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A Thermo-Viscoelastic Fractional Contact Problem with Normal Compliance and Coulomb's Friction

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This study concerns the analysis of a quasistatic frictional contact problem between a thermo-viscoelastic body and a thermally conductive foundation. The constitutive relation is built by a fractional Kelvin–Voigt model. The heat conduction is governed by time-fractional of temperature parameter θ. The contact is described by the normal compliance condition and the friction is described by Coulomb's law. We derive a variational formulation of the problem and prove the existence of a weak solution to the model by using the theory of monotone operator, Caputo derivative, Clark subdifferential, Galerkin method and Banach fixed point theorem.

Key words: Thermo-viscoelastic contact, fractional viscoelastic constitutive law, friction, Caputo derivative, Galerkin method, Banach fixed point theorem

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1. Introduction

General models for contact problems with friction of a viscoelastic body can be found in [2, 3, 18]. The mathematical model which describes the quasistatic frictional contact between an electro-viscoelastic body and a deformable conductive foundation was studied in [11]. A. Amassad et al. considered in [1] the modeling of quasistatic thermoviscoelastic problem with bilateral contact and with a slip rate dependent condition, they also proved the existence and uniqueness of the weak solution and studied the regularized version of the problem.

The foundation of the theory of fractional calculus was initiated by Gottfried Leibniz, Guillaume de l'Hôpital and Johann Bernoulli at the end of the 17th century [9]. After the publications of Joseph Liouville and Bernhard Riemann, several results on this theory were introduced in the middle of the 19th century, see [17].

Among the applications of fractional calculation there is the mechanical modelling of rubber-like materials. In this sense, [8, 16] are cited as references for the models that include specific materials having viscoelastic properties, where the fractional constitutive laws of Kelvin–Finger and the fractional model of Maxwell are taken into account.

In [7], the authors studied a general quasistatic frictionless contact problem for a viscoelastic body modeled by the fractional Kelvin–Voigt law and the contact condition described by the Clarke subdifferential of a nonconvex and nonsmooth functional.

Z. Zeng et al. [20, 21] introduced a class of generalized differential hemivariational inequalities involving the time fractional order derivative operator applied to a frictional contact problem.

The aim of the present paper is to study the solvability of a new mathematical model for a frictional contact problem between a thermo-viscoelastic body and a thermally conductive foundation. The novelty is in using the Kelvin–Voigt constitutive law with time-fractional as below

$$\sigma(t) = \mathcal{C}\varepsilon\left({}_0^C D_t^\alpha u(t)\right) + \mathcal{E}\varepsilon(u(t)) - \theta(t)\mathcal{M} \quad \text{in } \Omega \times (0, T). \tag{1.1}$$

Also, we model the Fourier law of heat conduction for a temperature field with a time-fractional as follows:

$${}_0^C D_t^\alpha \theta(t) + \operatorname{div} q(t) = q_0(t) \quad \text{in } \Omega \times (0, T), \tag{1.2}$$

which leads to a new and more sophisticated mathematical model.

The difficulty of solving this type of problem lies in the coupling of viscoelastic and thermal aspects with time fractional, also in the nonlinearity of the boundary conditions, which gives us a nonlinear variational and hemivariational inequalities.

We provide the variational analysis of the mechanical problem which leads to a coupled system of time fractional and we show the existence of a weak solution.

Our main result is based on Theorem 19 from [21], the fractional Caputo derivative, the Galerkin method and the Banach fixed point theorem.

The rest of the paper is organized as follows. In Section 2, we state the mechanical model of a thermo-viscoelastic fractional contact problem with normal compliance and Coulomb's friction. In Section 3, we review some basic mathematical notations, definitions and assumptions. We derive the variational formulation and present the main result of our problem. In Section 4, we prove our main existence result. Finally, in Appendix (Section 5), we recall some results: the Riemann–Liouville fractional integral, the Caputo derivative of order $0 < \alpha \leq 1$, the Clarke generalized directional derivative and the generalized gradient, which are useful in the proof of the main result.

2. Time-fractional contact problem

We consider a body made of a viscoelastic material, which occupies an open domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with a smooth boundary $\partial\Omega = \Gamma$ and a unit outward normal ν. This boundary is divided into three open disjoint parts Γ_D, Γ_N and Γ_C such that $\text{meas}(\Gamma_D) > 0$. Let $T > 0$ and $[0, T]$ be the time interval of interest. The body is submitted to the action of body forces of density f_0 and a volume heat flux of density q_0. It is also submitted to mechanical and thermal constants on the boundary. The body is clamped on Γ_D. The surface traction of density f_1 acts on $\Gamma_N \times (0,T)$. On $\Gamma_C \times (0,T)$, the body may come in frictional contact with the so-called foundation which is thermally conductive. We assume that the thermal potential is maintained fixed of θ_F. The normalized gap between $\Gamma_C \times (0,T)$ and the conductive foundation is denoted by g.

To simplify the notation, we denote by $\mathbb{S}^d$ the space of the second-order symmetric tensor on $\mathbb{R}^d$, "$\cdot$" and "$|\cdot|$" represent the inner product and the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ respectively. Thus,

$$u \cdot v = u_i v_i, \qquad \|v\|_{\mathbb{R}^d} = (v,v)^{\frac{1}{2}} \qquad \text{for all } u = (u_i),\ v = (v_i) \in \mathbb{R}^d,$$
$$\sigma \cdot \tau = \sigma_{ij}\tau_{ij}, \qquad \|\tau\|_{\mathbb{S}^d} = (\tau,\tau)^{\frac{1}{2}} \qquad \text{for all } \sigma = (\sigma_{ij}),\ \tau = (\tau_{ij}) \in \mathbb{S}^d.$$

We also use the notation u_ν and u_τ for the normal and tangential displacements, that is, $u_\nu = u \cdot \nu$ and $u_\tau = u - u_\nu \cdot \nu$. We denote by σ_ν and σ_τ the normal and tangential stress tensors given by $\sigma_\nu = \sigma\nu \cdot \nu$, $\sigma_\tau = \sigma\nu - \sigma_\nu \nu$.

