Journal of Mathematical Physics, Analysis, Geometry
2021, Vol. 17, No. 3, pp. 326–340
doi: https://doi.org/10.15407/mag17.03.326

The Peculiarity of Solving the Synthesis Problem for Linear Systems to a Non-Equilibrium Point

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The paper is devoted to the problem of the construction of a control which transfers a control system from any point to a given non-equilibrium point in a finite time. The construction of the control is based on the method of controllability function. The ambiguity of the solution of the equation that determines the controllability function is found, which leads to a number of interesting cases. The problem is solved for a linear system. The obtained results are illustrated on a model example.

Key words: controllability function method, linear system, non-equilibrium point

Mathematical Subject Classification 2010: 93B05, 93B11, 93C10

1. Introduction

1.1. Statement of the problem and a crucial point of research ideas. Let us consider the system of differential equations

$$\dot{x} = f(x,u), \quad x \in R^n,\ u \in \Omega = \{u \in R^r :\ \|u\| \le d\} \subset R^r. \tag{1.1}$$

The aim is to construct a control $u(t)$ for which the trajectory of a system starting at an arbitrary point $x_0 \in R^n$ transfers into a given non-equilibrium point $x_T \in R^n$ of system (1.1) in a finite time $T = T(x_0, x_T)$, which also satisfies a given restriction $\|u(t)\| \le d$ for all $t \in [0,T]$, where $d > 0$ is a given number. We are going to examine in detail the the following system:

$$\dot{x} = Ax + bu, \quad x \in R^n,\ u \in \Omega \subset R^m. \tag{1.2}$$

Assume that the condition

$$\operatorname{rank}(b, Ab, \dots, A^{n-1}b) = n \tag{1.3}$$

holds. As was shown in [10], if the condition (1.3) is not satisfied, then there are points of R^n from which the origin is not reachable in any finite time, and therefore it is impossible to solve the problem of synthesis. Since we are interested in the study of globally controlled systems, we suppose that (1.2) holds, $\|u(t)\| \leq d$ for all $t \in [0, T]$, and the eigenvalues of the matrix A from (1.3) have non-positive real parts. Let $u(t)$ be a control, which transfers the point x_0 to the point x_T in a finite time T along the trajectory $x(t)$ of system (1.2) according to the Cauchy formula

$$x(t) = e^{At}\Big(x_0 + \int\limits_0^t e^{-A\tau} bu(\tau)d\tau\Big).$$

Let us denote $x(T) = x_T$ and put $t = T$, which is the time of transferring to the point x_T. Then, due to the elementary transformations (including multiplication on the left side of the last identity by e^{-AT}), we obtain the equality

$$e^{-AT}x_T - x_0 = \int\limits_0^T e^{-A\tau} bu(\tau)d\tau.$$

As a result, we have

$$x_0 - e^{-AT}x_T = -\int\limits_0^T e^{-A\tau} bu(\tau)d\tau. \tag{1.4}$$

If T were given, then the problem of transferring to a stationary point or even non-equilibrium point would be reduced to the problem of transferring from a fixed point $(x_0 - e^{-AT}x_T)$ to zero. The difficulty is that we do not know T, and the left side of equality (1.4) depends on T. By virtue of identity (1.4), it is enough to find a path (trajectory) that connects $x_0 - e^{-AT}x_T$ and the origin in a finite time T. Since we need to find this time T, it is also clear that $x_0 - e^{-AT}x_T$ is not given; x_T is not an equilibrium point of the initial system (which, of course, is of interest to the authors). The control $u(t)$, which transfers $(x_0 - e^{-AT}x_T)$ to $(0; 0)$ in time T and satisfies the preassigned constraint, is constructed by the controllability function method.

