aligned}
P_{n_{k_r}}(T)(x) = P_{n_{k_r}}(T) S^{n_{k_i}} y_j = P_{n_{k_r}}(T) \frac{1}{2^{n_{k_i}}} y_{j+n_{k_i}} &= \sum_{\lambda=0}^{n_{k_r}} \frac{a_\lambda}{2^{n_{k_i}}} T^\lambda y_{j+n_{k_i}} \\
&= \frac{1}{2^{n_{k_i}}} \sum_{\lambda=0}^{j+n_{k_i-1}} a_\lambda 2^\lambda B^\lambda (y_{j+n_{k_i}}) + \frac{1}{2^{n_{k_i}}} \sum_{\lambda=j+n_{k_i}}^{n_{k_r}} a_\lambda 2^\lambda B^\lambda (y_{j+n_{k_i}}) \\
&= \frac{1}{2^{n_{k_i}}} \sum_{\lambda=0}^{j+n_{k_i}-1} a_\lambda 2^\lambda B^\lambda (y_{j+n_{k_i}}) + 0 \to 0
\end{aligned}
$$

because n_{k_r} is large enough for $i \geq j$. So, for all $x \in X$, $P_{n_{k_r}}(T)x \to 0$.

(2) For each $y \in Y \subset c_{00} \bigcap \mathcal{M}$, we must show that there exists a sequence $(x_{k_i})_{i=1}^{\infty}$ in $X := \bigcup_{j \geq 1} \{S^{n_{k_i}} y_j : i \geq j\}$ such that $x_{k_i}(y_j) \to 0$ and $P_{n_{k_i}}(T)x_{k_i} \to y$.

Since $y \in Y = (y_j)_{j=1}^{\infty}$, then there exists $j \geq 1$ such that $y = y_j$ and $y_j \in c_{00} \bigcap \mathcal{M}$. So, there exists k_i such that $y_j \in \mathcal{M}_{k_i}$ for $j \leq i$. But $\mathcal{M}_{k_i} = \mathcal{M}_{k_i - 1} \oplus L_{k_i}$, so there exists $x_{k_i} \in L_{k_i}$ such that $y_j = y_{j-1} + x_{k_i}$, and $L_{k_i} = S^{n_{k_i}}(\mathcal{M}_{k_i - 1})$ and $i \geq j$, thus $x_{k_i} \in X$. Hence, the existence of the sequence $(x_{k_i})_{i=1}^{\infty}$ in X is proved.

We have $x_{k_i} = S^{n_{k_i}}(y_j) = \frac{1}{2^{n_{k_i}}} y_{j+n_{k_i}}$ as $k_i \to \infty$ and $(y_j) \in c_{00}$ then $n_{k_i} \to \infty$ and $x_{k_i} \to 0$.

Show that $P_{n_{k_i}}(T)x_{k_i} \to y$. Since

$$
\begin{aligned}
P_{n_{k_i}}(T)x_{k_i} &= P_{n_{k_i}}(T)S^{n_{k_i}}(y_j) = P_{n_{k_i}}(T)y_{j+n_{k_i}} \frac{1}{2^{n_{k_i}}} \\
&= \sum_{\lambda=0}^{n_{k_i}} a_\lambda \frac{1}{2^{n_{k_i}}} T^\lambda (y_{j+n_{k_i}}) = \sum_{\lambda=0}^{n_{k_i}} \frac{1}{2^{n_{k_i}}} a_\lambda 2^\lambda B^\lambda (y_{j+n_{k_i}}) \\
&= \frac{a_0}{2^{n_{k_i}}} y_{j+n_{k_i}} + \frac{a_1}{2^{n_{k_i}-1}} y_{j+n_{k_i}-1} + \cdots + a_{n_{k_i}} y_j \quad \text{as } k_i \to \infty,
\end{aligned}
$$

then $P_{n_{k_i}}(T)x_{k_i} \to y_j = y$.

Thus the first two conditions of Theorem 4.2 are satisfied.