We denote by $u : \Omega\times]0,T[\to \mathbb{R}^d$ the displacement field, by $\sigma = (\sigma_{ij}) : \Omega \times (0,T) \to \mathbb{S}^d$, the stress tensor, and by $q = (q_i) : \Omega \times (0,T) \to \mathbb{R}^d$, the heat flux vector. Also, $\varepsilon(u) = (\varepsilon_{ij}(u)) = \dfrac{1}{2}(u_{i\cdot j} + u_{j\cdot i})$, $\mathcal{E} = (e_{ijkl})$, $\mathcal{M} = (m_{ij})$, $\mathcal{C} = (c_{ijkl})$ and $\mathcal{K} = (k_{ij})$ are respectively the linearized strain tensor, the elastic tensor, the thermal expansion tensor, the (fourth-order) viscosity tensor and the thermal conductivity tensor. Here and below "Div" and "div" denote the divergence operator for tensor and vector valued functions, i.e., $\operatorname{Div}\sigma = (\sigma_{ij.j})$ and $\operatorname{div} q = (q_{i,i})$.

The classical form of the mechanical fractional contact problem is stated as follows.

Problem (P): *Find a displacement field* $u : \Omega\times]0,T[\to \mathbb{R}^d$ *and a temperature field* $\theta : \Omega\times]0,T[\to \mathbb{R}$ *such that*

$$\sigma(t) = \mathcal{C}\varepsilon\left({}_0^C D_t^\alpha u(t)\right) + \mathcal{E}\varepsilon(u(t)) - \theta(t)\mathcal{M} \quad \text{in} \quad \Omega \times (0,T), \tag{2.1}$$
$$q(t) = -\mathcal{K}\nabla\theta(t) \quad \text{in} \quad \Omega \times (0,T), \tag{2.2}$$
$$\operatorname{Div}\sigma(t) + f_0(t) = 0 \quad \text{in} \quad \Omega \times (0,T), \tag{2.3}$$
$${}_0^C D_t^\alpha \theta(t) + \operatorname{div} q(t) = q_0(t) \quad \text{in} \quad \Omega \times (0,T), \tag{2.4}$$
$$u(t) = 0 \quad \text{on} \quad \Gamma_D \times (0,T), \tag{2.5}$$
$$\sigma(t)\nu = f_1(t) \quad \text{on} \quad \Gamma_N \times (0,T), \tag{2.6}$$
$$\theta(t) = 0 \quad \text{on} \quad (\Gamma_D \cup \Gamma_N) \times (0,T), \tag{2.7}$$
$$u(0,x) = u_0,\ \theta(0,x) = \theta_0 \quad \text{in} \quad \Omega, \tag{2.8}$$
$$-\sigma_\nu(t)(u(t) - g) = p_\nu(u_\nu(t) - g) \quad \text{on} \quad \Gamma_C \times (0,T), \tag{2.9}$$
$$\|\sigma_\tau(t)\| \le p_\tau(u_\nu(t) - g) \quad \text{on} \quad \Gamma_C \times (0,T), \tag{2.10}$$
$$\|\sigma_\tau(t)\| < p_\tau(u_\nu(t) - g) \Rightarrow u_\tau(t) = 0 \quad \text{on} \quad \Gamma_C \times (0,T), \tag{2.11}$$
$$\|\sigma_\tau(t)\| = p_\tau(u_\nu(t) - g) \Rightarrow \exists\lambda \ne 0\ \sigma_\tau(t) = -\lambda u_\tau(t) \quad \text{on} \quad \Gamma_C \times (0,T), \tag{2.12}$$

$$\frac{\partial q(t)}{\partial \nu} = k_c(u_\nu(t) - g)\phi_L(\theta(t) - \theta_F) \qquad on \quad \Gamma_C \times (0,T). \tag{2.13}$$

We now describe problem (2.1)–(2.13). First, equations (2.1) and (2.2) are the time-fractional Kelvin–Voigt thermo-viscoelastic constitutive law of Caputo type, see [19]. Equations (2.3)–(2.4) represent the equilibrium stress and the Fourier law of heat conduction with time-fractional. Conditions (2.5)–(2.7) are the displacement and thermal boundary conditions. The initial conditions are represented by equation (2.8). Moreover, equation (2.9) represents the normal compliance contact condition, where p_ν is a prescribed function. When it is positive, $u_\nu - g$ represents the penetration of the surface asperities into those of the foundation. The Coulomb law of friction is considered in (2.10)–(2.12), where p_τ is a prescribed nonnegative function, the so-called friction bound. Finally, the relation (2.13) represents a regularized thermal contact condition, where $\dfrac{\partial q}{\partial \nu}$ is the normal derivative of q such that

$$\phi_L(s) = \begin{cases} -L & \text{if } s < -L \\ s & \text{if } -L \le s \le L \\ L & \text{if } s > L \end{cases}, \qquad \begin{cases} k_c(r) = 0 & \text{if } r < 0 \\ k_c(r) > 0 & \text{if } r \ge 0 \end{cases},$$

where L is a large positive constant, see [11].