1.2. Brief history of the problem. In many cases, it is very difficult to find a control that solves the problem of time-optimal control. An interesting problem is to steer an arbitrary point to a given point in a finite time with the restriction for a control. Return sets whose points are transferable into themselves after a period of time were studied in detail in [5]. A number of interesting researches dealt with the case of attaining to a stationary point. But here we will study the problem of the construction of $u(t)$. That is, we will solve the so-called constructive problem of controllability, which satisfies the given restriction and transfers an arbitrary point to a given non-equilibrium point in a finite time using the controllability function method proposed by V. I. Korobov in [7]. Using this method, we can estimate the time of transferring one point to another and construct a trajectory. In the manuscript [8], the controllability function method is applied to various control problems, including the construction of finite-time stabilizing positional control for wave equations and linear quadratic systems. The general approach to the construction of a control for arbitrary linear systems is presented in [9]. In [2] and [3], a family of feedback controls with controllability functions represents the exact time of movement either for a single or a multivariable feedback. In [4], for a family of nonlinear control systems, a bounded finite-time stabilizing control is attained. The controllability function can be applied to a non-equilibrium point of linear system (1.2) (see [11], [12]). The studying of the problem of the construction of a constrained control, which transfers a system from the initial point to a given non-equilibrium point in a finite time, was initiated in the paper [6]. In this paper, we are aimed to solve this problem for arbitrary linear systems and for nonlinear systems of the form (1.1), which can be reduced to linear systems. The results are illustrated on the model example. While solving the equation that defines the controllability function, we discuss nuances and features not having been met before.

Now, we will very briefly outline the basic idea of the controllability function method. Following this approach, let us introduce the controllability function $\Theta(x)$ ($\Theta(x) > 0$ at $x \neq 0$ and $\Theta(0) = 0$) for system (1.1) such that the differential inequality

$$\sum_{i=1}^{n} \frac{\partial \Theta(x)}{\partial x_i} f_i(x, u(x)) \leq -\varphi(\Theta(x)) \tag{1.5}$$

holds, where

$$\varphi(\Theta) > 0 \quad \text{at } \Theta \neq 0, \qquad \varphi(0) = 0$$

and

$$\int_0^a \frac{d\Theta}{\varphi(\Theta)} < \infty, \quad a > 0.$$

Inequality (1.5) means that the control $u(x) = \widetilde{u}(x, \Theta(x))$ is chosen such that the trajectory follows the direction of decrease of the function $\Theta(x)$. Due to the properties of the function $\varphi(\Theta)$, this inequality ensures that the trajectory targets the origin in a finite time. It is obvious that for $\varphi(\Theta) = \beta\Theta^{1-\frac{1}{\alpha}}$, inequality (1.5), having the form

$$\sum_{i=1}^{n} \frac{\partial \Theta(x)}{\partial x_i} f_i(x, u(x)) \leq -\beta\Theta^{1-\frac{1}{\alpha}}(x), \quad \beta > 0, \ \alpha > 0, \tag{1.6}$$

is valid. According to [7–9], the time of the motion satisfies the estimate

$$T(x_0) \leq \frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}(x_0).$$

In particular, of great interest is the case when it is possible to give a precise movement time. This ensures if

$$\sum_{i=1}^{n} \frac{\partial \Theta(x)}{\partial x_i} f_i(x, u(x)) = -1, \tag{1.7}$$

then $T(x_0) = \Theta(x_0)$, i.e., the controllability function is the time of movement. If, in addition, the control is such that

$$\min_{u\in\Omega} \sum_{i=1}^{n} \frac{\partial \Theta(x)}{\partial x_i} f_i(x, u(x)) = -1, \tag{1.8}$$

then the time is optimal (see [1]), and denoting $\omega(x) = -\Theta(x)$, we obtain the Bellman equation

$$\max_{u\in\Omega} \sum_{i=1}^{n} \frac{\partial \omega(x)}{\partial x_i} f_i(x, u(x)) = 1.$$

1.3. Article outline. In Section 2, we collect some propositions, remarks and definitions obtained earlier, which can help to solve the problem of the construction of a control which transfers a given point to a non-equilibrium point in a finite time and satisfies the preassigned constraint. The construction of the control is based on the controllability function method [7,8]. Then we apply the results of Section 2 to the model example of a linear system in Section 3. Finally, in Section 4, we make some conclusions for the further development of the obtained results.

2. Auxiliary constructions and assertions

In order to construct a control $u(t)$ which transfers an arbitrary point to a given non-equilibrium point for a class of the linear systems (1.2) in a finite time T and satisfies $\|u(t)\| \le d$ for all $t \in [0, T]$, we use Theorem 1 from [6] and slightly reformulate it.

