Journal of Mathematical Physics, Analysis, Geometry
2021, Vol. 17, No. 1, pp. 95–115
doi: https://doi.org/10.15407/mag17.01.095

On a Spectral Inverse Problem in Perturbation Theory

V.A. Marchenko, A.V. Marchenko, and V.A. Zolotarev

We consider an inverse spectral problem for Sturm–Liouville operators $\widehat{H}_V$ defined on the interval $[a,b]$ by a certain potential $V \in L^2[a,b]$ and mixed separated boundary conditions. We show that if the L^1-norm of V is small enough, then there exists V_{app} such that $\|V - V_{\text{app}}\|_{L^2} = O(\|V\|^2_{L^1})$ and we indicate an algorithm to find V_{app}. The algorithm determines the Fourier coefficients of V_{app} with respect to eigenfunctions $\{\psi_{k,0}\}_{k=1}^{\infty}$ of the unperturbed operator $\hat{H}_0$ via eigenvalues $\{\lambda_{k,V}\}_{k=1}^{\infty}$ of the "perturbed" operator $\hat{H}_V$, the values of its eigenfunctions $\{\psi_{k,V}\}_{k=1}^{\infty}$ at the endpoints of $[a,b]$, and values of $\{\psi_{k,V}\}_{k=1}^{\infty}$ and their derivatives at the middle of $[a,b]$.

Key words: spectral theory, potential, inverse problem, perturbation theory

Mathematical Subject Classification 2010: 34A55, 34B24, 47A55

1. Introduction

The perturbation theory of self-adjoined operators compares two operators, one of which, $\widehat{H}_0$, is known and possesses good analytical properties (it is called the unperturbed operator), while the second $\widehat{H}_V = \widehat{H}_0 + \widehat{V}$, the perturbed one, differs from $\widehat{H}_0$ by a small term $\widehat{V}$ called the perturbation. In what follows we always denote objects related to a perturbed operator with an index $\alpha = V$ and those for unperturbed operator with an index $\alpha = 0$.

The direct problem of spectral theory is to find the spectral data (eigenvalues $\{E_{k,V}\}_{k=1}^{\infty}$ and corresponding eigenfunctions $\{\psi_{k,V}\}_{k=1}^{\infty}$) of $\widehat{H}_V$ given those of $\widehat{H}_0$ and the perturbation $\widehat{V}$. In particular, spectral perturbation theory investigates what impact small perturbations have on the spectral data, see, e.g., [1]. It is convenient to define the smallness of the perturbation in terms of an appropriate norm $\|\widehat{V}\|$ of $\widehat{V}$. The direct problem of the first order perturbation theory is then to find the spectral data of the perturbed operator up to terms of the order $o\left(\|\widehat{V}\|\right)$.

The inverse problem requires doing the opposite: given the spectral data of the perturbed operator $\widehat{H}_V$ one has to find the perturbation $\widehat{V}$. Note that if the complete set of spectral data, i.e., all the eigenvalues and eigenfunctions of $\widehat{H}_V$

are known, then the inverse problem is trivial, because we have by the spectral theorem (in a simple case of discrete spectrum)

$$\widehat{H}_V = \sum_k E_{k,V}\, \psi_{k,V} \otimes \psi_{k,V}. \tag{1.1}$$

In practice, however, the complete spectral data, i.e., $\{E_{k,V}\}_{k=1}^{\infty}$ and $\{\psi_{k,V}\}_{k=1}^{\infty}$, of the perturbed operator usually are not known. Therefore a significant part of the inverse problem studies consists in determining the properties of perturbation that can be recovered given some incomplete spectral data and specifying these data. In particular, the first order perturbation theory provides linear in $\widehat{V}$ relationships between the change in certain spectral data and the change of the operator itself (perturbation $\widehat{V}$). Of course one has to make certain assumptions about the perturbation.

This paper deals with a particular case of the problem formulated above where $\widehat{H}_V$ is the ordinary differential operator of second order defined on a finite interval of the real axis and having mixed boundary conditions (the Sturm–Liouville operator)

$$\widehat{H}_V = -\widehat{D}_x^2 + \widehat{V}, \tag{1.2}$$

where $\widehat{D}_x^2$ is the operator of the second derivative and $\widehat{V}$ is an operator of multiplication by a real-valued function V called the potential. We consider the case where the domains of $\widehat{H}_0$ and $\widehat{H}_V$ coincide.

We show, that the function V can be recovered up to the second order of magnitude in its norm provided certain partial spectral data of the perturbed operator are known. These are all the eigenvalues, the values of corresponding eigenfunctions at the ends of the interval and values of these eigenfunctions and their derivatives in the middle of the interval (unlike the complete of set of spectral data in (1.1)). The exact formulation of our results is given in the Main Theorem at the end of Section 3.

We want to stress that we do not consider in this paper the problem of existence of a Sturm–Liouville operator with given spectral data. This is a separate question. We will just assume that such an operator exists and we recover its potential V up to terms of the order $O(\|V\|^2)$.

2. Definitions, notations, and spectral properties of the unperturbed operator

We denote by $L^2[a,b]$ a Hilbert space of functions $f : [a,b] \to \mathbb{C}$ with the inner product

$$(f_1, f_2) := \frac{1}{(b-a)} \int_a^b f_1(x)\, \overline{f_2(x)}\, dx.$$

Consider the (unperturbed) Sturm–Liouville operator

$$\widehat{H}_0 = -\widehat{D}_x^2 \tag{2.1}$$

acting in this space, where $\widehat{D}_x^2$ is the operator of the second derivative with respect to x. The domain of $\widehat{H}_0$ consists of functions $y \in L^2[a,b]$ such that $y'' \in L^2[a,b]$ and

$$y'(a) - h_- y(a) = 0, \quad y'(b) + h_+ y(b) = 0, \quad h_\mp > 0. \tag{2.2}$$

Here and below we denote a derivative with respect to x by the apostrophe "$'$".

The operator $\widehat{H}_V$ of (1.2) is obtained from the operator $\widehat{H}_0$ by adding to it an operator $\widehat{V}$ of multiplication by a real-valued function

$$V \in L^2[a,b], \tag{2.3}$$

called the potential. The domain of $\widehat{H}_V$ coincides with that of $\widehat{H}_0$. Indeed, if $y'' \in L^2[a,b]$, then y is bounded on $[a,b]$, hence,

$$|(Vy)(x)| \le \max_{t\in[a,b]} |y(t)|\,|V(x)|,$$

hence, $Vy \in L^2[a,b]$ in view of (2.3).

Operators $\widehat{H}_\alpha$, $\alpha = 0, V$, are self-adjoint, bounded from below and have a simple discrete spectrum (see [2, Lemma 3.3.1], [3, Chap. V, Section 19, Theorem 5], or Section 4 of the paper). Denote their eigenvalues

$$E_{1,\alpha} < E_{2,\alpha} < \cdots. \tag{2.4}$$

It is convenient to introduce the spectral parameter λ related to E as

$$\lambda_{k,\alpha} = \sqrt{E_{k,\alpha}}, \quad k = 1,2,\ldots, \; \alpha = 0, V.$$

The eigenfunctions of $\widehat{H}_\alpha$ are non-zero solutions of the differential equation

$$-y''(x) + V(x)\,y(x) = \lambda_{k,\alpha}^2 y(x) \tag{2.5}$$

satisfying boundary conditions (2.2) which are the same for $\alpha = 0, V$ (recall that $\alpha = 0$ corresponds to $V = 0$).

On the other hand, for any $\lambda \in \mathbb{C}$ and $x \in [a,b]$ the analog

$$-y''(x) + V(x)\,y(x) = \lambda^2 y(x) \tag{2.6}$$

of (2.5) admits a unique solution φ_α satisfying the initial conditions

$$\varphi_\alpha(\lambda, a) = 1, \quad \varphi'_\alpha(\lambda, a) = h_-. \tag{2.7}$$

If, in addition, we have

$$\varphi'_\alpha(\lambda, b) + h_+ \varphi_\alpha(\lambda, b) = 0, \tag{2.8}$$

then φ_α satisfies both boundary conditions (2.2), hence, is the eigenfunction of $\widehat{H}_\alpha$ corresponding to the eigenvalue $E_{k,\alpha} = \lambda_{k,\alpha}^2$ of $\widehat{H}_\alpha$, $\alpha = 0, V$.