3. Variational formulation and the main result

To present the variational formulation of Problem **(P)**, we will use the notations

$$\begin{aligned} H &= \{v = (v_i) \mid v_i \in L^2(\Omega),\ i = 1,\dots,d\} = L^2(\Omega)^d, \\ H_1 &= \{v = (v_i) \mid v_i \in H^1(\Omega),\ i = 1,\dots,d\} = H^1(\Omega)^d, \\ \mathcal{H} &= \{\tau = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega),\ i,j = 1,\dots,d\}, \\ \mathcal{H}_1 &= \{\sigma \in \mathcal{H} \mid \operatorname{Div}\sigma \in H\}. \end{aligned}$$

These are real Hilbert spaces endowed with the inner products

$$\begin{aligned} (u,v)_H &= \int_\Omega u_i v_i dx, \qquad & (u,v)_{H_1} &= (u,v)_H + (\varepsilon(u) + \varepsilon(v))_{\mathcal{H}}, \\ (\sigma,\tau)_{\mathcal{H}} &= \int_\Omega \sigma_{ij}\tau_{ij} dx, \qquad & (\sigma,\tau)_{\mathcal{H}_1} &= (\sigma,\tau)_{\mathcal{H}} + (\operatorname{Div}\sigma + \operatorname{Div}\tau)_{\mathcal{H}} \end{aligned}$$

with the associated norms $\|\cdot\|_H$, $\|\cdot\|_{H_1}$, $\|\cdot\|_{\mathcal{H}}$, and $\|\cdot\|_{\mathcal{H}_1}$.

Also, for every real Hilbert space X, we use the classical notations for the spaces $L^p(0,T;X)$, $C(0,T;X)$ and $W^{k,p}(0,T;X)$, $1 \le p \le +\infty$ and $k = 1,2,\dots$

Keeping in mind the boundary conditions (2.5) and (2.7), we introduce the closed subspace of H_1 by

$$V = \{v \in H_1 \mid v = 0 \text{ on } \Gamma_D\}, \quad Q = \{\eta \in H^1(\Omega) \mid \eta = 0 \text{ on } \Gamma_D \cup \Gamma_N\},$$

endowed with the inner product and the norm given by

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \qquad \|v\|_V = (v, v)_V^{\frac{1}{2}},$$
$$(\theta, \eta)_Q = (\nabla\theta, \nabla\eta)_H, \qquad \|\eta\|_Q = (\eta, \eta)_Q^{\frac{1}{2}}.$$

Let V_{ad} be the set of admissible displacements defined by

$$V_{\text{ad}} = \{v \in V \mid v_\nu - g \leq 0 \text{ on } \Gamma_C\}.$$

Since $\text{meas}(\Gamma_D) > 0$, Korn's inequality

$$\|\varepsilon(v)\|_{\mathcal{H}} \geq c_K \|v\|_{H_1}, \quad \text{for all } v \in V, \tag{3.1}$$

holds, where $c_K > 0$ is a constant which depends only on Γ and Γ_D.

The following Frierichs–Poincaré inequality holds on Q:

$$\|\nabla\eta\|_H \geq c_P \|\eta\|_Q \quad \text{for all } \eta \in Q. \tag{3.2}$$

Moreover, by Sobolev's trace theorem, there exist constants c_d and c_t, which depend only on Ω, Γ_D and Γ_C, for all $v \in V$ and $\eta \in Q$, such that

$$\|v\|_{L^2(\Gamma_C)^d} \leq c_d \|v\|_V \quad \text{and} \quad \|\eta\|_{L^2(\Gamma_C)} \leq c_t \|\eta\|_Q. \tag{3.3}$$

Next, we define the following operators:

$$\begin{aligned}
a &: V \times V \to \mathbb{R}, & a(u, v) &:= (\mathcal{E}\varepsilon(u), \varepsilon(v))_{\mathcal{H}},\\
c &: V \times V \to \mathbb{R}, & c(u, v) &:= (\mathcal{C}\varepsilon(u), \varepsilon(v))_{\mathcal{H}},\\
d &: Q \times Q \to \mathbb{R}, & d(\theta, \eta) &:= (\mathcal{K}\nabla\theta, \nabla\eta)_H,\\
m &: Q \times V \to \mathbb{R}, & m(\theta, v) &:= (\mathcal{M}\theta, \varepsilon(v))_{\mathcal{H}}.
\end{aligned}$$

The mappings $j : V \times V \to \mathbb{R}$ and $\chi : V \times Q \times Q \to \mathbb{R}$ are defined by

$$j(u(t), v) := \int_{\Gamma_C} p_\nu(u_\nu(t) - g) v_\nu \, da + \int_{\Gamma_C} p_\tau(u_\tau(t) - g)\|v_\tau\| \, da, \tag{3.4}$$

$$\chi(u(t), \theta(t), \eta) := \int_{\Gamma_C} k_c(u_\nu(t) - g)\phi_L(\theta(t) - \theta_F)\eta \, da \tag{3.5}$$

for all v in V and η in Q.

Now we assume the following assumptions.

1. a) The operators a, c, and d are bilinear and satisfy the usual property of symmetry

$$e_{ijkl} = e_{jikl} = e_{lkij} \in L^\infty(\Omega), \quad c_{ijkl} = c_{jikl} = c_{lkij} \in L^\infty(\Omega),$$
$$k_{ij} = k_{ji} \in L^\infty(\Omega);$$

   b) the operators c, a and d satisfy the property of ellipticity, i.e., there exist positive constants m_c, m_a and m_d such that

$$c(v, v) \geq m_c \|v\|_V^2, \quad a(v, v) \geq m_a \|v\|_V^2, \quad \text{and} \quad d(\eta, \eta) \geq m_d \|\eta\|_Q^2.$$

2. The operators a, c, d and m satisfy the usual property of boundedness

$$|a(u,v)| \le M_a \|u\|_V \|v\|_V, \qquad |c(u,v)| \le M_c \|u\|_V \|v\|_V,$$
$$|d(\theta,\eta)| \le M_d \|\theta\|_Q \|\eta\|_Q, \qquad |m(\theta,v)| \le M_m \|\theta\|_Q \|v\|_V,$$

where $M_a, M_c, M_d, M_m > 0$.