Proposition 2.1. *Let the eigenvalues of the matrix A from (1.2) have non-positive real parts and the condition*

$$\operatorname{rank}(b, Ab, \dots, A^{n-1}b) = n$$

hold. Assume that T is one of positive solutions of the equation

$$2a_0 T - \left(N^{-1}(T) z(T), z(T)\right) = 0. \tag{2.1}$$

Here

$$0 < a_0 \le 2d^2 \min\left\{l,\, \gamma/\|b\|^2\right\}, \quad l > 0,\ \gamma > 0, \tag{2.2}$$

$N^{-1}(T)$ is the inverse matrix to the matrix function

$$N(T) = \int_0^T \left(1 - \frac{t}{T}\right) e^{-At} b b^* e^{-A^* t} dt, \quad T > 0, \tag{2.3}$$

and

$$z(T) = \left(x_0 - e^{-AT} x_T\right),$$

where $x_0 \in R^n$ *is a given initial point and* $x_T \in R^n$ *is a non-equilibrium point of system* (1.2).

Then the control $u(t)$ *of the form*

$$u(t) = -\frac{1}{2}\, b^* N^{-1}(T-t) z(t), \quad t \in [0, T], \tag{2.4}$$

transfers the point x_0 *to the point* x_T *in the finite time* T *along the trajectory*

$$x(t) = z(t) + e^{-A(T-t)} x_T, \tag{2.5}$$

where $z(t)$ *is the solution of the Cauchy problem*

$$\begin{cases} \dot{z} = \Big(A - \frac{1}{2} b b^* N^{-1}(T-t)\Big) z, \\ z(0) = x_0 - e^{-AT} x_T \end{cases} \tag{2.6}$$

and satisfies the preassigned constraint $\|u(t)\| \le d$ *for all* $t \in [0, T]$.

The proof of this proposition is carried out by using the controllability function method and is based on the results obtained in [9]. The proofs of the key points can be found in [6]. Here special attention is paid to the peculiarities of solving the synthesis problem of linear systems for a non-equilibrium point.

Remark 2.2. Firstly, it should be noted that for the canonical linear system, the function

$\Phi(\Theta, z) := 2a_0 \Theta - (N^{-1}(\Theta) z, z)$ is a polynomial with respect to Θ and z.

For the controllability function, the natural way to define it is the implicit form with using the controllability function method which is more preferable than the traditionally used explicit specification of the Lyapunov function. Note that if in differential inequality (1.6), $\alpha = \infty$, i.e., the finiteness of the attain time is not required and the target point is stationary, then the equation

$$\Phi(\Theta, z) := 2a_0 \Theta - (N^{-1}(\Theta) z, z) = 0, \quad z \in R^n \setminus \{0\}, \tag{2.7}$$

turns into the explicit equation $\Theta(z) = V(z)$, where $V(z)$ is the Lyapunov function for the closed system. If targeting occurs in a finite time, we get the Bellman equation.

Remark 2.3. It is obvious that equation (2.1) can be easily rewritten in the form of (2.7) at $\Theta = T$, $z = x_0 - e^{-AT} x_T$.

Remark 2.4. As is well known (see, for example, [7]), in the problem of targeting a stationary point for linear systems, equation (2.7) has only one positive solution $\Theta(z)$. In [6], the authors assumed that for any positive number a_0 determined from condition (2.2), the positive root of $\Phi(\Theta, z) = 0$ is also unique. But we discovered the appearance of several positive roots of equation (2.7) (or (2.1) which is the same) and this fact motivated us to study new problems related to the construction of an unstable synthesis and prompted the writing of this paper.

Let us recall some definitions.

Definition 2.5. An equilibrium solution x_e to an autonomous system of a first-order ordinary differential equation is called stable if for each $\epsilon > 0$, there exists $\delta > 0$ such that for any solution $x(t)$ satisfying $\|x(t_0) - x_e\| < \delta$, we have $\|x(t) - x_e\| < \epsilon$ for all $t \geq t_0$.

Definition 2.6. We are talking about an unstable synthesis if there exists $\epsilon > 0$ such that for every $\delta > 0$ there exists a solution $x(t)$ such that $\|x(t_0) - x_e\| < \delta$ and there exists $t \geq t_0$ such that we have $\|x(t) - x_e\| > \epsilon$.

The above proposition guaranties that $\Theta(\cdot)$ is the time of motion from x_0 to x_T. Note that solving the synthesis problem, we have to solve a nonlinear equation. In addition, solving an implicit equation that determines one of the phase coordinates makes the situation significantly more complicated. The function Θ defined by equation (2.7) is a solution of the partial differential equation (1.7). In the case of time-optimal control, the solution to equation (1.8) is also determined by an equation similar to (2.7). Thus, we do not directly solve (1.7) since searching a solution to the partial differential equation is reduced to finding the controllability function or the moment minproblem, as a result of which Θ is defined implicitly. The described here method for finding a trajectory does not require an explicit form of Θ.