It remains two show that T is not an $\mathcal{M}$ convex-cyclic operator. Suppose that T is an $\mathcal{M}$ convex-cyclic operator with $\mathcal{M}$ convex-cyclic vector $x \in \mathcal{M}$. For an arbitrary k_i, there exists m such that $P_{n_{k_i}}(T)x \in \mathcal{M}$, where

$$
m \geq \sum_{\lambda=0}^{k_i} n_\lambda. \tag{5.1}
$$

One can choose $l_i > k_i$ such that

$$
n_{l_i+1} - \sum_{\mu=0}^{l_i} n_{k_\mu} > 2m. \tag{5.2}
$$

This implies that

$$
\sum_{\mu=0}^{l_i} n_\mu \leq \sum_{\mu=0}^{l_i} n_\mu + m < n_{l_i+1} - m,
$$

so

$$
\sum_{\mu=0}^{l_i} n_\mu < n_{l_i+1} - m < n_{l_i+1} - m + \sum_{\mu=0}^{k_i} n_\mu. \tag{5.3}
$$

From Equation (5.1), we get that $-m + \sum_{\mu=0}^{k_i} n_\mu$ has a negative value, so

$$
\sum_{\mu=0}^{l_i} n_\mu < n_{l_i+1} - m < n_{l_i+1} - m + \sum_{\mu=0}^{k_i} n_\mu < n_{l_i+1}. \tag{5.4}
$$

Equation (5.4) gives us

$$\sum_{\mu=0}^{l_i} n_\mu + m < n_{l_i+1} < n_{l_i+1} + m,$$

so there exists some positive integer r such that

$$\sum_{\mu=0}^{l_i} n_\mu + m \leq r \leq n_{l_i+1} + m, \tag{5.5}$$

that is,

$$\sum_{\mu=0}^{l_i} n_\mu \leq r - m \leq n_{l_i+1}.$$

Now, from Remark 4.3, if $x \in \mathcal{M}$, then $x(r-m) = 0$ for all

$$\sum_{\mu=0}^{l_i} n_\mu \leq r - m \leq n_{l_i+1}.$$

So, for r satisfying Equation (5.5), we get

$$P_{n_{k_i}}(T)x(r) = \sum_{\lambda=0}^{n_{k_i}} a_\lambda T^\lambda x(r) = \sum_{\lambda=0}^{n_{k_i}} a_\lambda 2^\lambda x(r-\lambda) = 0.$$

By the construction of $\mathcal{M}$, it is easy two see that $\mathcal{M} = \bigoplus_{q=0}^{\infty} S^{n_{k_q}} \mathcal{M}_{l_1}$ for some increasing sequence $\left(n_{k_q}\right)_{q=0}^{\infty}$ such that $n_{k_0} = 0$ and

$$n_{k_q+1} - n_q - \sum_{\mu=0}^{l_i} n_\mu > 2m, \quad q \geq 0.$$

Now, for all

$$n_{k_q} \leq r \leq n_{k_q} + \sum_{\mu=0}^{k_i} n_\mu, \quad q \geq 1,$$

we have

$$P_{n_{k_i}}(T)x(r) = \sum_{\lambda=0}^{n_{k_i}} a_\lambda T^\lambda x(r) = \sum_{\lambda=0}^{n_{k_i}} a_\lambda 2^\lambda x(r-\lambda) \in \operatorname{Lin}\{e_0, e_1, \cdots, e_\kappa\} \subseteq \mathcal{M},$$

where $0 \leq r - \lambda \leq \sum_{\mu=0}^{k_i} n_\mu = \kappa$ for all $\lambda = 0, 1, \cdots, n_{k_q}$.
For some $x \in X$, we get $P_{n_{k_i}}(T)x \notin \mathcal{M}$, i.e., $x \in X$, but $x \notin P_{n_{k_i}}(T)^{-1}(M)$. $\square$

To explain the above proof numerically, let $n_k = \{0, 1, 3, 9, 27, \cdots\}$, $\quad e_4 \in X$ since $2^4 e_4 = S^3(e_1)$ as $3 \geq 1$. We have

$$\begin{aligned} P_{n_{k_i}}(T)(e_4) &= 2^4 a_0 e_4 + a_1 T e_4 + a_2 T^2 e_4 + \cdots + a_{n_k} T^{n_k} e_4 \\ &= \alpha_0 e_4 + \alpha_1 e_3 + \alpha_2 e_2 + \alpha_3 e_1 + \alpha_4 e_0 + 0 + 0 + \cdots + 0 \\ &\in \operatorname{Lin}\{e_0, e_1, e_2, e_3, e_4\}, \quad \text{but} \quad e_2 \notin \mathcal{M}. \end{aligned}$$

Example 5.3. Let $\lambda \in \mathbb{C}$ such that $|\lambda| > 1$, and consider $T := \lambda B$, where B is a backward shift on ℓ^2. Let $\mathcal{M}$ be a subspace of ℓ^2 consisting of all sequences with zeros on even entries: $\mathcal{M} = \left\{\{a_n\}_{n=0}^{\infty} \in \ell^2 : a_{2k} = 0 \text{ for all } k\right\}$. Then T is a convex-cyclic operator for $\mathcal{M}$.