3. The forces, the traction and the heat flux satisfy

$$f_0 \in C\left(0,T; L^2(\Omega)\right), \quad f_1 \in C\big(0,T; L^2(\Gamma_N)^d\big), \quad \text{and} \quad q_0 \in C\left(0,T; L^2(\Omega)\right).$$

4. The gap function, the initial conditions and the thermal potential satisfy

$$g \ge 0,\ g \in L^\infty(\Gamma_C), \quad u_0 \in V_{ad}, \quad \theta_0 \in Q, \quad \text{and} \quad \theta_F \in L^2\big(0,T; L^2(\Gamma_C)\big).$$

5. The coefficient of heat exchange $k_c : \Gamma_C \times \mathbb{R} \to \mathbb{R}^+$ satisfies the conditions:

   a) there exists $M_{k_c} > 0$ such that $|k_c(x,u)| < M_{k_c}$ for all $u \in \mathbb{R}$ and $x \in \Gamma_3$, such that $x \mapsto k_c(x,u)$ is measurable on Γ_C for all $u \in \mathbb{R}$ and vanishes for all $u \le 0$ and a.a. $x \in \Gamma_C$;

   b) there exists $L_{k_c} > 0$ such that

$$|k_c(x,u_1) - k_c(x,u_2)| \le L_{k_c} |u_1 - u_2| \quad \text{for all } u_1, u_2 \in \mathbb{R}.$$

6. The normal compliance function p_ν and the friction bound p_τ satisfy the following hypotheses for $\delta = \nu, \tau$:

   a) $p_\delta : \Gamma_C \times \mathbb{R} \to \mathbb{R}_+$;

   b) $x \to p_\delta(x,u)$ is measurable on Γ_C for all $u \in \mathbb{R}$;

   c) $x \to p_\delta(x,u) = 0$ for $u \le 0$ and a.a. $x \in \Gamma_C$;

   d) there exits $L_\delta > 0$ such that

$$|p_\delta(\cdot,u) - p_\delta(\cdot,v)| \le L_\delta |u - v| \quad \text{for all } u, v \in \mathbb{R}_+.$$

7. The functional j satisfies

$$\|\partial j(u(t),v)\|_{V^*} \le m_j(1 + \|u\|_V + \|v\|_V) \text{ for all } u \in V, v \in V \text{ and a.a. } t \in (0,T)$$

with $m_j \ge 0$.

Using Riesz's representation theorem, we conclude that there exist the elements $f \in V$ and $q_t \in Q$ given by

$$(f(t),v)_V = \int_\Omega f_0(t) \cdot v \, dx + \int_{\Gamma_N} f_1(t) \cdot v \, da \qquad \text{for all } v \in V, \tag{3.6}$$

$$(q_t(t),\eta)_Q = \int_\Omega q_0(t) \cdot \eta \, dx \qquad \text{for all } \eta \in Q. \tag{3.7}$$

From all these assumptions and notations, we obtain the following variational formulation of Problem **(P)**.

Problem (PV): *Find a displacement field $u:\Omega\times]0,T[\to\mathbb{R}^d$ and a temperature field $\theta:\Omega\times]0,T[\to\mathbb{R}$ such that for a.a. $t\in]0,T[$, $v\in V$, $\eta\in Q$, and $\alpha\in]0,1[$, we have*

$$\begin{aligned} c\left({}_0^C D_t^\alpha u(t), v-u(t)\right)+a\big(u(t),v-u(t)\big)-m(\theta(t),v-u(t)) \\ +j(u(t),v)-j(u(t),u(t))\geq (f(t),v-u(t)), \quad (3.8)\\ \big({}_0^C D_t^\alpha \theta(t),\eta)_Q+d(\theta(t),\eta\big)+\chi(u(t),\theta(t),\eta)=(q_t(t),\eta), \quad (3.9)\\ u(0)=u_0,\ \theta(0)=\theta_0. \quad (3.10)\end{aligned}$$

In the following theorem, we state the solvability of Problem **(PV)**.

Theorem 3.1. *Let (3.4)–(3.5), Assumptions 1–7 and the conditions*

$$m_a>c_d^2(L_\nu+L_\tau) \quad \text{and} \quad m_d>M_{k_c}c_t^2 \tag{3.11}$$

*hold. Then Problem **(PV)** has at least one solution*

$$(u,\theta)\in W^{1,2}(0,T;V)\times W^{1,2}(0,T;Q). \tag{3.12}$$

4. Existence of the weak solution

The proof of Theorem 3.1 will be carried out in several steps and it is based on the argument for the monotone operator, the Caputo derivative, the Clarke subdifferential, the Galerkin method and the Banach fixed point theorem.

First, let $\beta\in L^2(0,T;V)$ be given by

$$(\beta(t),v-u_\beta(t))=m\left(\theta_\beta(t),v-u_\beta(t)\right), \tag{4.1}$$

and we consider the following problem.

Problem (PV1): *Find a displacement field $u_\beta:\Omega\times]0,T[\to\mathbb{R}^d$ such that for a.a. $t\in]0,T[$, $v\in V$, and $\alpha\in]0,1[$, we have*

$$\begin{aligned} c\left({}_0^C D_t^\alpha u_\beta(t), v-u_\beta(t)\right)+a(u_\beta(t),v-u_\beta(t))-(\beta(t),v-u_\beta(t)) \\ +j(u_\beta(t),v)-j(u_\beta(t),u_\beta(t))\geq (f(t),v-u_\beta(t)), \quad (4.2)\\ u_\beta(0)=u_0. \quad (4.3)\end{aligned}$$

We have the following result.

Lemma 4.1. *For all $v\in V$ and a.a. $t\in]0,T[$, Problem **(PV1)** has at least one solution $u_\beta\in W^{1,2}(0,T;V)$.*

Proof. Using Riesz's representation theorem, we define the functional

$$(f_\beta(t),v)_V=(f(t),v)+(\beta(t),v) \quad \text{for all } v\in V. \tag{4.4}$$

Problem **(PV1)** can be written as follows:

$$c\left({}_0^C D_t^\alpha u_\beta(t), v-u_\beta(t)\right)+a(u_\beta(t),v-u_\beta(t))+j(u_\beta(t),v)$$

$$-j(u_\beta(t), u_\beta(t)) \geq (f_\beta(t), v - u_\beta(t)), \tag{4.5}$$

$$u_\beta(0) = u_0. \tag{4.6}$$

It is easy to see that under Assumption 1 the operator c is bilinear continuous and coercive.