Briefly, finding the trajectory leading from x_0 to x_T is reduced to the following steps: calculating the derivative of the function Θ along a trajectory of the original system; and, as a result of the first step, we arrive at a system of equations, the order of which is increased by 1; the searched trajectory is the solution of the obtained Cauchy problem. All steps are illustrated in detail by an example (see Section 3).

Remark 2.7. In equation (2.7), a number a_0 can be chosen such that the control satisfies the preassigned constraint. It is seen that the less is d, the less is a_0. The general result for finding a_0 is given in [9]. In the case of the canonical system, the result has the form:

Proposition 2.8 ([6, Corollary 1])**.** *Let $x_0 \in R^n$ be an arbitrary point and let $x_T \in R^n$ be a non-equilibrium point of system (1.2), where*

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Assume that a number a_0 satisfies the condition

$$0 < a_0 \leq \frac{2d^2}{(N^{-1}(T)b, b)}, \tag{2.8}$$

and $T = T(x_0, x_T)$ is a positive root of equation (2.1).

Then the control defined by (2.4)*, where* $z(t)$ *is the solution of the Cauchy problem* (2.6)*, transfers the point* x_0 *to the point* x_T *in the time* T *along the trajectory defined by* (2.5) *and satisfies the preassigned constraint*

$$|u(t)| \leq d \quad \textit{for all } t \in [0, T].$$

3. The model example

Consider the problem of the construction of a restricted control which transfers the point $x_0 = (x_{10}, x_{20})^T$ to the non-equilibrium point $x_T = (x_{11}, x_{21})^T$ along the trajectory $x(t)$ of the two-dimensional system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases}, \quad \|u\| \leq 1. \tag{3.1}$$

Let us find in this case the matrices $N(\Theta)$, $N^{-1}(\Theta)$:

$$N(\Theta) = \int_0^\Theta \left(1 - \frac{t}{\Theta}\right) \begin{pmatrix} t^2 & -t \\ -t & 1 \end{pmatrix} dt = \begin{pmatrix} \frac{\Theta^3}{12} & -\frac{\Theta^2}{6} \\ -\frac{\Theta^2}{6} & \frac{\Theta}{2} \end{pmatrix},$$

$$N^{-1}(\Theta) = \begin{pmatrix} \frac{36}{\Theta^3} & \frac{12}{\Theta^2} \\ \frac{12}{\Theta^2} & \frac{6}{\Theta} \end{pmatrix}.$$

As mentioned above, the equation for determining the controllability function $\Theta(z)$ for all $z \neq 0$ has the form (2.7), where $a_0 = \frac{1}{3}$, which follows from condition (2.8).

That is, the controllability function is defined as a positive solution of the equation, which takes the form

$$\frac{2}{3}\Theta^4 - 6\Theta^2 z_2^2 - 24\Theta z_1 z_2 - 36 z_1^2 = 0. \tag{3.2}$$

The control solving the main problem of synthesis and satisfying the given constraint has the form

$$\tilde{u}(z) = -\frac{6z_1}{\Theta^2(z)} - \frac{3z_2}{\Theta(z)}, \tag{3.3}$$

where $z = (z_1, z_2)^T$. The control transfers an arbitrary point $(x_0 - e^{-AT}x_T) \in R^2$ to the origin in the time $\Theta(x_0 - e^{-AT}x_T)$ and satisfies the restriction $\|u(z)\| \leq 1$. However, as T is still unknown, then $x_0 - e^{-AT}x_T$ is also unknown, and hence the time is to be found. Equation (2.1) takes the form

$$\frac{2}{3}T^4 - p_1 T^2 - p_2 T - p_3 = 0, \tag{3.4}$$

where

$$p_1 = 6(x_{20} - x_{2k})(x_{20} + 3x_{2k}) + 36x_{2k}^2,$$
$$p_2 = 24(x_{10} - x_{1k})(x_{20} + 2x_{2k}),$$

$$p_3 = 36(x_{10} + x_{1k})^2.$$

Let T be the only positive root or one of positive roots of equation (3.4). Then

$$x_0 - e^{-AT}x_T = \begin{pmatrix} x_{10}-x_{1k} + \Theta x_{2k} \\ x_{10}-x_{2k} \end{pmatrix} := \begin{pmatrix} z_{10} \\ z_{20} \end{pmatrix}. \tag{3.5}$$

