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Stability of Complex Functional Equations in 2-Banach Spaces

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In the paper, we obtain some results for the Hyers–Ulam stability of the following functional equations:

$$\mathsf{q}(x+iy)+\mathsf{q}(x-iy)+\mathsf{q}(y+ix)+\mathsf{q}(y-ix)=2\mathsf{q}(x)+2\mathsf{q}(y)$$

and

$$\mathsf{q}(x+iy)+\mathsf{q}(x-iy)+\mathsf{q}(y+ix)+\mathsf{q}(y-ix)=0$$

in the setting of 2-Banach spaces.

Key words: 2-normed spaces, 2-Banach space, Hyers–Ulam–Rassias stability, additive mapping, quadratic equation

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1. Introduction

The problem of the stability of functional equation was motivated by a classical question of Ulam [51] put in 1940,'When is it true that the solution of an equation differing slightly from a given one must of necessity be close to the solution of the given equation?'

If the problem accepts a solution, we can say that the given equation is stable. Ulam was the first to raise the stability problem of group homomorphisms.

Let G and (H,d) be a group and a metric group respectively. Given a real number $\varepsilon>0$. Does there exist a positive real number δ such that if $f:G\to H$ satisfies the inequality

$$d\left[f(x,y),f(x)f(y)\right]<\delta$$

for all $x,y\in G$, then there exists a homomorphism $F:G\to H$ with

$$d\left[f(x),F(x)\right]<\varepsilon$$

for all $x\in G$?

The first affirmative partial answer to Ulam's question was given by Hyers [23] in 1941. Ulam's question and Hyers' result became the basis for the so-called stability theory of functional equations in Hyers–Ulam sense. In 1978, Rassias [44] provided a generalization of Hyers's theorem which allows the Cauchy difference to be unbounded.

In 1990, Rassias during the 27^{th} International Symposium on Functional Equations asked the question whether the theorem holds for $p \geq 1$. In 1991, Gajda [19] answered Rassias' question provided an affirmative solution for $p > 1$ in view of his result by defining the formula as

$$T(x) := \begin{cases} \lim_{n\to\infty} \frac{1}{2^n} f(2^n x) & \text{if } p < 1, \\ \lim_{n\to\infty} 2^n f(\frac{x}{2^n}) & \text{if } p > 1. \end{cases}$$

It was proved by Gajda [19], as well as Rassias et al. [45], that one cannot prove the Rassias type theorem when $p = 1$. In 1994, Găvruta [20] provided a further generalization of Rassias' theorem in which he replaced the bound by a general control function $\phi(x, y)$ for the existence of a unique linear mapping.

A popular basic equation in the theory of functional equations is the Cauchy functional equation

$$\mathsf{q}(x + y) = \mathsf{q}(x) + \mathsf{q}(y). \tag{1.1}$$

In addition to this equation, its three sisters

$$\begin{aligned} \mathsf{q}(x + y) &= \mathsf{q}(xy), \\ \mathsf{q}(xy) &= \mathsf{q}(x) + \mathsf{q}(y), \\ \mathsf{q}(xy) &= \mathsf{q}(x)\mathsf{q}(y) \end{aligned}$$

were introduced by Cauchy (see [10]). Cauchy carefully analyzed equation (1.1) under the assumptions that the unknown function q is a continuous function from $\mathbb{R}$ to $\mathbb{R}$ and the variables x and y are arbitrary real numbers.

A common path of studying (1.1) is to impose various types of "regularity" conditions on the unknown function. It turns out that in the specific case, where $f : \mathbb{R} \to \mathbb{R}$, each of these conditions implies the existence of some $c \in \mathbb{R}$ such that $\mathsf{q}(x) = cx$ for all $x \in \mathbb{R}$, and this fact has been proved in various ways. For example, Cauchy [10] assumed that q is continuous, Darboux showed that q may be either monotone [14] or bounded on an interval [15]. Fréchet [17], Blumberg [8], Banach [6], Sierpiński [46, 47], Kac [28], Alexiwicz–Orlicz [5], and Figiel [16] assumed that q is Lebesgue measurable. Ostrowski [41] and Kestelman [31] assumed that q is bounded from one side on a measurable set of positive measure. Mehdi [39] assumed that q is bounded above on a second category Baire set. In 1905, Hamel [22] introduced a Hamel basis and showed that there are nonlinear solutions to (1.1).

More studies and applications of equation (1.1) can be found in the books of Aczél [4], Aczél–Dhombres [3], Czerwik [49], Járai [25], Kuzma [32] and Kannappan [29].

Similarly, the functional equation

$$\mathsf{q}(x + y) + \mathsf{q}(x - y) = 2\mathsf{q}(x) + 2\mathsf{q}(y) \tag{1.2}$$

is called a quadratic functional equation. It is easy to see that the quadratic function $\mathsf{q}(x) = x^2$ is a solution of the quadratic functional equation. A mapping $q : P_1 \to P_2$ is called quadratic if q satisfies the quadratic functional equation

$$\mathsf{q}(x+y) + \mathsf{q}(x-y) = 2\mathsf{q}(x) + 2\mathsf{q}(y)$$

for all $x, y \in P_1$. F. Skof [48] was the first author who studied the generalized Hyers–Ulam stability of the quadratic functional equation. Cholewa [12] found that the result of F. Skof [48] is still valid if a domain normed space is replaced by an Abelian group. Czerwik [13] further generalized Skof's result.