The function

$$Q_\alpha(\lambda) = \varphi'_\alpha(\lambda, b) + h_+ \varphi_\alpha(\lambda, b), \qquad \alpha = 0, V, \qquad \lambda \in \mathbb{C}, \tag{2.9}$$

is called the characteristic function of $\widehat{H}_\alpha$. The squares of its zeros $\{\lambda_{k,\alpha}\}_{k=1}^{\infty}$ are eigenvalues (2.4) of $\widehat{H}_\alpha$, and corresponding solutions $\{\varphi_\alpha(\lambda_{k,\alpha},x)\}_{k=1}^{\infty}$ of (2.6)–(2.8) form an orthogonal (but not an orthonormal because of normalization (2.7)) basis in $L^2[a,b]$. They are related to the complete system $\{\psi_{k,\alpha}\}_{k=1}^{\infty}$ of orthonormal eigenfunctions of $\widehat{H}_\alpha$ by an equality

$$\psi_{k,\alpha} = \frac{\varphi_{k,\alpha}}{\|\varphi_{k,\alpha}\|_{L^2[a,b]}}, \quad \varphi_{k,\alpha}(x) = \varphi_\alpha(\lambda_{k,\alpha},x). \tag{2.10}$$

It is easy to verify that the substitution

$$x = a + (b-a)\,x_1, \quad x \in [a,b],\ x_1 \in [0,1], \tag{2.11}$$

transforms unitarily the space $L^2[a,b]$ in $L^2[0,1]$ and the operator $\widehat{H}_V$ into the operator

$$\widehat{H}_{V_1} = -\widehat{D}^2_{x_1} + V_1\,(x_1)\,, \quad V_1\,(x_1) = (b-a)^2\, V\,(a+(b-a)\,x_1)$$

with the boundary conditions

$$y'\,(0) - (b-a)\,h_-y\,(0) = 0, \quad y'\,(1) + (b-a)\,h_+y\,(1) = 0. \tag{2.12}$$

This observation allows us to confine ourselves to the interval $[0,1]$. Note for the future purpose that eigenvalues of the operator $\widehat{H}_{V_1}$ are

$$E_{k,V_1} = (b-a)^2\, E_{k,V} \tag{2.13}$$

and its eigenfunctions are

$$\psi_{k,V_1}\,(x_1) = \psi_{k,V}\,(a+(b-a)\,x_1)\,. \tag{2.14}$$

Analogous formulas are true for the unperturbed operator.

In what follows we will consider without loss of generality the Sturm-Liouville operators (1.2) acting in $L^2[0,1]$ with the domain consisting of twice differentiable functions y with $y'' \in L^2[0,1]$ satisfying boundary conditions

$$y'\,(0) - h_-y\,(0) = 0, \quad y'\,(1) + h_+y\,(1) = 0, \quad h_\mp > 0. \tag{2.15}$$

Note that the condition $h_\mp > 0$ excludes the Dirichlet and the von Neumann boundary conditions, however, our results can be extended to these cases as well.

We will remind now some basic properties of the spectral data of the unperturbed operator $\widehat{H}_0$. For more details and proofs see, e.g., [2,3].

Direct calculations show that the operator $\widehat{H}_0$ is positively defined. Therefore all its eigenvalues $E_{k,0}$, $k = 1,2,\ldots$ are strictly positive and the corresponding $\lambda_{k,0} := \sqrt{E_{k,0}}$, $k = 1,2,\ldots$ are real.

The equation (2.6) is now

$$-\,y''\,(x) = \lambda^2 y\,(x)\,, \quad E = \lambda^2, \tag{2.16}$$

and its solution for $\lambda \neq 0$ (real, if λ is real) and initial data (cf. (2.8))

$$\varphi_0(\lambda, 0) = 1, \quad \varphi_0'(\lambda, 0) = h_- \tag{2.17}$$

are

$$\varphi_0(\lambda, x) = \cos \lambda x + h_- \frac{\sin \lambda x}{\lambda} = \frac{1}{2} C_+(\lambda, x), \tag{2.18}$$

where

$$C_\pm(\lambda, x) := \left(e^{i\lambda x} B_-(\lambda) \pm e^{-i\lambda x} B_-(-\lambda)\right), \quad B_\pm(\lambda) := 1 + h_\pm / i\lambda. \tag{2.19}$$

The characteristic function (2.20) of $\widehat{H}_0$ is

$$Q_0(\lambda) = \varphi_0'(\lambda, 1) + h_+ \varphi_0(\lambda, 1) = p \cos \lambda - \left(\lambda - q^2/\lambda\right) \sin \lambda, \tag{2.20}$$

where we denote

$$p = h_- + h_+ > 0, \ q = \sqrt{h_- \, h_+} > 0. \tag{2.21}$$

The function Q_0 is an even entire function of λ. It is not hard to find (see, e.g., [2,3] and Section 4 of the paper) that

- all zeros $\{\lambda_{k,0}\}_{k=1}^{\infty}$ of Q_0 are real and simple;
- there is one and only one zero $\lambda_{k,0}$ in the interval $((k-1)\pi, k\pi)$, $k = 1, 2, \ldots$;
- if k is large enough then we have the bounds

$$k\pi + \frac{h_- + h_+}{k\pi} - \frac{(h_- + h_+) \cdot h_- \cdot h_+}{((k+1)\pi)^3} < \lambda_{k,0} < k\pi + \frac{h_- + h_+}{k\pi}. \tag{2.22}$$

To compare spectral data of the perturbed and the unperturbed operators one first needs to establish a one to one correspondence between their eigenvalues. The explicit construction of the correspondence is given in Section 4. Here and in the next section we will just give certain facts that allow us to formulate the main result of the paper, see Theorem 3.1 in the next section.

We shall see below that a convenient measure of the perturbation "smallness" is the norm

$$\|V\|_{L^1[0,1]} := \int_0^1 |V(x)| \, dx \leq \|V\|_{L^2[0,1]}. \tag{2.23}$$

It is also technically convenient to define an equivalent (for small $\|V\|_{L^1}$) quantity

$$\sigma := \|V\|_{L^1[0,1]} \exp\left\{\|V\|_{L^1[0,1]}\right\}, \tag{2.24}$$

which appears naturally in our estimates. In the rest of the text we assume that the perturbation $\widehat{V}$ is "small enough", namely

$$0 \leq \sigma < \min\left\{\frac{R_1 \lambda_{1,0}}{2(1+h_-)(1+\sigma+h_+)}, \rho\right\}, \tag{2.25}$$

where ρ is the positive root of the quadratic equation

$$\left(1+h_{-}\right)\rho\left(1+\rho+h_{+}\right)-p=0, \tag{2.26}$$

and R_1 is the positive root of the quadratic equation

$$R_1\left(R_1+\lambda_{1,0}+h_{-}+h_{+}+2+h_{-}h_{+}\right)-\lambda_{1,0}=0. \tag{2.27}$$

Note that the condition (2.25) is sufficient but not necessary for our considerations to be valid.

3. The main theorem

The central part in the recovering of the potential V is the comparison of two functions φ_0 and φ_V defined by (2.7). Consider their difference

$$z=\varphi_V-\varphi_0 \tag{3.1}$$

that satisfies the equation

$$\begin{aligned}-z''\left(\lambda,x\right)-\lambda^2 z\left(\lambda,x\right)&=-V\left(x\right)\varphi_V\left(\lambda,x\right)\\&=-V\left(x\right)z\left(\lambda,x\right)-V\left(x\right)\varphi_0\left(\lambda,x\right)\end{aligned} \tag{3.2}$$

and the zero initial conditions

$$z\left(\lambda,0\right)=0,\quad z'\left(\lambda,0\right)=0.$$

Using the variation of parameters method to solve the equation, we obtain

$$\begin{aligned}z\left(\lambda,x\right)=&\int_0^x\frac{\sin\lambda\left(x-t\right)}{\lambda}V\left(t\right)\varphi_0\left(\lambda,t\right)\,dt\\&+\int_0^x\frac{\sin\lambda\left(x-t\right)}{\lambda}V\left(t\right)z\left(\lambda,t\right)\,dt,\end{aligned} \tag{3.3}$$

and, after the differentiation in x,

$$\begin{aligned}z'\left(\lambda,x\right)=&\int_0^x\cos\lambda\left(x-t\right)V\left(t\right)\varphi_0\left(\lambda,t\right)\,dt\\&+\int_0^x\cos\lambda\left(x-t\right)V\left(t\right)z\left(\lambda,t\right)dt.\end{aligned} \tag{3.4}$$

It follows from (3.3), (3.4) that for small V the second terms on the right are of the higher order in V than the first terms (see Lemma 4.2 for the proof). Omitting these terms, we obtain the system

$$\begin{aligned}\varphi_V\left(\lambda,x\right)&=\varphi_0\left(\lambda,x\right)+\int_0^x\frac{\sin\lambda\left(x-t\right)}{\lambda}V\left(t\right)\varphi_0\left(\lambda,t\right)\,dt,\\\varphi_V'\left(\lambda,x\right)&=\varphi_0'\left(\lambda,x\right)+\int_0^x\cos\lambda\left(x-t\right)V\left(t\right)\varphi_0\left(\lambda,t\right)\,dt,\end{aligned} \tag{3.5}$$

relating the unknown perturbation V to certain known entities:

$$\varphi_0(\lambda, t), \quad \frac{\sin\lambda(x-t)}{\lambda}, \quad \cos\lambda(x-t),$$

and, for $\lambda = \lambda_{k,V}$, to the spectral data

$$\left(\lambda_{k,V}, \varphi_V(\lambda_{k,V}, x), \varphi_V^{'}(\lambda_{k,V}, x)\right)$$

of the perturbed operator, which we assume to be observable.