By Assumptions 1a) and 2, the operator a is bilinear and continuous.

From Assumption 3, (3.6), (4.4), and the regularity of β, we have that $f_\beta \in L^2(0, T; V)$.

It is clear from Assumption 6d) that j is a locally Lipschitz function.

We combine these results of the operators c, a, j and the function f_β with Assumption 7 and using the result provided by Theorem 19 in [21], we find that Problem **(PV1)** has at least one solution $u_\beta \in W^{1,2}(0, T; V)$. □

Here and below c_1, c_2 and c_s denote positive generic constants whose values may change from line to line.

In the second step, we use the displacement field u_β obtained in Lemma 4.1 to prove the existence result for the temperature field θ_β of the following problem.

Problem (PV2): *Find a temperature field $\theta_\beta : \Omega\times]0, T[\to \mathbb{R}$ such that for a.a. $t \in]0, T[$, $\eta \in Q$, and $\alpha \in]0, 1[$, we have*

$$\left({}_0^C D_t^\alpha \theta_\beta(t), \eta\right) + d(\theta_\beta(t), \eta) + \chi\left(u_\beta(t), \theta_\beta(t), \eta\right) = \left(q_t(t), \eta\right), \tag{4.7}$$

$$\theta_\beta(0) = \theta_0. \tag{4.8}$$

Lemma 4.2. *For all $\eta \in Q$ and a.a. $t \in]0, T[$ Problem* **(PV2)** *has at least one solution $\theta_\beta \in W^{1,2}(0, T; Q)$.*

Proof. We will implement the Galerkin approximation method. For $k = 1, 2, \ldots$, let (w_k) be a K^{th} mode consisting of the eigenfunctions of $-\Delta$ such that $(w_k)_{k\geq 1}$ forms a Hilbertian basis of $H^1(\Omega)$.

We are to find a function $\theta_{\beta_n} :]0, T[\to H^1(\Omega)$ of the form

$$\theta_{\beta_n}(t) := \sum_{i=1}^{n} x_n^i(t) w_i. \tag{4.9}$$

We denote by F_n the vector space generated by $w_1, w_2, \ldots, w_n$.

Whence $\theta_{\beta_n} \in F_n$ and $\theta_{\beta_n} \to \theta_\beta$ in Q.

For each integer $n \geq 1$, consider the following approximate problem: Find $\theta_{\beta_n} \in L^2(0, T; F_n)$ such that ${}_0^C D_t^\alpha \theta_{\beta_n} \in L^2(0, T; F_n)$ and

$$\left({}_0^C D_t^\alpha \theta_{\beta_n}(t), w_k\right) + d\left(\theta_{\beta_n}(t), w_k\right) + \chi\left(u_\beta(t), \theta_{\beta_n}(t), w_k\right) = \left(q_t(t), w_k\right), \tag{4.10}$$

$$\theta_{\beta_n}(0) = \theta_0. \tag{4.11}$$

Using (4.9), we have

$$\left({}_0^C D_t^\alpha \theta_{\beta_n}(t), w_k\right)_Q = {}_0^C D_t^\alpha x_n^i(t), \tag{4.12}$$

$$d\left(\theta_{\beta_n}(t), w_k\right) = \mathcal{K} x_n^i(t), \tag{4.13}$$

$$\chi\left(u_\beta(t),\theta_{\beta_n}(t),w_k\right)=\chi\left(u_\beta(t),\sum_{i=1}^{n}x_n^i(t)w_i,w_k\right), \tag{4.14}$$

$$(q_t(t),w_k)=q_t^k(t). \tag{4.15}$$

Then (4.10)–(4.11) can be written as follows:

$$ {}_0^C D_t^\alpha x_n^i(t)=h\left(t,x_n^i(t)\right), \tag{4.16}$$

$$x_n^i(0)=(\theta_0,w_i), \tag{4.17}$$

where

$$h\left(t,x_n^i(t)\right)=q_t^k(t)-\mathcal{K}x_n^i(t)-\chi\left(u_\beta(t),\sum_{i=1}^{n}x_n^i(t)w_i,w_k\right). \tag{4.18}$$

Due to Assumption 2, we find

$$\left|\mathcal{K}x_{n_1}^i(t)-\mathcal{K}x_{n_2}^i(t)\right|\le M_d\left|x_{n_1}^i(t)-x_{n_2}^i(t)\right|, \tag{4.19}$$

and by (3.5), Assumptions 5 and 6, we have that

$$\left|\chi\left(u_\beta(t),\sum_{i=1}^{n}x_{n_1}^i(t)w_i,w_k\right)-\chi\left(u_\beta(t),\sum_{i=1}^{n}x_{n_2}^i(t)w_i,w_k\right)\right|$$
$$\le M_{k_c}L_{k_c}\operatorname{meas}(\Gamma_C)\left|x_{n_1}^i(t)-x_{n_2}^i(t)\right|. \tag{4.20}$$

Combining this inequality with (4.18)-(4.19), we see that there exists a positive constant c_s such that

$$\left|h\left(t,x_{n_1}^i(t)\right)-h\left(t,x_{n_2}^i(t)\right)\right|\le c_s\left|x_{n_1}^i(t)-x_{n_2}^i(t)\right|. \tag{4.21}$$

Then, by a standard method for fractional calculus (see Proposition 4.6 in [12]), there exists a unique absolutely continuous function $x_n(t)=\left(x_n^1(t),x_n^2(t),\ldots,x_n^n(t)\right)$ on $[0,T_*)$ that satisfies the system of fractional ordinary differential equation (4.16)-(4.17).