Proof. We will apply Theorem 4.1 to give the alternative proof. Let $X = Y$ be the subsets of $\mathcal{M}$ consisting of all finite sequences, i.e., the sequences that have only a finite number of non-zero entries. This clearly is a dense subset of $\mathcal{M}$. Let $P_k(T) := T^{2k}$, where $\{P_k\}_{k>1}$ is a sequence of convex polynomials.

Now let us check that all three conditions of Theorem 4.1 hold. Let $x \in X$. Since x has finitely many non-zero entries, then $P_k(T)x$ will be zero for large enough k. Thus condition 1 holds.

Let $y \in Y$ and define $x_k := \frac{1}{\lambda^{2k}} S^{2k} y$, where S is a forward shift operator on ℓ^2. Each x_k is in $\mathcal{M}$ since the even entries of y are shifted by S^{2k} into the even entries of x_k. We have $\|x_k\| = \frac{1}{|\lambda|^{2k}}\|y\|$, and thus it follows that $x_k \to 0$ since $|\lambda| > 1$. Also, because

$$P_k(T)(x_k) := T^{2k}(x_k) = (\lambda B)^{2k}(x_k) = (\lambda B)^{2k} \frac{1}{\lambda^{2k}} S^{2k} y = y,$$

we have that condition 2 holds.

Condition 3 also holds. It follows from the fact that if a vector has zeros on all even positions, then it will also have zero entries on all even positions after the application of the backward shift any even number of times. Therefore T is a convex-cyclic operator on $\mathcal{M}$. $\square$

Remark 5.4. There is a relation between $\mathcal{M}$ convex-cyclic operators and invariant subspaces. As we see in Theorem 3.2, $\mathcal{M}$ is invariant for $P_k(T)$ for all k, also in Example 2.3 it is invariant for $P_k(T)$ whenever $k = 1$. But the converse is not true, i.e., if for all k, $P_k(T)$ is an $\mathcal{M}$ convex-cyclic operator, then $\mathcal{M}$ does not need to be invariant under $P_k(T)$ as in Example 5.3 for $k = 1$. But the subspace $\mathcal{M}$ is invariant under the $\mathcal{M}$ convex-cyclic operator for T^{2k}.

The following two lemmas show that an $\mathcal{M}$ convex-cyclic operator does not need to be $\mathcal{M}$ convex-cyclic transitive. We use the arguments similar to those in [12]. Remind that $\mathcal{M}$ is a closed linear subspace generated by the set $\{e_j : n_k \leq j \leq m_k \text{ for some } k \geq 1\}$ for given (arbitrary) increasing sequences of positive integers $(n_k)_{k=1}^{\infty}$ and $(m_k)_{k=1}^{\infty}$ such that $n_k < m_k < n_{k+1}$.

Lemma 5.5. *If* $\sup_{k\geq 1}(m_k - n_k) = \infty$, *then* $T = 2B$ *is an* $\mathcal{M}$ *convex-cyclic operator.*

Proof. Let $Y = (y_j)_{j=1}^{\infty} \subset c_{00} \cap \mathcal{M}$ be a dense subset of $\mathcal{M}$. Since $\sup_{k\geq 1}(m_k - n_k) = \infty$, for y_1 there exist k_1 and N_1 such that

$$|y_1| < n_{k_1} < N_1 < N_1 + |y_1| < m_{k_1}.$$

By induction, it is easy to see that there exist increasing sequences $(N_j)_{j=1}^{\infty}$ $(k_j)_{j=1}^{\infty}$ such that for every fixed $j > 1$ we have:

1. $|y_j| < n_{k_j} < N_j < N_j + |y_i| < m_{k_j}$ for all $1 \le i \le j$,

2. $N_j - N_i > n_{k_j}$ for all $1 \le i < j$.

Let

$$X = \bigcup_{j \ge 1} \{S^{N_i} y_j : i \ge j\}.$$

It is clear that $X \subset \mathcal{M}$. Verify that T, X, Y and $(N_j)_{j=1}^{\infty}$ satisfy all conditions of Theorem 4.2, and hence T is an $\mathcal{M}$ convex-cyclic operator. Owing to Example 5.2, conditions (1) and (2) hold. It is enough to check that

$$X \subset \bigcap_{j=1}^{\infty} P_{N_j}(T)^{-1}(\mathcal{M}).$$

Let $x = S^{N_i} y_j \in X$, where $i \ge j$. For every l, consider $P_{N_l}(T)(x)$. If $l > i$, then

$$\begin{aligned} p_{N_l}(T)(x) &= \sum_{\lambda=0}^{N_l} a_\lambda T^\lambda x = \sum_{\lambda=0}^{N_{i-1}} a_\lambda T^\lambda x + a_{N_i} T^{N_i} x + \sum_{\lambda=N_{i+1}}^{N_l} a_\lambda T^\lambda x \\ &= \sum_{\lambda=0}^{N_{i-1}} a_\lambda T^\lambda x + a_{N_i} y_i + 0, \end{aligned}$$

and since T is a backward shift operator and $x = S^{N_i} y_j$,

$$p_{N_l}(T)(x) = \sum_{\lambda=0}^{N_{l-1}} a_\lambda S^{N_i - \lambda} y_j + a_{N_i} y_j.$$