The Cauchy problem (2.6) takes the form

$$\begin{cases} \dot{z}_1 = z_2, \\ \dot{z}_2 = -\dfrac{6z_1}{(T-t)^2} - \dfrac{3z_2}{T-t}, \\ z_1(0) = z_{10}, \quad z_2(0) = z_{20}. \end{cases} \tag{3.6}$$

Then the control $u(t)$, which transfers the point $x_0 - e^{-AT}x_T$ to the origin, has the form

$$u(t) = -\frac{6z_1(t)}{(T-t)^2} - \frac{3z_2(t)}{(T-t)}. \tag{3.7}$$

In order to obtain an analytical solution, we reduce the system of differential equations to an equation of Euler type. For this purpose, we proceed to the second-order differential equation by virtue of system (3.6):

$$\ddot{z}_1 = \dot{z}_2 = -\frac{6z_1}{(T-t)^2} - \frac{3z_2}{T-t} = -\frac{6z_1}{(T-t)^2} - \frac{3\dot{z}_1}{T-t}, \tag{3.8}$$

i.e.,

$$\ddot{z}_1 + \frac{3\dot{z}_1}{T-t} + \frac{6z_1}{(T-t)^2} = 0.$$

Put

$$z_1(t) = (T-t)^\lambda. \tag{3.9}$$

Then

$$\dot{z}_1(t) = -\lambda(T-t)^{\lambda-1} \tag{3.10}$$

and we have

$$\ddot{z}_1 = \lambda(\lambda-1)(T-t)^{\lambda-2}. \tag{3.11}$$

Let us substitute (3.9)–(3.11) into equation (3.8) to obtain

$$\lambda^2 - 4\lambda + 6 = 0.$$

Thus, we get

$$z_1(t) = e^{2t}(C_1 \sin\sqrt{2}t + C_2 \cos\sqrt{2}t), \tag{3.12}$$

$$z_2(t) = e^{2t}\left((2C_1 - \sqrt{2}C_2)\sin\sqrt{2}t + (\sqrt{2}C_1 + 2C_2)\cos\sqrt{2}t\right), \tag{3.13}$$

where $C_i, i = \overline{1,2}$ are the constants determined from the Cauchy condition. Using (2.5), we obtain the trajectory

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} (1-T+t) + z_1(t) \\ 1 + z_2(t) \end{pmatrix} \tag{3.14}$$

along which the control (3.7) transfers the point x_0 to the point x_T in a finite time T.

Remark 3.1. It should be noted that the trajectory (3.14) is constructed in an explicit form and, in addition, the control (3.7) after substitution of equalities (3.12), (3.13) is obtained in an explicit form.

In order to illustrate the possibilities of applying the theory from Section 2 to various problems and to show the peculiarities of this approach, we plot the graph of the phase trajectory and the graph of the control leading from any initial point x_0 to the given point x_T.

Thus we are to consider a few cases for the linear system (3.1):

Case (i): $x_0 = (1; 1), \qquad x_T = (1; 1);$

Case (ii): $x_0 = (0; 0), \qquad x_T = (0; 0);$

Case (iii): $x_0 = (0.1; 1), \quad x_T = (1; 1).$

Case (i). Let us consider the trajectory going from the point $(1; 1)$ and coming back to the same point according to the algorithm described earlier. Equation (3.4) takes the form

$$\frac{2}{3}T^4 - 36T^2 = 0,$$

and

$$T = T(x_0, x_T) = 7.348469228349534`$$

is the unique positive solution of this equation. The phase trajectory of the system transferring a point into itself in the time $T = 7.348469228349534`$ and the control are shown in Figures 3.1 and 3.2, respectively.

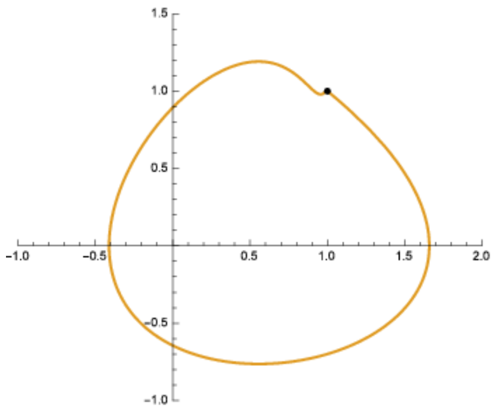

Fig. 3.1: The phase trajectory of system (3.1) on the plane x_1Ox_2.