Estimates: Multiply (4.10) by $x_n^i(t)$, sum for $i=1,\ldots,n$ and the fact that $\theta_{\beta_n}\mapsto\frac{1}{2}\|\theta_{\beta_n}\|_Q^2$ is a convex functional to obtain

$$ {}_0^C D_t^\alpha\left(\frac{1}{2}(\|\theta_{\beta_n}\|_Q^2\right)\le\left({}_0^C D_t^\alpha\theta_{\beta_n},\theta_{\beta_n}\right)+d\left(\theta_{\beta_n},\theta_{\beta_n}\right)$$
$$+\chi\left(u_\beta,\theta_{\beta_n},\theta_{\beta_n}\right)=\left(q_t,\theta_{\beta_n}\right). \tag{4.22}$$

After some calculus, for all $\epsilon>0$, we have

$$m_d\|\theta_{\beta_n}(t)\|_Q^2\le\left|d\left(\theta_{\beta_n},\theta_{\beta_n}\right)\right|, \tag{4.23}$$

$$\left|\chi\left(u_\beta,\theta_{\beta_n},\theta_{\beta_n}\right)\right|\le\frac{M_{k_c}^2M_L^2}{2\epsilon}+\frac{\epsilon c_t^2}{2}\|\theta_{\beta_n}\|_Q^2, \tag{4.24}$$

$$|(q_t, \theta_{\beta_n})| \leq \frac{1}{2\epsilon} \|q_t\|_Q^2 + \frac{\epsilon}{2} \|\theta_{\beta_n}\|_Q^2 . \tag{4.25}$$

So,

$${}_0^C D_t^\alpha \left(\frac{1}{2} (\|\theta_{\beta_n}\|_Q^2 \right) + c_1 \|\theta_{\beta_n}\|_Q^2 \leq c_2 \|q_t\|_Q^2 . \tag{4.26}$$

Applying Proposition 5.3(ii) to inequality (4.26), we obtain

$$\|\theta_{\beta_n}\|_Q^2 + \frac{2c_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|\theta_{\beta_n}(s)\|_Q^2 \, ds \leq c_s \left(\|q_t\|_Q^2 + \|\theta_0\|_Q^2 \right) . \tag{4.27}$$

Consequently, we find that $T_* = +\infty$.

Let $\eta \in Q$, with $\|\eta\|_Q \leq 1$, and write $\eta = \eta_1 + \eta_2$, where $\eta_1 \in spam\, \{w_k\}_{k=1}^n$ are orthogonal in Q,

$$\|\eta_1\|_Q \leq \|\eta\|_Q \leq 1. \tag{4.28}$$

Using (4.10), we conclude that

$$\left({}_0^C D_t^\alpha \theta_{\beta_n}, \eta_1 \right) + d\left(\theta_{\beta_n}, \eta_1\right) + \chi\left(u_\beta, \theta_{\beta_n}, \eta_1\right) = \left(q_t, \eta_1\right) . \tag{4.29}$$

Similarly to (4.23)–(4.25), we have

$$|d\left(\theta_{\beta_n}, \eta_1\right)| \leq M_d \|\theta_{\beta_n}(t)\|_Q , \tag{4.30}$$

$$|\chi\left(u_\beta, \theta_{\beta_n}, \eta_1\right)| \leq M_{k_c} M_L c_t, \tag{4.31}$$

$$|(q_t, \eta_1)| \leq \|q_t\|_Q . \tag{4.32}$$

Thus,

$$\left\| {}_0^C D_t^\alpha \theta_{\beta_n} \right\|_{Q^*} \leq c_1 + c_2 \|\theta_{\beta_n}\|_Q + \|q_t\|_Q . \tag{4.33}$$

By inequality (4.27), there exists a positive constant c_s such that

$$\left\| {}_0^C D_t^\alpha \theta_{\beta_n} \right\|_{L^2(0,T;Q^*)} \leq c_s. \tag{4.34}$$

Passage to the limit: Let $\{\tau_n\}$ be a sequence such that $\tau_n \to 0$, as $n \to \infty$.

Using the previous estimates and applying the compactness result (see Theorem 4.2 in [13]) for the Caputo derivative, there exists a subsequence $\theta_{\beta_{\tau_n}}$ and $\theta_\beta \in L^2(0,T;Q)$ such that

$$\theta_{\beta_{\tau_n}} \to \theta_\beta \quad \text{strongly in } L^2(0,T;Q), \tag{4.35}$$

and

$${}_0^C D_t^\alpha \theta_{\beta_{\tau_n}} \rightharpoonup {}_0^C D_t^\alpha \theta_\beta \quad \text{weakly in } L^2(0,T;Q^*). \tag{4.36}$$

Then

$$d\left(\theta_{\beta_{\tau_n}}, \eta\right) \to d\left(\theta_\beta, \eta\right) \quad \text{in } \mathbb{R}, \tag{4.37}$$

$$\left({}_0^C D_t^\alpha \theta_{\beta_{\tau_n}}, \eta\right) \to \left({}_0^C D_t^\alpha \theta_\beta, \eta\right) \quad \text{in } \mathbb{R}. \tag{4.38}$$

By (3.5) and Assumption 5, we obtain

$$|\chi\left(u_\beta, \theta_{\beta_n}, \eta\right)| \leq M_{k_c} L_{k_c} \|\eta\|_{L^2(\Gamma_C)}. \tag{4.39}$$

Since $\{\chi\left(u_\beta, \theta_{\beta_n}, \eta\right)\}_{n=1}^{\infty}$ is bounded in $\mathbb{R}$, we may pass to a subsequence if it is necessary. For $\eta = \theta_\beta - \theta_{\beta_{\tau_n}}$, by using (3.5), we have

$$\left|\chi\left(u_\beta, \theta_\beta, \theta_\beta - \theta_{\beta_{\tau_n}}\right) - \chi\left(u_\beta, \theta_{\beta_n}, \theta_\beta - \theta_{\beta_{\tau_n}}\right)\right| \leq c_t^2 M_{k_c} L_{k_c} \left\|\theta_\beta - \theta_{\beta_{\tau_n}}\right\|_Q^2. \tag{4.40}$$