It is convenient to introduce the function

$$\widetilde{V}(\lambda, x) := \int_0^x e^{i\lambda t} V(t)\, dt \tag{3.6}$$

allowing us to write (3.5) as

$$\begin{aligned} C_-(\lambda, x)\widetilde{V}(0, x) + e^{i\lambda x} B_-(-\lambda)\widetilde{V}(-2\lambda, x) - e^{-i\lambda x} B_-(\lambda)\widetilde{V}(2\lambda, x) \\ = 4i\lambda\left\{\varphi_V(\lambda, x) - \varphi_0(\lambda, x)\right\}, \\ C_+(\lambda, x)\widetilde{V}(0, x) + e^{i\lambda x} B_-(-\lambda)\widetilde{V}(-2\lambda, x) + e^{-i\lambda x} B_-(\lambda)\widetilde{V}(2\lambda, x) \\ = 4\lambda\left\{{\varphi_V}'(\lambda, x) - {\varphi_0}'(\lambda, x)\right\}, \end{aligned} \tag{3.7}$$

where $C_\pm(\lambda, x)$ and $B_\pm(\lambda)$ are defined in (2.19).

These equations can be resolved with respect to $\widetilde{V}(\pm 2\lambda, x)$. We shall do it a bit later and meanwhile explain the use of $\widetilde{V}(\pm 2\lambda, x)$.

It follows from (2.10) and (2.18), (2.19) that we have for every $k = 1, 2, \ldots$

$$\begin{aligned} V_{k,0} :&= \int_0^1 V(t)\,\psi_{k,0}(t)\, dt \\ &= \left(\|\varphi_{k,0}\|_{L^2[0,1]}\right)^{-1} \int_0^1 V(t)\, C_+(\lambda_{k,0}, t)\, dt \\ &= \Re\{\widetilde{V}(\lambda_{k,0}, 1)\, B_-(\lambda_{k,0})\}\, \|\varphi_{k,0}\|_{L^2[0,1]}^{-1} \end{aligned} \tag{3.8}$$

with $\varphi_{k,0}$ defined in (2.10). Taking into account that their norms $\|\varphi_{k,0}\|_{L^2[0,1]}$, $k = 1, 2, \ldots$, can be calculated explicitly using (2.18), we conclude that the collection $\{\widetilde{V}(\lambda_{k,0}, 1)\}_{k=1}^\infty$ determines uniquely the Fourier coefficients $\{V_{k,0}\}_{k=1}^\infty$ of the potential with respect to the complete system $\{\psi_{k,0}\}_{k=1}^\infty$ of the eigenfunctions of $\widehat{H}_0$, see (2.10). This indicates that (3.7) with $\lambda = \lambda_{k,0}$, $k = 1, 2, \ldots.$ could be used to recover the potential up to the terms of the order $o(\|V\|)$.

However, in order to use this indication, we need to settle the following items:

(i) Equations (3.7) with $\lambda = \lambda_{k,0}$, $k = 1, 2, \ldots$, contain $\widetilde{V}(2\lambda_{k,0}, x)$, $k = 1, 2, \ldots$ but not $\widetilde{V}(\lambda_{k,0}, x)$, $k = 1, 2, \ldots$, hence, determine the integrals

$$\int_0^1 V(t)\,\Psi_{k,0}(x)\, dx, \quad k = 1, 2, \ldots,$$

where the functions

$$\Psi_{k,0}(x) = \varphi(2\lambda_{k,0}, x) \left(\int_0^1 \varphi^2(2\lambda_{k,0}, x)\, dx \right)^{-1/2}, \quad k = 1, 2, \ldots,$$

are not, eigenfunctions of $\widehat{H}_0$, since $(2\lambda_{k,0})^2$, $k = 1, 2, \ldots$, are not, the eigenvalues of $\widehat{H}_0$. According to (2.22), these numbers are asymptotically close to the "half" $\{(2\lambda_{k,0})^2\}_{k=1}^{\infty}$ of the set $\{(\lambda_{k,0})^2\}_{k=1}^{\infty}$ of all eigenvalues of $\widehat{H}_0$, hence, functions $\{\Psi_{k,0}\}_{k=1}^{\infty}$ cannot form, even asymptotically, a complete system.

(ii) The right hand sides of (3.7) with $\lambda = \lambda_{k,0}$, $k = 1, 2, \ldots$, contain $\varphi_V(\lambda_{k,0}, x)$ and $\varphi'_V(\lambda_{k,0}, x)$ but not eigenfunctions $\varphi_V(\lambda_{k,V}, x)$ of the perturbed operator $\widehat{H}_V$ and their derivatives assumed to be known.

(iii) Equations (3.7) contain the term $\widetilde{V}(0, x) = \int_0^x V(t)\, dt$.

Let us consider the items of the above list. This requires certain estimates that are proved in Section 4.

Item (i) of the list can be settled by using an observation presented at the end of Section 2. Indeed, applying the argument leading to (2.9)–(2.15) to the intervals $[0, 1]$ and $[0, 1/2]$ instead of $[a, b]$ and $[0, 1]$, we find that the collection $\left\{(2\lambda_{k,0})^2\right\}_{k=1}^{\infty}$ is the spectrum of the (unperturbed) self-adjoint operator $\widehat{H}_{0,-}$ defined by the operation $-\frac{d^2}{dx^2}$ and appropriate boundary conditions (see (2.12) and (3.10)) on the interval $[0, 1/2]$. Likewise, the same collection is the spectrum of the self-adjoint operator $\widehat{H}_{0,+}$ defined by the operation $-\frac{d^2}{dx^2}$ and appropriate boundary conditions (see (2.12)) on the interval $[1/2, 1]$. It suffices to transform $L^2[0, 1/2]$ unitarily to $L^2[1/2, 1]$ by the shift $[0, 1/2] \ni x \to x + 1/2 \in [1/2, 1]$.

Thus, we can use the analogs of (3.7), (3.8) for the intervals $[0, 1/2]$ and $[1/2, 1]$ to find all the Fourier coefficients of the restrictions

$$V_- = V|_{[0,1/2]}, \quad V_+ = V|_{[1/2,1]} \tag{3.9}$$

of V with respect to the complete sets $\{\psi_{k,0}^{\pm}\}_{k=1}^{\infty}$ of eigenfunctions of operators $\widehat{H}_{0,\pm}$. According to (3.7), (3.8), this requires collections $\{\widetilde{V}(2\lambda_{k,0}, 1/2)\}_{k=1}^{\infty}$ and $\{\widetilde{V}(2\lambda_{k,0}, 1)\}_{k=1}^{\infty}$ which, in turn, are determined by the collections

$$\{\varphi_V(\lambda_{k,V}, 1/2)\}_{k=1}^{\infty}, \quad \{\varphi'_V(\lambda_{k,V}, 1/2)\}_{k=1}^{\infty}, \quad \text{and} \{\varphi_V(\lambda_{k,V}, 1)\}_{k=1}^{\infty}.$$

These collections and the spectrum $\{E_{k,V} = \lambda_{k,V}^2\}_{k=1}^{\infty}$ of $\widehat{H}_V$ form the set of spectral data needed to restore V up to the terms of the order $o(V)$.