Case (ii). Unlike in the previous case, the construction of the trajectory going from the origin and coming back to the origin is split into two stages. Since we are searching for a nontrivial trajectory, we choose an arbitrary point to which we will go from the origin. It is obvious that the hit time depends on the chosen point. For definiteness, we set it (1; 1). That is, we move from (0; 0) to (1; 1),

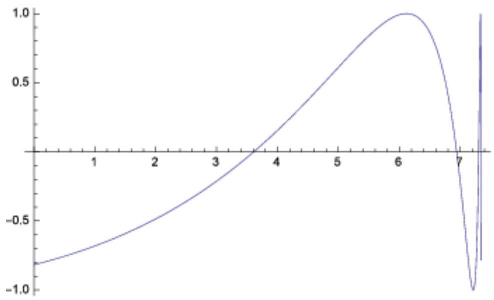

Fig. 3.2: The graph of the control on the trajectory.

then from (1; 1) to (0; 0). Equation (3.4) for the first stage takes the form

$$\frac{2}{3}T^4 - 18T^2 + 48T - 36 = 0,$$

and

$$T = T(x_0, x_T) = 3$$

is the unique positive solution of this equation. Thus, the control

$$u(t) = -\frac{6e^{2t}\left(-\frac{5}{\sqrt{2}}\sin\sqrt{2}t + 2\cos\sqrt{2}t\right)}{(3-t)^2} - \frac{3e^{2t}\left(\frac{4-10\sqrt{2}}{2}\sin\sqrt{2}t - \cos\sqrt{2}t\right)}{(3-t)}$$

transfers the point x_0 to the point x_T in the time $T = 3$ along the trajectory (3.14) of the form

$$x(t) = \begin{pmatrix} t-2+e^{2t}\left(-\frac{5}{\sqrt{2}}\sin\sqrt{2}t + 2\cos\sqrt{2}t\right) \\ 1+e^{2t}\left(\frac{4-10\sqrt{2}}{2}\sin\sqrt{2}t - \cos\sqrt{2}t\right) \end{pmatrix}.$$

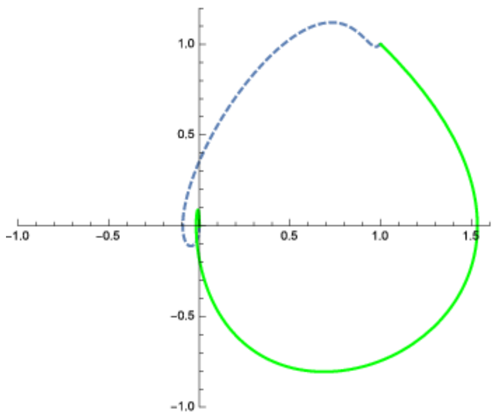

Fig. 3.3: The combined phase trajectories of system (3.1) on the plane x_1Ox_2

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Absolutely similarly is constructed the trajectory from point (1; 1) to point (0; 0). We only note here that the unique positive solution of the characteristic equation for **Case (ii)**. from (1; 1) to (0; 0) is $T = 4.451198623210905`$. In Figure 3.3, there are the graphs of the trajectories for two stages, where a dashed line is the phase trajectory of system (3.1) on the plane x_1Ox_2 from (0; 0) to (1; 1); a solid line is the phase trajectory from (1; 1) to (0; 0).

In **Case (iii)**, equation (3.4) takes the form

$$\frac{2}{3}T^4 - 36(T - 0.9)^2 = 0. \tag{3.15}$$

Somewhat unexpected and interesting is the fact of the appearance of three different positive roots of the characteristic equation, while in all the examples considered earlier (including outside the scope of this paper) the solution was unique. So, there are three different positive roots:

$$T_1 = 6.298425478895819`,$$
$$T_2 = 0.8105867095252739`,$$
$$T_3 = 1.0500437494537151`.$$

All the corresponding trajectories and controls are found similarly to the previously considered cases for (3.5)–(3.14). Below all the trajectories are combined in one figure and we get the shape resembling a penguin.