It follows from

$$N_i - \lambda \ge N_i - N_{i-1} > n_{k_i} \quad \text{and} \quad N_i - N_{i-1} < N_i - \lambda + |y_j| < N_i + |y_j| < m_{k_i}$$

that $n_{k_i} < N_i - \lambda < m_{m_k}$. Hence

$$p_{N_l}(T)(x) = \sum_{\lambda=0}^{N_{l-1}} a_\lambda S^{N_i - \lambda} y_j + a_{N_i} y_j \in \mathrm{Lin}\{e_r : n_{k_i} \le r \le m_{k_i}\} + \mathcal{M} \subseteq \mathcal{M},$$

that is, $p_{N_l}(T)(x) \in \mathcal{M}$. Thus, $x \in p_{N_l}(T)^{-1}(\mathcal{M})$.

If $l < i$, then

$$p_{N_l}(T)(x) = \sum_{\lambda=0}^{N_l} a_\lambda T^\lambda x = \sum_{\lambda=0}^{N_l} a_\lambda S^{-\lambda} x = \sum_{\lambda=0}^{N_l} a_\lambda S^{N_i - \lambda} y_j.$$

But

$$N_i - \lambda \ge N_i - N_l > n_{k_i} \quad \text{and} \quad N_i - N_l \le N_i - \lambda + |y_l| < N_i + |y_l| < m_{k_i}.$$

So, $n_{k_i} < N_i - \lambda < m_{k_i}$. Hence

$$p_{N_l}(T)(x) = \sum_{\lambda=0}^{N_l} a_\lambda S^{N_i-\lambda} y_j \in \mathrm{Lin}\{e_r : n_{k_i} \le r \le m_{k_i}\} \subset \mathcal{M},$$

and thus $x \in p_{N_l}(T)^{-1}(\mathcal{M})$. If $l = i$, then

$$\begin{aligned} p_{N_l}(T)(x) &= \sum_{\lambda=0}^{N_l} a_\lambda T^\lambda x = \sum_{\lambda=0}^{N_{l-1}} a_\lambda T^\lambda x + a_{N_l} T^{N_l} x \\ &= \sum_{\lambda=0}^{N_{l-1}} a_\lambda S^{N_i \lambda}(y_j) + a_{N_l} S^{N_i - N_l}(y_j) = \sum_{\lambda=0}^{N_{l-1}} a_\lambda S^{N_i-\lambda}(y_j) + a_{N_l}(y_j). \end{aligned}$$

Since

$$N_i - \lambda \ge N_i - N_{l-1} > N_{k_i} \quad \text{and} \quad N_i - N_{l-1} \le N_i - \lambda + |y_l| < N_i + |y_l| < m_{k_i},$$

then $n_{k_i} < N_i - \lambda < m_{k_i}$.
Hence

$$p_{N_l}(T)(x) = \sum_{\lambda=0}^{N_{l-1}} a_\lambda S^{N_i-\lambda}(y_j) + a_{N_l}(y_j)\,\mathrm{Lin}\{e_r : n_{k_i} \le r \le m_{k_i}\} + \mathcal{M} \subset \mathcal{M}.$$

That is, $x \in p_{N_l}(T)^{-1}(\mathcal{M})$. Consequently, T satisfies all conditions of Theorem 4.2, so T is an $\mathcal{M}$ convex-cyclic operator. □

Lemma 5.6. *If* $\sup_{k\ge1}(n_{k+1}-m_k) = \infty$, *then* $T = 2B$ *is not* $\mathcal{M}$ *convex-cyclic transitive.*

Proof. Suppose that $T = 2B$ is $\mathcal{M}$ convex-cyclic transitive. Let U and V be two non-empty open subsets of $\mathcal{M}$ and suppose that there exists a positive number m such that $U \cap P_m(T)^{-1}V$ contains an open subset W of $\mathcal{M}$. For $x \in W$, there exists $\epsilon > 0$ such that if $y \in \mathcal{M}$ and $\|x - y\| < \epsilon$, then $y \in W$ since W is open.