By the compactness of trace $\gamma : Q \to L^2(\Gamma_C)$, it follows from the weak convergence of $\theta_{\beta_{\tau_n}}$ that

$$\theta_{\beta_{\tau_n}} \to \theta_\beta \quad \text{strongly in } L^2(0,T;L^2(\Omega)). \tag{4.41}$$

Then

$$\chi\left(u_\beta,\ \theta_{\beta_{\tau_n}}, \eta\right) \to \chi\left(u_\beta, \theta_\beta, \eta\right) \quad \text{in } \mathbb{R}. \tag{4.42}$$

The lemma is proved. □

In the last step, for the function $\beta \in L^2(0,T;V)$ and the function θ_β obtained in Lemma 4.2, we consider the operator $\Lambda : L^2(0,T;V) \to L^2(0,T;V)$ defined by

$$(\Lambda\beta(t), v)_V := m\left(\theta_\beta(t), v\right) \quad \text{for all } v \in V \text{ and } t \in]0,T[. \tag{4.43}$$

We have the following lemma.

Lemma 4.3. *For $\beta \in L^2(0,T;V)$, the function $\Lambda\beta :]0,T[\to Q$ is continuous. Moreover, there exists a unique element $\beta^* \in L^2(0,T;V)$ such that $\Lambda\beta^* = \beta^*$.*

Proof. Let $\beta \in L^2(0,T;V)$ and $t_1, t_2 \in]0,T[$. Using (4.1) and Assumption 2, we deduce that

$$\|\Lambda\beta(t_1) - \Lambda\beta(t_2)\|_V \leq c_s \left\|\theta_\beta(t_1) - \theta_\beta(t_2)\right\|_Q. \tag{4.44}$$

Since $\theta_\beta \in L^2(0,T;Q)$, we conclude that $\Lambda\beta \in C(0,T;V)$.

Let now $\beta_1,\ \beta_2 \in L^2(0,T;V)$. Similarly to (4.44), we get

$$\|\Lambda\beta_1(t) - \Lambda\beta_2(t)\|_V \leq c_s \left\|\theta_{\beta_1}(t) - \theta_{\beta_2}(t)\right\|_Q. \tag{4.45}$$

Therefore, from (4.2), we obtain

$$\begin{aligned} c\left({}_0^C D_t^\alpha u_{\beta_1}(t) - {}_0^C D_t^\alpha u_{\beta_2}(t), u_{\beta_1}(t) - u_{\beta_2}(t)\right) & \\ + a\left(u_{\beta_1}(t) - u_{\beta_2}(t), u_{\beta_1}(t) - u_{\beta_2}(t)\right) & \\ + j\left(u_{\beta_1}(t), u_{\beta_1}(t)\right) - j\left(u_{\beta_1}(t), u_{\beta_2}(t)\right) & \\ + j\left(u_{\beta_2}(t), u_{\beta_2}(t)\right) - j\left(u_{\beta_2}(t), u_{\beta_1}(t)\right) \leq 0. & \end{aligned} \tag{4.46}$$

From (3.5) and Assumption 6, we have

$$\begin{aligned} &|j(u_{\beta_1}(t), u_{\beta_1}(t)) - j(u_{\beta_1}(t), u_{\beta_2}(t)) + j(u_{\beta_2}(t), u_{\beta_2}(t)) \\ &\qquad -j(u_{\beta_2}(t), u_{\beta_1}(t))| \leq c_d^2(L_\tau + L_\nu)\, \|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V^2 . \end{aligned} \tag{4.47}$$

By Definition 5.2, we deduce

$$\left\| {}_0^C D_t^\alpha u_{\beta_1}(t) - {}_0^C D_t^\alpha u_{\beta_2}(t) \right\|_V \leq \frac{T^{1-\alpha}}{\Gamma(\alpha)} \|\dot{u}_{\beta_1}(t) - \dot{u}_{\beta_2}(t)\|_V . \tag{4.48}$$

Combining (4.46)–(4.48), integrating from 0 to t and using the Gronwall inequality, we conclude that there exists $c_s > 0$ such that

$$\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_{L^2(0,T;V)} \leq c_s \|\beta_1(t) - \beta_2(t)\|_{L^2(0,T;V)} \tag{4.49}$$

with the condition $m_a > c_d^2(L_\nu + L_\tau)$. Using (4.7), we have

$$\begin{aligned} &\left({}_0^C D_t^\alpha \theta_{\beta_1}(t) - {}_0^C D_t^\alpha \theta_{\beta_2}(t), \theta_{\beta_1}(t) - \theta_{\beta_2}(t) \right) + d\left(\theta_{\beta_1}(t) - \theta_{\beta_2}(t), \theta_{\beta_1}(t) - \theta_{\beta_2}(t)\right) \\ &\quad \chi(u_{\beta_1}(t), \theta_{\beta_1}(t), \theta_{\beta_1}(t) - \theta_{\beta_2}(t)) - \chi(u_{\beta_2}(t), \theta_{\beta_2}(t), \theta_{\beta_1}(t) - \theta_{\beta_2}(t)) = 0. \end{aligned} \tag{4.50}$$

By (3.5) and Assumption 5, we conclude

$$\begin{aligned} &|\chi(u_{\beta_1}(t), \theta_{\beta_1}(t), \theta_{\beta_1}(t) - \theta_{\beta_2}(t)) - \chi(u_{\beta_2}(t), \theta_{\beta_2}(t), \theta_{\beta_1}(t) - \theta_{\beta_2}(t))| \\ &\qquad \leq M_{k_c} c_t^2 \|\theta_{\beta_1}(t) - \theta_{\beta_2}(t)\|_Q^2 \\ &\qquad\quad + L_{k_c} L c_t c_d \|\theta_{\beta_1}(t) - \theta_{\beta_2}(t)\|_Q \|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V . \end{aligned} \tag{4.51}$$

In the same way as above, after some calculations we get that there exists $c_s > 0$ such that

$$\|\theta_{\beta_1}(t) - \theta_{\beta_2}(t)\|_{L^2(0,T;Q)} \leq c_s \|u_{\beta_1}(t) - u_{\beta_2}(t)\|_{L^2(0,T;V)} \tag{4.52}$$

with the condition $m_d > M_{k_c} c_t^2$.