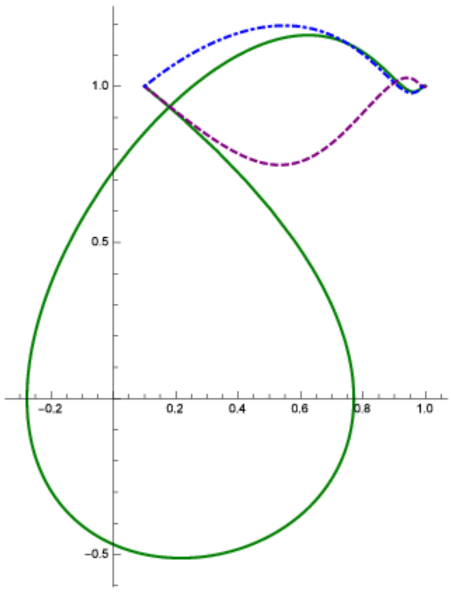

Fig. 3.4: The combined phase trajectories of system (3.1) on the plane x_1Ox_2.

In Figure 3.4, there are the graphs of the trajectories which correspond to different positive roots: a solid line is the trajectory at T_1, a dashed dotted line is the trajectory at T_2 and a dashed line is the trajectory at T_3.

Remark 3.2. In this example, there are already three positive roots of the equation that defines the controllability function, since the left side of equation (3.15) (or (2.1) or (2.7), or (3.4) which are the same) is multi-valued. So, the problem of choosing a positive root arises. Exactly, the problem of choosing a more preferable positive root. After making the choice, we "discard" the remaining positive roots and the corresponding trajectories. For definiteness, we may focus on searching, for example, the minimum root each time.

Remark 3.3. Let us take various starting points from the area that contains (0,1; 1), and we target them to the given non-equilibrium endpoint (1; 1). The result of the constructing the trajectories is shown in Figure 3.5. This example for the canonical system shows that the curves may intersect (see the top of the figure) and may have self-intersection (as, for example, in the case of the trajectory drawn by a solid line).

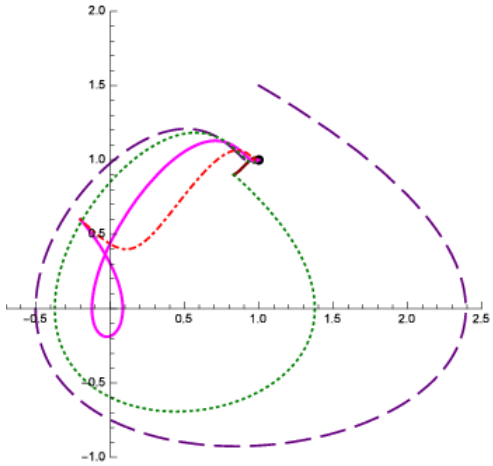

Fig. 3.5: The combined phase trajectories of system (3.1) on the plane x_1Ox_2 from the area that contains (0.1; 1) to (1; 1).

Remark 3.4. In addition, we found the equation of the curve dividing the whole area into parts, where equation (3.15) (or (2.1) or (2.7), or (3.4) which are the same) has one root, two roots, three roots and multiple roots. The graph of this curve (having the shape of butterfly wings) and its analytical form are seen in Figure 3.6.

Solving the equation for $x_1 = 1$, we find one positive root and the zero root of multiplicity 2 (we discard it, because only positive roots are of interest). The straight line $x_1 = 1$ divides the picture into two parts. When we solve the equation, a pair of complex conjugate roots and one positive root appear on the left and on the right of $x_1 = 1$. Moreover, if $\operatorname{Re} z > 0$ is on the left, then $\operatorname{Re} z < 0$ is on the right of $x_1 = 1$. Note that closer to the border (butterfly shape) from either side $\operatorname{Im} z \to 0$ takes place. We can observe an interesting antisymmetry. For example, if inside the "wing", on the left of $x_1 = 1$, while solving the equation, we obtain three different positive roots and one negative, then the situation on the right is exactly opposite (i.e., only one positive and three negative roots).

Now let us consider the most interesting area of Figure 3.7.