Since $\sup_{k\ge1}(n_{k+1} - m_k) = \infty$, then $\exists j \in \mathbb{Z}^+$such that $n_{j+1} - m_j > m$. It follows from $\|e_{n_{j+1}}\| \le 1$ and $\frac{1}{2}\epsilon < \epsilon$ that $\|\frac{1}{2}\epsilon e_{n_{j+1}}\| \le \epsilon$. So, $\|x-(x+\frac{1}{2}\epsilon e_{n_{j+1}})\| < \epsilon$. Now consider $y := x + e_{n_{j+1}}\frac{1}{2}\epsilon \in \mathcal{M}$ and $\|x - y\| < \epsilon$. Since $y \in \mathcal{M}$, then $y \in W \subset U \cap P_m(T)^{-1}V$, and $y \in P_m(T)^{-1}V$ which means $P_m(T)(y) \in V$ and

$$P_m(T)(y) = P_m(T)\left(x + \frac{1}{2}\epsilon P_m(T) e_{n_{j+1}}\right) = P_m(T)(x) + \frac{1}{2}\epsilon P_m(T) e_{n_{j+1}} \in V \subset \mathcal{M}.$$

Then we get that $P_m(T)e_{n_{j+1}} \in \mathcal{M}$, which is a contradiction because

$$P_m(T)e_{n_{j+1}} = \sum_{\lambda=0}^{m} a_\lambda T^\lambda(e_{n_{j+1}}) = \sum_{\lambda=0}^{m} a_\lambda e_{n_{j+1}-\lambda} \in \ \mathrm{Lin}\{e_r : n_j \le r \le m_j\} \subseteq \mathcal{M}.$$

That is, $n_{j+1} - \lambda \le m_j$ and $n_{j+1} - m \le n_{j+1} - \lambda \le m_j$. This contradicts the fact that $n_{j+1} - m > m_j$. □

Proposition 5.7. *If*

$$\sup_{k\geq 1}(m_k - n_k) = \sup_{k\geq 1}(n_{k+1} - m_k) = \infty,$$

then $T = 2B$ is an $\mathcal{M}$ convex-cyclic operator, but T is not $\mathcal{M}$ convex-cyclic transitive.

6. Open questions

We end our paper with some open questions on $\mathcal{M}$ convex-cyclic operators.

Bourdon and Feldman [6] proved for hypercyclic operators that somewhere dense orbits are everywhere dense. Thus it is natural to ask:

Open question 1. If any orbit of an $\mathcal{M}$ convex-cyclic operator in $\mathcal{M}$ is somewhere dense, then does it imply that it is everywhere dense in $\mathcal{M}$?

Open question 2. Does there exist an $\mathcal{M}$ convex-cyclic operator for $\mathcal{M}$ such that $P_k(T)(M) \subset M$ does not hold for any k?

H. Rezaei [13] introduced convex-cyclic operators, and N. Feldman and P. McGuire [7] characterized completely such operators on finite dimensional vector spaces. We can also ask:

Open question 3. If $\mathcal{H}$ is a finite dimensional vector space, then is there an $\mathcal{M}$ convex-cyclic operator for every subspace of $\mathcal{H}$?

Open question 4. If $T : \mathcal{H} \to \mathcal{H}$ is an $\mathcal{M}$ convex-cyclic operator acting on a Hilbert space $\mathcal{H}$, then is T^m an $\mathcal{M}$ convex-cyclic operator for every integer $m > 1$?

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Про підпросторові опукло-циклічні оператори

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Нехай $\mathcal{H}$ є нескінченновимірним дійсним або комплексним гільбертовим простором. Уведено спеціальний тип обмеженого лінійного оператора T і досліджено його важливий зв'язок із проблемою інваріантного підпростору в $\mathcal{H}$: оператор T називається підпросторово опукло-циклічним для підпростору $\mathcal{M}$ якщо існує вектор, орбіта якого відносно T перетинає підпростір $\mathcal{M}$ у відносно щільній множині. Надано достатню умову для того, щоб підпросторово опукло-циклічний транзитивний оператор T був підпросторово опукло-циклічним. Також надано спеціальний тип критерію Китаї, пов'язаного з інваріантними підпросторами, з якого витікає підпросторова опукло-циклічність. Наприкінці наводиться контрприклад підпросторово опукло-циклічного оператора, що не є підпросторово опукло-циклічним транзитивним.

Ключові слова: ергодичні динамічні системи, опукло-циклічні оператори, критерій Китаї, опукло-циклічні транзитивні оператори