We pass now to the details of the above scheme. Consider the operator $\widehat{H}_{0,-}$ defined on the interval $[0, 1/2]$ by the operation $-\frac{d^2}{dx^2}$ and the boundary conditions (cf. (2.12))

$$y'(0) - 2h_- y(0) = 0, \quad y'(1/2) + 2h_+ y(1/2) = 0. \tag{3.10}$$

Its spectrum is $\left\{ (2\lambda_{k,0})^2 \right\}_{k=1}^{\infty}$ and if $\varphi_0^-(\lambda, x)$ is the unique solution of (2.6) satisfying the initial conditions

$$\varphi_0^-(2\lambda_{k,0}, 0) = 1, \qquad \frac{\partial}{\partial x}\varphi_0^-(\lambda, x)\bigg|_{x=0} = 2h_-$$

(cf. (2.15), (2.17), and (3.10)), then its orthonormal eigenfunctions (cf. (2.14) and (2.18))

$$\psi_{k,0}^-(x) = \frac{\varphi_{k,0}^-(x)}{\|\varphi_{k,0}^-\|_{L^2[0,1/2]}}, \ \varphi_{k,0}^-(x) = \varphi_0^-(\lambda_{k,0}, x), \quad k = 1, 2, \dots,$$

$$\varphi_0^-(\lambda, x) = \left\{ e^{2i\lambda x} B_-(\lambda) + e^{-2i\lambda x} B_-(-\lambda) \right\} / 2 \tag{3.11}$$

form an orthonormal basis in $L^2[0, 1/2]$. Thus, we can write the $L^2_{[0,1/2]}$-converging series (see (2.18), (3.6) and (3.11))

$$\sum_{k=1}^{\infty} V_{k,0}^- \psi_{k,0}^-(x), \tag{3.12}$$

where

$$\begin{aligned} V_{k,0}^- &:= \int_0^{\frac{1}{2}} V_-(t)\, \psi_{k,0}^-(t)\, dt \\ &= \left(\|\varphi_{k,0}^-\|_{L^2[0,1/2]} \right)^{-1} \Re\{ B_-(\lambda_{k,0})\, \widetilde{V}(2\lambda_{k,0}, 1/2) \} \end{aligned} \tag{3.13}$$

that can be viewed as a "candidate" for the first order approximation of the restriction $V_- = V|_{[0,1/2]}$

Since the norm $\|\varphi_{k,0}^-\|_{L^2[0,1/2]}$ can be easily found by using the analogs of (2.14) and (2.18) for $[0, 1/2]$, we get the formula for V_- in terms of $\{\widetilde{V}(\pm 2\lambda_{k,0}, 1/2)\}_{k=1}^{\infty}$.

Analogous formula holds for $V_+ = V|_{[1/2,1]}$. Indeed, if we know $\widetilde{V}(2\lambda_{k,0}, 1/2)$ and $\widetilde{V}(2\lambda_{k,0}, 1)$, then we can also find

$$\int_{1/2}^{1} e^{\pm 2i\lambda_{k,0}(t-1/2)} V(t)\, dt = e^{i\lambda_{k,0}} \left(\widetilde{V}(\pm 2\lambda_{k,0}, 1) - \widetilde{V}(\pm 2\lambda_{k,0}, 1/2) \right),$$

and use then the analogs of (2.14) and (2.18) for $[1/2, 1]$ to calculate the Fourier coefficients $\{\widetilde{V}_{k,0}^+\}$ of V_+ of (3.9) with respect to the complete orthonormal system $\{\psi_{k,0}^+\}_{k=1}^{\infty}$ of eigenfunctions of the operator $\widehat{H}_{0,+}$.

Consider now item (ii) of the above list, i.e., the fact that the right hand sides of (3.7) with $\lambda = \lambda_{k,0}$, $k = 1, 2, \dots$, contain $\varphi_V(\lambda_{k,0}, x)$, $k = 1, 2, \dots$, but not the eigenfunctions $\varphi_V(\lambda_{k,V}, x)$, $k = 1, 2, \dots$, of the perturbed operator $\widehat{H}_V$ (see (2.5)–(2.8)) assumed to be known. To this purpose set $\lambda = \lambda_{k,V}$ in (3.7) and estimate the corresponding errors in the Fourier coefficients (3.13). To this end we will use Lemma 4.4 (see (4.17)) implying:

$$\left| \widetilde{V}(\pm 2\lambda_{k,V}, x) - \widetilde{V}(\pm 2\lambda_{k,0}, x) \right| = \left| \int_0^x \left(e^{2i(\lambda_{k,V} - \lambda_{k,0})t} - 1 \right) e^{2i\lambda_{k,0} t} V(t)\, dt \right|$$

$$\leq 2\left|\lambda_{k,V}-\lambda_{k,0}\right|\int_0^x |V(t)|\,dt \leq 4\sigma^2(1+h_-)(1+\sigma+h_+)/\lambda_{k,0}.$$

Hence, we have

$$\begin{aligned}2V^-_{k,0}\|\varphi^-_{k,0}\|_{L^2[0,1/2]} &= B_-(\lambda_{k,0})\,\widetilde{V}(2\lambda_{k,0},1/2)+B_-(-\lambda_{k,0})\,\widetilde{V}(-2\lambda_{k,0},1/2)\\ &= B_-(\lambda_{k,0})\,\widetilde{V}(2\lambda_{k,V},1/2)+B_-(-\lambda_{k,0})\,\widetilde{V}(-2\lambda_{k,V},1/2)+D_k,\end{aligned}$$

where

$$\begin{aligned}|D_k| &\leq |B_-(\lambda_{k,0})|\left|\widetilde{V}(2\lambda_{k,0},1/2)-\widetilde{V}(2\lambda_{k,V},1/2)\right|\\ &\quad+|B_-(-\lambda_{k,0})|\left|\widetilde{V}(-2\lambda_{k,0},1/2)-\widetilde{V}(-2\lambda_{k,V},1/2)\right|\\ &\leq 8\sigma^2(1+h_-)(1+\sigma+h_+)\,|B_-(\lambda_{k,0})|/\lambda_{k,0}\\ &\leq 8\sigma^2(1+h_-)(1+\sigma+h_+)(1+h_-/\lambda_{k,0})/\lambda_{k,0}.\end{aligned}$$

Thus, the squared $L^2[0,1/2]$ norm of the error in the right-hand side of (3.12) due to the replacement $\lambda_{k,0}$ by $\lambda_{k,V}$ is

$$\begin{aligned}\frac{1}{4}\sum_{k=1}^{\infty}|D_k|^2\|\varphi^-_{k,0}\|^{-2} &\leq 16\sigma^4(1+h_-)^2(1+\sigma+h_+)^2\sum_{k=1}^{\infty}\left(\frac{(1+h_-/\lambda_{k,0}))}{\lambda_{k,0}\|\varphi^-_{k,0}\|}\right)^2\\ &\leq 16\sigma^4(1+h_-)^2(1+\sigma+h_+)^2(1+h_-/\lambda_{1,0})^2\sum_{k=1}^{\infty}\left(\lambda_{k,0}\|\varphi^-_{k,0}\|\right)^{-2}.\end{aligned}$$

It follows from (2.22) that $\lambda_{k,0} > (k-1)\pi$. Besides, it can be verified by explicit integration that

$$\|\varphi^-_{k,0}\|^2_{L^2[0,1/2]} \geq 1/8.$$

These bounds and (2.24) imply that the norm of the error generated by replacement $\lambda_{k,0}$ by $\lambda_{k,V}$ is $O(\|V\|^2_{L^1[0,1/2]})$ and thus can be neglected.

An analogous estimate holds for $V_+ = V|_{[1/2,1]}$ and the corresponding Fourier coefficients. Thus, we settled the second item of the list given after formula (3.8).

The last item (iii) of the above list can be settled by considering the limit of equations (3.6) as $\lambda = \lambda_{k,V} \to +\infty$. Indeed, the second of these equations and (2.19) yields

$$\widetilde{V}(0,x) = \frac{4\lambda\left(\varphi'_V(\lambda,x)-\varphi'_0(\lambda,x)\right)-2\Re\left(e^{i\lambda x}B_-(-\lambda)\,\widetilde{V}(-2\lambda,x)\right)}{2\Re(e^{i\lambda x}B_-(\lambda))}.$$

Set here $\lambda = 2\lambda_{k,V}$ and use again (2.19) to write the denominator as

$$e^{i2\lambda_{k,V}x}B_-(2\lambda_{k,V})+e^{-i2\lambda_{k,V}x}B_-(-2\lambda_{k,V}) = 2\cos 2\lambda_{k,V}x+\frac{h_-}{\lambda_{k,V}}\sin 2\lambda_{k,V}x.$$

It follows then from (2.22), (4.17), the above formula with $x = 1, 1/2$, and $k \to \infty$ that

$$2\cos(2\lambda_{k,V})+\frac{h_-}{\lambda_{k,V}}\sin(2\lambda_{k,V}) = 2+o\left(k^{-1}\right),$$

$$2\cos\left(\lambda_{k,V}\right) + \frac{h_-}{\lambda_{k,V}} \sin\left(\lambda_{k,V}\right) = 2\left(-1\right)^k + o\left(k^{-1}\right).$$

These asymptotic relations and our basic assumption (2.2), implying in view of the Riemann–Lebesgue lemma that $\lim_{\lambda\to\infty} \widetilde{V}\left(\pm 2\lambda, x\right) = 0$, lead to the formula

$$\widetilde{V}\left(0, x\right) = \lim_{k\to\infty} \left(-1\right)^{2x} 2\lambda_{k,V} \left\{\varphi_V'\left(\lambda_{k,V}, x\right) - \varphi_0'\left(\lambda_{k,V}, x\right)\right\}, \; x = 1/2, 1. \tag{3.14}$$

Note that:

- In what follows we will need the values $\widetilde{V}\left(0, x\right)$ for $x = 1/2, 1$ only and we assume that for these cases the quantities $\varphi_V(\lambda_{k,V}, x)$ and $\varphi_V'(\lambda_{k,V}, x)$ are observed ($\varphi_V'(\lambda_{k,V}, 1)$ can be found from $\varphi_V(\lambda_{k,V}, 1)$ and boundary condition for $x = 1$).