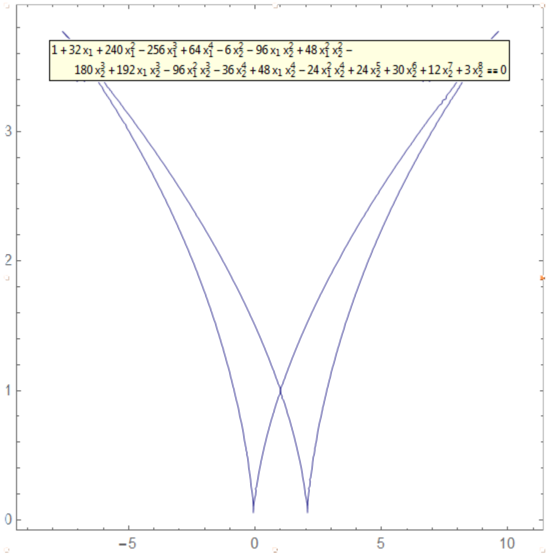

Fig. 3.6: The graph consists of two curvilinear triangles symmetrical with respect to the straight line $x_1 = 1$ and has the shape of butterfly wings

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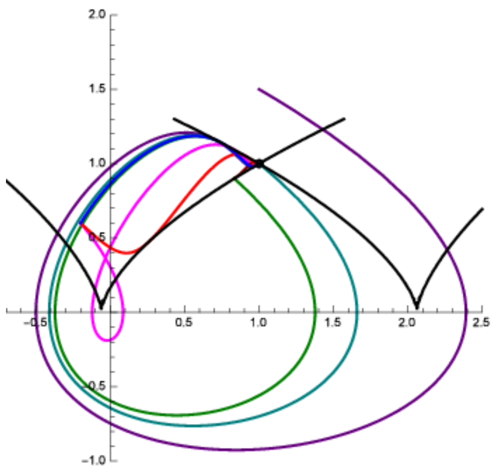

Fig. 3.7: Overlay of the graphs of phase trajectories of system (3.1) and enlarged fragment of Figure 3.6

Inside the curvilinear triangle (the left wing of a butterfly) shown in Figure 3.8, equation (3.15) has three different positive roots, two different positive roots on its borders, and one positive root outside the triangle.

4. Conclusion

The results of Section 2 can be developed for arbitrary linear systems and for nonlinear systems of the form (1.1), which can be reduced to linear systems. Recall that solving a problem of synthesis requires solving a nonlinear equation.

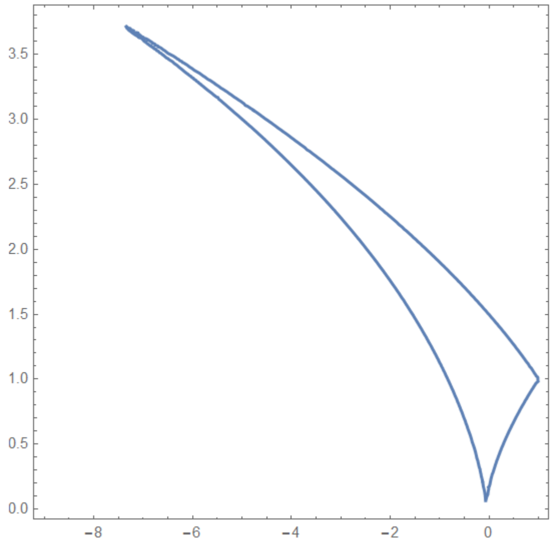

Fig. 3.8: Part of the Figure 3.6

In addition, the search of a solution to the implicit equation that defines one of the phase coordinates is more complicated. In the present paper, multiple roots determine the controllability function and cause the appearance of two, three and more trajectories that transfer a control system from the initial point to a given non-equilibrium point in a finite time. It essentially differs from getting to a stationary point. The ambiguity of solutions of the equation that determines the controllability function leads to a number of interesting cases related to the multi-valued synthesis, which require additional research.

Acknowledgment. The authors are very grateful to Larissa Fardigola for valuable comments.

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Received November 9, 2020.

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Особливість розв'язання проблеми синтезу в точку нерівноваги для лінійних систем

Valeriy Korobov and Kateryna Stiepanova

Статтю присвячено проблемі побудови керування, яке переводить лінійну систему з довільної точки в задану точку нерівноваги за скінченний час. Побудова цього керування основана на методі функції керованості. Виявлено неоднозначність розв'язку рівняння, яке визначає функцію керованості, що призводить до ряду цікавих питань. Проблему розв'язано для лінійних систем. Одержані результати проілюстровано модельними прикладами.

Ключові слова: метод функції керованості, лінійна система, точка нерівноваги