Combining (4.45), (4.49) and (4.52), we obtain

$$\|\Lambda\beta_1(t) - \Lambda\beta_2(t)\|_{L^2(0,T;V)} \leq c_s \|\beta_1(t) - \beta_2(t)\|_{L^2(0,T;Q)} . \tag{4.53}$$

Reiterating this inequality n times leads to

$$\|\Lambda^n\beta_1(t) - \Lambda^n\beta_2(t)\|_{L^2(0,T;V)} \leq \frac{(c_s)^n}{n!} \|\beta_1(t) - \beta_2(t)\|_{L^2(0,T;V)} , \tag{4.54}$$

which implies that for n sufficiently large the power Λ^n of Λ is a contraction in $L^2(0,T;V)$. Therefore, there exists a unique element $\beta^* \in L^2(0,T;V)$ such that $\Lambda\beta^* = \beta^*$. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let $\beta^* \in L^2(0,T;V)$ be a fixed point of the operator Λ. Denote by u_{β^*} a solution of Problem **(PV1)** and let θ_{β^*} be a solution of Problem **(PV2)** for $\beta = \beta^*$. Using the definition of Λ, (4.1)–(4.3) and (4.7)–(4.8), we find that $(u_{\beta^*}, \theta_{\beta^*})$ is a solution of Problem **(PV)**. □

5. Appendix

In this section, we recall some known definitions and properties on nonlinear analysis and fractional calculus, which can be found in [4, 10, 14, 15].

Definition 5.1 (The Riemann–Liouville fractional integral)**.** Let X be a Banach space and $(0, T)$ be a finite time interval. The Riemann-Liouville fractional integral of order $\alpha > 0$ for a given function $f \in L^1(0, T; X)$ is defined by

$$ {}_0I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds \quad \text{for all } t \in (0, T), $$

where $\Gamma(\cdot)$ stands for the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt$.

To complement the definition, we set ${}_0I_t^0 = I$, where I is the identity operator, which means that ${}_0I_t^0 f(t) = f(t)$ for a.a. $t \in (0, T)$.

Definition 5.2 (The Caputo derivative of order, $0 < \alpha \leq 1$)**.** Let X be a Banach space, $0 < \alpha \leq 1$ and $(0, T)$ be a finite time interval. For a given function $f \in AC(0, T; W)$, the Caputo fractional derivative of f is defined by

$$ {}_0^C D_t^\alpha f(t) = {}_0I_t^{1-\alpha} f'(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)\, ds \quad \text{for all } t \in (0, T). $$

The notation $AC(0, T; X)$ refers to the space of all absolutely continuous functions from $(0, T)$ into X.

It is obvious that if $\alpha = 1$, then the Caputo derivative reduces to the classical first-order derivative, that is, we have

$$ {}_0^C D_t^1 f(t) = I f'(t) = f'(t) \quad \text{for a.a. } t \in (0, T). $$

Proposition 5.3. *Let X be a Banach space and $\alpha, \beta > 0$. Then the following statements hold:*

(i) *for $y \in L^1(0, T; X)$, we have*

$$ {}_0I_t^\alpha\, {}_0I_t^\beta y(t) = {}_0I_t^{\alpha+\beta} y(t) \quad \textit{for a.a. } t \in (0, T); $$

(ii) *for $y \in AC(0, T; X)$ and $\alpha \in (0, \alpha]$, we have*

$$ {}_0I_t^\alpha\, {}_0^C D_t^\alpha y(t) = y(t) - y(0) \quad \textit{for a.a. } t \in (0, T); $$

(iii) *for $y \in L^1(0, T; X)$, we have*

$$ {}_0^C D_t^\alpha\, {}_0I_t^\alpha y(t) = y(t) \quad \textit{for a.a. } t \in (0, T). $$

Definition 5.4 (The Clarke generalized directional derivative and the generalized gradient)**.** Let $J : X \to \mathbb{R}$ be a locally Lipschitz function. We denote by

$J^0(u,v)$ the Clarke generalized directional derivative of J at the point $x \in X$ in the direction $y \in X$ defined by

$$J^0(x,y) = \limsup_{\substack{\lambda \to 0^+ \\ z \to x}} \frac{J(z+\lambda y) - J(z)}{\lambda}.$$

The generalized gradient of $J : X \to \mathbb{R}$ at $x \in X$ is defined by

$$\partial J(x) = \left\{ \xi \in X^* \mid \forall y \in X \;\; J^0(x,y) \geq \langle \xi, y \rangle_{X^*,X} \right\}.$$

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Задача термов'язкопружного контакту з тертям із нормальним та кулонівським тертям

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Дослідження стосується аналізу задачі квазістатичного контакту з тертям між термов'язкопружним тілом і термопровідною основою. Рівняння стану побудоване з використанням моделі Кельвіна–Фойгта з дробовою похідною. Теплопровідність моделюється дробовою похідною відносно часу температурного параметру θ. Контакт описується за припущеннями нормальної піддатливості та кулонівського тертя. Ми отримуємо варіаційне формулювання задачі і доводимо існування слабкого розв'язку для моделі, використовуючи теорію монотонного оператора, похідну Капуто, субдиференціал Кларка, метод Гальоркіна та теорему Банаха про нерухому точку.

Ключові слова: термов'язкопружний контакт, нормальна піддатливість, кулонівське тертя, похідна Капуто, слабкий розв'язок, метод Гальоркіна, теорема Банаха про нерухому точку