- We do not discuss in this paper the existence of the limit (3.14).This is a problem of the existence of a perturbation $\widehat{V}$ corresponding to observed spectral data.

We will express now $\widetilde{V}\left(2\lambda_{k,V}, x\right)$, $x = 1/2, 1$, via the observable spectral data. Writing (3.7) in the matrix form we get for each $\lambda_{k,0}$, $k = 1, 2, \ldots$.

$$M \begin{pmatrix} \widetilde{V}\left(2\lambda, x\right) \\ \widetilde{V}\left(-2\lambda, x\right) \end{pmatrix} = \begin{pmatrix} R_1\left(x\right) \\ R_2\left(x\right) \end{pmatrix} + O\left(\sigma^2\right),$$

where

$$M = \begin{pmatrix} -e^{-i\lambda x} B_-\left(\lambda\right) & e^{i\lambda x} B_-\left(-\lambda\right) \\ e^{-i\lambda x} B_-\left(\lambda\right) & e^{i\lambda x} B_-\left(-\lambda\right) \end{pmatrix},$$

and

$$\begin{pmatrix} R_1\left(x\right) \\ R_2\left(x\right) \end{pmatrix} = \begin{pmatrix} 4i\lambda\left\{\varphi_V\left(\lambda, x\right) - \varphi_0\left(\lambda, x\right)\right\} - C_-(\lambda, x)\widetilde{V}\left(0, x\right) \\ 4\lambda\left\{\varphi_V'\left(\lambda, x\right) - \varphi_0'\left(\lambda, x\right)\right\} - C_+(\lambda, x)\widetilde{V}\left(0, x\right) \end{pmatrix}, \tag{3.15}$$

where $C_\pm(\lambda, x)$ are given by (2.19). We are only interested in values $\lambda = \lambda_{k,V}$ and $x = 1/2, 1$ and for these values all the terms in (3.15) are known, since they are either observable or calculable.

We have

$$M^{-1} = \frac{1}{2B_-\left(\lambda\right) B_-\left(-\lambda\right)} \begin{pmatrix} -e^{i\lambda x} B_-\left(-\lambda\right) & e^{i\lambda x} B_-\left(-\lambda\right) \\ e^{-i\lambda x} B_-\left(\lambda\right) & e^{-i\lambda x} B_-\left(\lambda\right) \end{pmatrix}. \tag{3.16}$$

A direct calculation and (2.19) shows that the Hilbert-Schmidt norm of M^{-1} does not exceed 2. Therefore we get $M^{-1}O\left(\sigma^2\right) = O\left(\sigma^2\right)$ and then

$$\begin{pmatrix} \widetilde{V}\left(2\lambda, x\right) \\ \widetilde{V}\left(-2\lambda, x\right) \end{pmatrix} = M^{-1} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} + O\left(\sigma^2\right). \tag{3.17}$$

The above allows us now to formulate the main theorem.

Theorem 3.1. *Let $\widehat{H}_0$ be the self-adjoined Sturm-Liouville operator acting in $L^2[0,1]$ and defined by the operation $-\frac{d^2}{dx^2}$ and the boundary conditions (2.2), $\widehat{H}_V$ be the self-adjoined Sturm–Liouville operator $-\frac{d^2}{dx^2}+\widehat{V}$ with the same domain, where $\widehat{V}$ is the operator of multiplication by a real-valued function $V \in L^2[0,1]$. If the norm $\|V\|_{L^1[0,1]}$ satisfies (2.25)–(2.27), then we have*

$$V = V_{app} + O\left(\|V\|^2_{L^1[0,1]}\right), \tag{3.18}$$

where V_{app} can be explicitly calculated (see (3.19)) given the following partial spectral data of the perturbed operator $\widehat{H}_V$:

(i) *all the eigenvalues $\{E_{k,V} = \lambda^2_{k,V}\}_{k=1}^{\infty}$ of the operator $\widehat{H}_V$,*

(ii) *the values of its eigenfunctions $\{\varphi_V(\lambda_{k,V}, x)\}_{k=1}^{\infty}$ at the points $x = 0, 1/2, 1$,*

(iii) *the values of the derivatives of its eigenfunctions $\{\varphi'_V(\lambda_{k,V}, 1/2)\}_{k=1}^{\infty}$ at the point $x = 1/2$ (see (2.5)–(2.8)).*

The function V_{app} is given by the L^2-converging series

$$V_{app}(x) = \begin{cases} \sum\limits_{k=1}^{\infty} V^-_{k,0}\psi^-_{k,0}(x), & 0 \le x \le \frac{1}{2} \\ \sum\limits_{k=1}^{\infty} V^+_{k,0}\psi^+_{k,0}(x), & \frac{1}{2} \le x \le 1 \end{cases}, \tag{3.19}$$

where $\{\psi^{\pm}_{k,0}\}_{k=1}^{\infty}$ are eigenfunctions of the operator $-\frac{d^2}{dx^2}$ defined on the intervals $[0,1/2]$ and $[1/2,1]$ respectively and boundary conditions $y' - 2h_-y = 0$ at the left end of the intervals and $y' + 2h_+y = 0$ at their right ends. The coefficients $\{V^{\pm}_{k,0}\}_{k=1}^{\infty}$ can be calculated by formula (3.13) and its analog for $\widehat{H}_{0,+}$ and integrals $\widetilde{V}(2\lambda, x)$ of (3.6) for $x = 1, 1/2$ by using formulas (3.14)–(3.17).

Remark 3.2. The following assertions hold:

(i) It follows from formulas (2.11)–(2.14), that the theorem holds for any finite interval with appropriate changes of constants.

(ii) It follows from the proof of the theorem that in order to find the potential V_{app} it suffices to know the eigenvalues of the perturbed operator but not the values of its eigenfunctions at the points $(0, 1/2, 1)$ and their derivative at the point $1/2$, but the ratio of these numbers and the values of the eigenfunctions at $x = 0$, thus four sets of numbers are required.

4. Auxiliary results

We will start with proving estimates for the function z introduced in (3.1).

Lemma 4.1. *Assume that the perturbation $\widehat{V}$ is small enough, so that conditions (2.25)–(2.27) hold. Consider the function z introduced in (3.1). We have for any $\lambda = \alpha + i\beta \in \mathbb{C}$:*

$$\begin{aligned} |z(\lambda, x)| &\leq \sigma(1 + h_-)\cosh\beta x, \\ \left|z'(\lambda, x)\right| &\leq \sigma(1 + \sigma)(1 + h_-)\cosh\beta x. \end{aligned} \tag{4.1}$$

Proof. It is possible to prove the above bounds by using transformation operators and following [2, Section 2]. We present here a direct proof based on equations (3.3)–(3.4). Write the equation (3.3) in symbolic form

$$\left(I - \widehat{K}\right) z = \widehat{K}\varphi_0, \tag{4.2}$$

where

$$(\widehat{K}u)(x) = \int_0^x \frac{\sin\lambda(x - t)}{\lambda} V(t)\, u(t)\, dt,$$

i.e., $\widehat{K}$ is a Volterra type operator. Thus, the operator $I - \widehat{K}$ is invertible and its inverse is given by the series $\sum_0^\infty \widehat{K}^k$. We will need below a certain bound on the norm of $\left(I - \widehat{K}\right)^{-1}$. To this end we consider individual terms of the series. Write the kernel of $\widehat{K}^k$, which is of the Volterra type for any k, in the form $K_k(x, t, \lambda)\, V(t)$. Then for $\widehat{K}^k = \widehat{K}\,\widehat{K}^{k-1}$, we have

$$K_k(x, t, \lambda) = \int_t^x K_1(x, s, \lambda)\, V(s)\, K_{k-1}(s, t, \lambda)\; ds, \tag{4.3}$$

where

$$K_1(x, s, \lambda) = \frac{\sin\lambda(x - s)}{\lambda} \quad \text{for } \lambda \neq 0. \tag{4.4}$$

Let us prove the estimate

$$|K_k(x, t, \lambda)| \leq \frac{\sigma^{k-1}(x)}{(k-1)!}\frac{(x - t)^k}{k^k}\cosh\beta(x - t), \quad \lambda = \alpha + i\beta \in \mathbb{C},\ k \geq 1, \tag{4.5}$$

where

$$\sigma(x) = \int_0^x |V(x)|\, dx, \tag{4.6}$$

hence, we have

$$\sigma = \sigma(1)\; e^{\sigma(1)} \tag{4.7}$$

for σ of (2.24). From (4.4), we have

$$|K_1(x, t, \lambda)| \leq |x - t|\cosh\beta(x - t), \ \lambda = \alpha + i\beta \in \mathbb{C} \setminus \{0\}, \tag{4.8}$$

i.e., (4.5) for $k = 1$ and $\lambda \neq 0$. Next, (4.3) and the induction assumptions (4.5) and (4.8) yield

$$\begin{aligned} &|K_{k+1}(x, t, \lambda)| \\ &\leq \frac{1}{k^k(k-1)!}\int_t^x (x - s)(s - t)^k\sigma^{k-1}(s)\,|V(s)|\cosh\beta(x - s)\cosh\beta(s - t)\; ds. \end{aligned}$$

Note that

1. for any $s \in [t, x]$ and $\beta \in \mathbb{R}$ we have

$$\cosh \beta (x-s) \cosh \beta (s-t) \leq \cosh \beta (x-t) .$$

2. the function $f(s) = (x-s)(s-t)^k$ is strictly positive for $t < s < x$, $f(t) = f(x) = 0$ and

$$\max_{s\in[t,x]} f(s) = f(s_0) = \frac{k^k}{(k+1)^{k+1}}(x-t)^{k+1}, \;\; s_0 = (kx+s)/(k+1).$$

This and (4.6) imply that

$$\begin{aligned} |K_{k+1}(x,t,\lambda)| &\leq \frac{(x-t)^{k+1}}{(k-1)!}\frac{\cosh\beta(x-t)}{(k+1)^{k+1}}\int_t^x \sigma^{k-1}(s)\,|V(s)|\,ds \\ &= \frac{(x-t)^{k+1}}{k!}\frac{\sigma^k(x)}{k!}\frac{\cosh\beta(x-t)}{(k+1)^{k+1}}, \end{aligned}$$

i.e., (4.5) for $k+1$ and $\lambda \neq 0$.

For the case $\lambda = 0$, we have $K_1(x,s,\lambda) = (x-t)$, hence, factors with cosh are replaced by 1 in all above formulas and the rest of the proof is still valid. This yields (4.5) for all $k \geq 1$ and all $\lambda \in \mathbb{C}$.

The kernel of the operator $\left(I - \widehat{K}\right)^{-1}\widehat{K}$ is

$$\left(\sum_1^\infty K_k(x,t,\lambda)\right) V(t)\,,$$

where the series converges absolutely and uniformly in $x, t \in [0,1]$ and λ belonging to any compact subset of $\mathbb{C}$. According (4.5)–(4.8), we have

$$\begin{aligned} \left|\sum_1^\infty K_k(x,t,\lambda)\right| &\leq \sum_{n=1}^\infty \frac{(x-t)^n}{n^n}\frac{\sigma^{n-1}(x)}{(n-1)!}\cosh\beta(x-t) \\ &\leq (x-t)\exp\{\sigma(x)\}\cosh\beta(x-t)\,. \end{aligned} \tag{4.9}$$

Now we can get an estimate for $z(\lambda, x)$. From (4.2), we have

$$z(\lambda,x) = \left(\left(I-\widehat{K}\right)^{-1}\widehat{K}\varphi_0\right)(\lambda,x) = \int_0^x \left\{\sum_1^\infty K_k(x,t,\lambda)\right\} V(t)\,\varphi_0(\lambda,t)\,dt.$$

This, the bound (see (2.18))

$$|\varphi_0(\lambda,t)| = |\cos\lambda t + h_- \sin\lambda t/\lambda| \leq (1+h_-)\cosh\beta t, \tag{4.10}$$

and (4.9) yields the first inequality in (4.1):

$$|z(\lambda,x)| \leq \int_0^x (x-t)\,e^{\sigma(x)}\,(1+h_-)\,|V(t)|\cosh\beta(x-t)\cosh\beta t\,dt$$

$$\leq (1+h_-)\, e^{\sigma(x)} \cosh \beta x \int_0^x |V(t)|\; dt = (1+h_-)\, \sigma(x)\, e^{\sigma(x)} \cosh \beta x.$$

Note that since $\cosh(\beta x)$ and $\sigma(x)$ do not decrease in $x \geq 0$, we can immediately get the uniform bound

$$|z(\lambda, x)| \leq (1+h_-)\, \sigma(1)\, e^{\sigma(1)} \cosh \beta = (1+h_-)\, \sigma \cosh \beta.$$

The only term depending on V in this formula is $\sigma = \sigma(1)\, e^{\sigma(1)}$ (see (4.6)–(4.7)), thus it is a natural measure of the perturbation smallness (see (2.24)).

To obtain the second inequality in (4.1), we consider the terms in the right-hand side of (3.4). By using again (4.10), we get for the first term

$$\begin{aligned}\left| \int_0^x V(t)\, \varphi_0(\lambda, t) \cos \lambda (x-t)\, dt \right| &\leq (1+h_-) \cosh \beta x \int_0^x |V(t)|\, dt \\ &= \sigma(x)\, (1+h_-) \cosh \beta x \leq \sigma (1+h_-) \cosh \beta x\end{aligned}$$

and for the second term

$$\begin{aligned}\left| \int_0^x V(t)\, z(\lambda, x) \cos \lambda (x-t)\, dt \right| &\leq (1+h_-) \cosh \beta x \int_0^x \sigma(t)\, e^{\sigma(t)}\, |V(t)|\; dt \\ &\leq \sigma^2(x)\, e^{2\sigma(x)}\, (1+h_-) \cosh \beta x \\ &\leq \sigma^2 (1+h_-) \cosh \beta x. \end{aligned} \tag{4.11}$$

and, as a result, the second bound in (4.1):

$$\begin{aligned}\left| z'(\lambda, x) \right| &\leq \sigma (1+h_-) \cosh \beta x + \sigma^2 (1+h_-) \cosh \beta x \\ &= \sigma (1+\sigma)(1+h_-) \cosh \beta x.\end{aligned}$$

For $\lambda = 0$, the formula (3.4) is

$$z'(0, x) = \int_0^x V(t)\, \varphi_0(0, x)\, dt + \int_0^x V(t)\, z(0, x)\, dt,$$

and it leads to the bounds (4.1) for $\lambda = 0$ by a simple version of the above argument. □

An immediate consequence of the lemma is

Lemma 4.2. *Assume that the perturbation $\widehat{V}$ is small enough, so that conditions (2.25)–(2.27) hold. Then we have for any $\lambda \in \mathbb{R}$*

$$\begin{aligned}\left| \int_0^x \cos \lambda (x-t)\, V(t)\, z(\lambda, t)\; dt \right| &\leq (1+h_-)\, \sigma^2, \\ \left| \int_0^x \frac{\sin \lambda (x-t)}{\lambda} V(t)\, z(\lambda, t)\; dt \right| &\leq (1+h_-)\, \sigma^2. \end{aligned} \tag{4.12}$$

Proof. The first inequality coincides with (4.11) and is proved in Lemma 4.1. The second inequality follows directly from (4.1). Indeed, we have

$$\left|\int_0^x \frac{\sin\lambda\,(x-t)}{\lambda} V\,(t)\,z\,(\lambda,t)\,dt\right| \leq \left|\int_0^x V\,(t)\,z\,(\lambda,t)\,dt\right|$$
$$\leq (1+h_-)\,\sigma\int_0^x |V\,(t)|\,dt \leq (1+h_-)\,\sigma^2. \qquad \square$$

The lemma provides the justification of the passage from (3.3)–(3.4) to (3.5).

Lemma 4.3. *Assume that the perturbation* $\widehat{V}$ *is small enough, so that conditions* (2.25)–(2.27) *hold. Then the characteristic function* Q_V (2.9) *of the perturbed operator* $\widehat{H}_V$ *has*

(i) *a single zero in every interval* $(k\pi, (k+1)\pi)$, $k \in \mathbb{Z}$,

(ii) *no other zeros.*

Proof. Note first of all that since the operator $\widehat{H}_V$ is self-adjoint, all his eigenvalues $\{E_{k,V}\}_{k=1}^{\infty}$ are real, hence, all the roots $\lambda_{k,V}$ of the characteristic equation $Q_V\,(\lambda) = 0$ are located either on the real or on the imaginary axis. Due to the symmetry of $Q_V\,(\lambda)$ it suffices to consider only the right half-plane.

We will follow the proof of Lemma 1.3.1 in [2] and use the Rouché theorem according to which if two functions f and g are analytic in a simply connected closed domain G with a piece-wise smooth boundary ∂G and satisfy the inequality

$$|f\,(\lambda)\,| > |g\,(\lambda)| > 0, \quad \lambda \in \partial G,$$

then f and $f+g$ have the same number of zeroes (counting multiplicity) inside ∂G. Set $f = Q_0$, $g = Q_V - Q_0$ and consider the collection $\{G_k\}_{k=0}^{\infty}$ of rectangles with vertices at $\big(k\pi \pm il, (k+1)\pi \pm il\big)$, $l > 0$, hence, ∂G_k consists of four intervals $\big(k\pi + i\beta\big)$, $\big((k+1)\pi + i\beta\big)$, $\big(\alpha \pm il\big)$, parametrized by real numbers α and β with $k\pi \leq \alpha \leq (k+1)\,\pi$, $-l \leq \beta \leq l$.

We will prove that if l is large enough, then

$$|Q_0\,(\lambda)| > |Q_V\,(\lambda) - Q_0\,(\lambda)|\,, \quad \lambda \in \partial G_k. \tag{4.13}$$

Since Q_0 has a single zero in every domain G_k (see (2.22)), the same is true for Q_V by the Rouché theorem, thus this single zero is located in $(k\pi, (k+1)\pi)$.

We are left with the proof of (4.13). We will begin with the proof of the bound

$$|Q_0\,(\lambda)| \geq p\cosh\beta, \quad \lambda \in \partial G_k. \tag{4.14}$$

By using (2.20)–(2.21), we obtain for every vertical interval $\{\lambda = k\pi + i\beta,\ |\beta| \leq l\}$:

$$|Q_0\,(k\pi + i\beta)| = \left|p\cos\,(k\pi + i\beta) - \left((k\pi + i\beta) - \frac{q^2}{(k\pi + i\beta)}\right)\sin\,(k\pi + i\beta)\right|$$
$$= \left|p\cosh\beta + \beta\left(1 + \frac{q^2}{(k\pi)^2 + \beta^2}\right)\sinh\beta\right.$$

$$
\begin{aligned}
&-ik\pi\left(1-\frac{q^2}{(k\pi)^2+\beta^2}\right)\sinh\beta\Bigg| \\
&\geq p\cosh\beta+\beta\sinh\beta\geq p\cosh\beta,
\end{aligned}
$$

while for every horizontal interval $[k\pi \pm il, (k+1)\pi \pm il]$, $\lambda = t \pm il$, $l > 0$, $k\pi \leq t \leq (k+1)\pi$ we have

$$
\begin{aligned}
|Q_0(\lambda)| &= |p\cos\lambda-(\lambda-q^2/\lambda)\sin\lambda| \geq |(\lambda-q^2/\lambda)|\,|\sin\lambda|-p|\cos\lambda| \\
&= |(t\pm il)-q^2/(t\pm il)|\,|\sin(t\pm il)|-p|\cos(t\pm il)| \\
&\geq |l-q^2/l|\sqrt{\cosh^2 l-\cos^2 t}-p\sqrt{\sinh^2 l+\cos^2 t} \\
&\geq |l-q^2/l|\sinh l-p\cosh l.
\end{aligned}
$$

Since

$$\lim_{l\to+\infty}\left(l-q^2l^{-1}\right)\tanh(l)=+\infty,$$

we get for sufficiently large l

$$\left|l-q^2/l\right|\sinh l-p\cosh l>p\cosh l.$$

The above proves (4.14) for the whole contour ∂G_k.

Next from (2.9) and (3.1), it follows that

$$
\begin{aligned}
Q_V(\lambda)-Q_0(\lambda) &= \varphi_V'(\lambda,1)+h_+\varphi_V(\lambda,1)-\left\{\varphi_0'(\lambda,1)+h_+\varphi_0(\lambda,1)\right\} \\
&= z'(\lambda,1)+h_+z(\lambda,1).
\end{aligned}
$$

This and (4.1) yield

$$
\begin{aligned}
|Q_V(\lambda)-Q_0(\lambda)| &\leq \left|z'(\lambda,1)\right|+h_+|z(\lambda,1)| \\
&\leq \sigma(1+h_-)(1+\sigma+h_+)\cosh\beta
\end{aligned}
\tag{4.15}
$$

and taking into account (2.25)–(2.26) according to which $\sigma \leq \rho \leq p$, we obtain

$$|Q_V(\lambda)-Q_0(\lambda)|<p\,\cosh\beta\leq|Q_0(\lambda)|\,,\quad \lambda\in\partial G_k. \tag{4.16}$$

We conclude that if l is large enough, the Rouché theorem is applicable, hence Q_V has a single zero in every domain G_k, $k = 0, 1, \ldots$.

Set now $\lambda = i\beta$. According to (4.16), we have

$$|Q_0(\lambda)|>|Q_V(\lambda)-Q_0(\lambda)|\,,$$

and since Q_0 has no pure imaginary zeros, Q_V has the same property, hence, the perturbed operator $\widehat{H}_V$ is positively definite. □

Lemma 4.4. *Assume that the perturbation $\widehat{V}$ is small enough, so that conditions (2.25)–(2.27) hold. Then the zeros of Q_V and Q_0 located in the every interval $(k\pi, (k+1)\pi)$, $k = 0, 1, \ldots$ satisfy the bound*

$$|\lambda_{k,V}-\lambda_{k,0}|\leq 2\sigma(1+h_-)(1+\sigma+h_+)/\lambda_{k,0}. \tag{4.17}$$

Proof. The idea is to prove a bound $|Q_0(\lambda)| \geq \text{const}\,|\lambda - \lambda_{k,0}|$ for real λ's close to $\lambda_{k,0}$ and combine it with (4.15). This will enable us to find a small interval containing $\lambda_{k,0}$, and such that the function Q_V takes different signs at its endpoints. This and Lemma 4.3 will imply that Q_V has a unique zero inside the interval.

It is convenient to denote

$$D_k(\lambda) := \frac{\partial^k Q_0(\lambda)}{\partial \lambda^k}. \tag{4.18}$$

Let us prove first that

$$|D_1(\lambda_{k,0})| \geq \lambda_{k,0} + q^2/\lambda_{k,0} \geq \lambda_{k,0}. \tag{4.19}$$

Denoting

$$f(\lambda) = \lambda - h_0 h_1/\lambda = \lambda - q^2/\lambda, \tag{4.20}$$

we write Q_0 of (2.20)–(2.21) as

$$Q_0(\lambda) = p\cos\lambda - f(\lambda)\sin\lambda, \tag{4.21}$$

hence,

$$D_1(\lambda) = -\sin\lambda\left(p + f'(\lambda) + f(\lambda)\cot\lambda\right)$$

and

$$(D_1(\lambda))^2 = \sin^2\lambda\left(p + f'(\lambda) + f(\lambda)\cot\lambda\right)^2 = \frac{\left(p + f'(\lambda) + f(\lambda)\cot\lambda\right)^2}{1 + \cot^2\lambda}.$$

It follows from (4.21) that $\cot\lambda_{k,0} = p^{-1} f(\lambda_{k,0})$, hence,

$$\begin{aligned}(D_1(\lambda_{k,0}))^2 &= \frac{\left(p^2 + f^2(\lambda_{k,0}) + pf'(\lambda_{k,0})\right)^2}{p^2 + f^2(\lambda_{k,0})}\\ &= p^2 + f^2(\lambda_{k,0}) + 2pf'(\lambda_{k,0}) + \frac{(pf'(\lambda_{k,0}))^2}{p^2 + f^2(\lambda_{k,0})}.\end{aligned}$$

Now, since $pf'(\lambda) = p\left(1 + q^2/\lambda^2\right) > 0$ by (4.20), we get

$$(D_1(\lambda_{k,0}))^2 > p^2 + \left(\lambda_{k,0} - q^2/\lambda_{k,0}\right)^2$$

and using then the definitions for p and q from (2.21), we obtain

$$\begin{aligned}p^2 + \left(\lambda_{k,0} - q^2/\lambda_{k,0}\right)^2 &= (h_- - h_+)^2 + \lambda_{k,0}^2 + 2h_-h_+ + (h_-h_+)^2/\lambda_{k,0}^2\\ &\geq \left(\lambda_{k,0} + q^2/\lambda_{k,0}\right)^2.\end{aligned}$$

hence, (4.19).

Next, we will prove that if $|\lambda - \lambda_{k,0}|$ is small enough, then

$$|Q_0(\lambda)| \geq \frac{|\lambda - \lambda_{k,0}|}{2}|D_1(\lambda_{k,0})| > \frac{|\lambda - \lambda_{k,0}|}{2}\lambda_{k,0}, \tag{4.22}$$

where we used (4.18)–(4.19) to obtain the second inequality.

Consider the Taylor expansion for Q_0 at $\lambda_{k,0}$:

$$Q_0\left(\lambda\right) = Q_0\left(\lambda_{k,0}\right) + \left(\lambda - \lambda_{k,0}\right) D_1\left(\lambda_{k,0}\right) + \frac{\left(\lambda - \lambda_{k,0}\right)^2}{2} D_2\left(\mu\right),\ \mu \in \left[\lambda, \lambda_{k,0}\right].$$

Taking into account that $Q_0\left(\lambda_{k,0}\right) = 0$, we obtain in view of (4.19)

$$\begin{aligned} |Q_0\left(\lambda\right)| &= |\lambda - \lambda_{k,0}||D_1\left(\lambda_{k,0}\right)| \left|1 + \frac{\left(\lambda - \lambda_{k,0}\right)}{2} \frac{D_2\left(\mu\right)}{D_1\left(\lambda_{k,0}\right)}\right| \\ &\geq \lambda_{k,0}|\lambda - \lambda_{k,0}| \left|1 + \frac{\left(\lambda - \lambda_{k,0}\right)}{2} \frac{D_2\left(\mu\right)}{D_1\left(\lambda_{k,0}\right)}\right|. \end{aligned}$$

Let us find now a conditions for $|\lambda - \lambda_{k,0}|$ under which the last factor of the second line is larger than $1/2$, thus

$$|Q_0\left(\lambda\right)| \geq \lambda_{k,0}\left|\lambda - \lambda_{k,0}\right|/2. \tag{4.23}$$

It suffices to have

$$\left|\frac{\left(\lambda - \lambda_{k,0}\right)}{2} \frac{D_2\left(\mu\right)}{D_1\left(\lambda_{k,0}\right)}\right| \leq 1/2. \tag{4.24}$$

Inequality (4.19) provides a lower estimate for the denominator $D_1\left(\lambda_{k,0}\right)$. Furthermore, write (2.20) as

$$Q_0\left(\lambda\right) = p\,\cos\lambda - \lambda\sin\lambda + q^2\int_0^1 \cos\lambda\xi d\xi,$$

implying

$$\begin{aligned} |D_2\left(\mu\right)| &:= \left|\frac{\partial^2 Q_0\left(\mu\right)}{\partial\mu^2}\right| \\ &= \left|-p\,\cos\mu - 2\cos\mu + \mu\sin\mu - q^2\int_0^1 \xi^2\cos\mu\xi d\xi\right| \\ &\leq p + 2 + \mu + q^2 \leq |\lambda - \lambda_{k,0}| + \lambda_{k,0} + p + 2 + q^2. \end{aligned}$$

Plugging this bound and (4.19) in (4.24), we get a condition on $|\lambda - \lambda_{k,0}|$ that is sufficient for (4.23) to hold:

$$|\lambda - \lambda_{k,0}| \leq \frac{\lambda_{k,0}}{|\lambda - \lambda_{k,0}| + \lambda_{k,0} + p + 2 + q^2}.$$

In turn, this inequality, hence, (4.24) holds if

$$\lambda \in [\lambda_-, \lambda_+], \quad \lambda_\pm = \lambda_{k,0} \pm R_k,$$

where R_k is a positive root of the quadratic equation (cf. (2.27))

$$R_k = \frac{\lambda_{k,0}}{R_k + \lambda_{k,0} + p + 2 + q^2}.$$

Combining (4.15) and (4.23) and taking into account that $\lambda_{1,0} \leq \lambda_{k,0}$, we find that condition (cf. (2.25))

$$\sigma(1+h_-)(1+\sigma+h_+) < \frac{R_k \lambda_{k,0}}{2},$$

leads to the inequality

$$|Q_V(\lambda_\pm) - Q_0(\lambda_\pm)| \leq \sigma(1+h_-)(1+\sigma+h_+) < \frac{R_k \lambda_{k,0}}{2} \leq |Q_0(\lambda_\pm)|$$

implying that $\operatorname{sign}(Q_V(\lambda_\pm)) = \operatorname{sign}(Q_0(\lambda_\pm))$.

Now set $\tau_- = \max\{(k-1)\pi, \lambda_-\}$, $\tau_+ = \min\{k\pi, \lambda_+\}$ to have

$$\lambda_{k,0} \in [\tau_-, \tau_+] = [\lambda_-, \lambda_+] \cap [(k-1)\pi, k\pi].$$

According to (4.16), $\operatorname{sign}(Q(V, k\pi)) = \operatorname{sign}(Q(0, k\pi))$ for all k, therefore

$$\operatorname{sign}(Q_V(\tau_\pm)) = \operatorname{sign}(Q_0(\tau_\pm)).$$

This, (4.19) and (4.23) yield that both functions Q_0 and Q_V have a single root in an interval $[\mu_-, \mu_+] \subset [(k-1)\pi, k\pi]$.

The requirement for σ is easy to make uniform in k. Indeed, it suffices to note that $R_k \lambda_{k,0}$ is monotone increasing in k and set

$$\sigma < \frac{R_1 \lambda_{1,0}}{2(1+h_-)(1+\sigma+h_+)}.$$

Since $\lambda_{k,V} \in [\tau_-, \tau_+] = [\lambda_-, \lambda_+] \cap [(k-1)\pi, k\pi]$, we get $|\lambda_{k,V} - \lambda_{k,0}| < R_k$ and repeating the same calculations we obtain

$$\begin{aligned} 0 = |Q_V(\lambda_{k,V})| &= |Q_0(\lambda_{k,V}) + Q_V(\lambda_{k,V}) - Q_0(\lambda_{k,V})| \\ &\geq |Q_0(\lambda_{k,V})| - |Q_V(\lambda_{k,V}) - Q_0(\lambda_{k,V})| \\ &\geq \frac{1}{2} |\lambda_{k,V} - \lambda_{k,0}| \, \lambda_{k,0} - \sigma(1+h_-)(1+\sigma+h_+), \end{aligned}$$

hence, (4.17). □

Acknowledgment. It is our pleasure to thank L. Pastur for his interest to the paper.

References

[1] T. Kato, *Perturbation theory for linear operators*, Springer, Heidelberg, 1995.

[2] V.A. Marchenko, *Sturm-Liouville Operators and Applications*, Birkhauser, Basel, 1986.

[3] M.A. Naimark, *Linear Differential Operators*, Dover Publications, New York, 2012.

Received February 20, 2021.

V.A. Marchenko,

B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine, 47 Nauky Ave., Kharkiv, 61103, Ukraine,
E-mail: marchenko@ilt.kharkov.ua

A.V. Marchenko,

Moody's Analytics, 5001 Yonge Street, Siute 1300, Box 172, Toronto, Ontario, M2N 6P6, Canada,
E-mail: avmarch@mail.com

V.A. Zolotarev,

B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine, 47 Nauky Ave., Kharkiv, 61103, Ukraine,
E-mail: zolotarev@ilt.kharkov.ua

Про обернену спектральну задачу в теорії збурень

V.A. Marchenko, A.V. Marchenko, and V.A. Zolotarev

Ми розглядаємо обернену спектральну задачу для операторів Штурма–Ліувілля $\widehat{H}_V$ визначених на інтервалі $[a, b]$ деяким потенціалом $V \in L^2[a, b]$ та змішаними розділеними крайовими умовами. Ми доводимо, що якщо L^1-норма V є досить малою, то існує V_{app} такий, що $\|V - V_{\text{app}}\|_{L^2} = O(\|V\|^2_{L^1})$, і ми вказуємо алгоритм для пошуку V_{app}. Цей алгоритм визначає коефіцієнти Фур'є V_{app} відносно власних функцій $\{\psi_{k,0}\}_{k=1}^{\infty}$ незбуреного оператора $\widehat{H}_0$ через власні значення $\{\lambda_{k,V}\}_{k=1}^{\infty}$ збуреного оператора $\widehat{H}_V$, значення його власних функцій $\{\psi_{k,V}\}_{k=1}^{\infty}$ в кінцях відрізку $[a, b]$ і величини $\{\psi_{k,V}\}_{k=1}^{\infty}$ та їх похідні в середині $[a, b]$.

Ключові слова: спектральна теорія, потенціал, обернена задача, теорія збурень