Kannappan [30] solved the following functional equation:

$$\mathsf{q}(x+y+z) + \mathsf{q}(x) + \mathsf{q}(y) + \mathsf{q}(z) = \mathsf{q}(x+y) + \mathsf{q}(y+z) + \mathsf{q}(z+x) \tag{1.3}$$

and proved that a function on a real vector space is a solution of (1.3) if and only if there exists a symmetric biadditive function P and an additive function R such that $\mathsf{q}(x) = P(x,x) + R(x)$ for any x.

Jung [26] proved the Hyers–Ulam–Rassias stability of the quadratic equation of a new type

$$\mathsf{q}(x-y-z) + \mathsf{q}(x) + \mathsf{q}(y) + \mathsf{q}(z) = \mathsf{q}(x-y) + \mathsf{q}(y+z) + \mathsf{q}(z-x).$$

Thereafter, many authors studied stability problems of this type of equation (see [24, 34, 42, 50]).

During the last four decades, many results concerning the Hyers-Ulam stability of important functional equations have been obtained by several mathematicians (see [1, 2, 27, 35, 40, 43] and references therein).

In this paper, we discuss the Hyers–Ulam stability of the following additive functional equation:

$$\mathsf{q}(x+iy) + \mathsf{q}(x-iy) + \mathsf{q}(y+ix) + \mathsf{q}(y-ix) = 2\mathsf{q}(x) + 2\mathsf{q}(y), \tag{1.4}$$

where $\mathsf{q}((1+i)x) = (1+i)\mathsf{q}(x)$, whose solution is an additive mapping and the Hyers–Ulam stability of the quadratic functional equation

$$\mathsf{q}(x+iy) + \mathsf{q}(x-iy) + \mathsf{q}(y+ix) + \mathsf{q}(y-ix) = 0, \tag{1.5}$$

where $\mathsf{q}((1+i)x) = 2i\mathsf{q}(x)$, whose solution is a quadratic mapping.

So, in this paper, following equations (1.1) and (1.2), we consider equations (1.4) and (1.5) in a complex plane. We also prove that both equations (1.4) and (1.5) can be reduced to equations (1.1) and (1.2). Equation (1.1) can be reduced to equation (1.4) by assuming $\mathsf{q}(ix) = i\mathsf{q}(x)$ (see Proposition 2.7) and in a similar way we can reduce (1.2) to equation (1.5) by keeping the assumption of $\mathsf{q}(ix) = -\mathsf{q}(x)$ (see Proposition 2.9). Thus, by using the concept of Găvruta, we investigate some stability problems for a complex additive type functional equation and a complex quadratic type functional equation by considering a complex 2-normed space as a domain and a complex 2-Banach space as a co-domain.

2. Preliminaries

In this section, we give some basic definitions and results to be used in the sequel.

The concept of a linear 2-normed space was introduced by Gähler [18] defined as follows:

Definition 2.1. Let X be a real linear space of dimension greater than 1 and $\|\cdot,\cdot\| : X \times X \to [0,\infty)$ be a function satisfying the following four conditions:

(N_1) $\|x,y\| = 0$ if and only if x and y are linearly dependent in X;

(N_2) $\|x,y\| = \|y,x\|$;

(N_3) $\|x,\alpha y\| = |\alpha|\|x,y\|$;

(N_4) $\|x,y+z\| \leq \|x,y\| + \|x,z\|$

for all $x,y,z \in X$ and $\alpha \in \mathbb{R}$. Then the function $\|\cdot,\cdot\|$ is called a 2-norm on X and the pair $(X,\|\cdot,\cdot\|)$ is a called a 2-normed space.

Example 2.2. Let $X = \mathbb{R}^2$ and $\|\cdot,\cdot\| : X \times X \to \mathbb{R}_+$ be defined by

$$\|x,y\| = |x_1y_2 - x_2y_1|$$

for all $x = (x_1,x_2)$, $y = (y_1,y_2) \in \mathbb{R}^2$. Then the function $\|\cdot,\cdot\|$ is a 2-norm on $\mathbb{R}^2$.

Proof. For all $x,y,z \in X$, we have.

1. Here $\|x,y\| = 0$ if and only if $|x_1y_2 - x_2y_1| = 0$ implies that x and y are linearly dependent for all $x = (x_1,x_2), y = (y_1,y_2) \in \mathbb{R}^2$.

2. It is easy to see that $\|x,y\| = \|y,x\|$.

3. For some $\alpha \in \mathbb{R}$, we have

$$\|\alpha x,y\| = |\alpha x_1y_2 - \alpha x_2y_1| = |\alpha||x_1y_2 - x_2y_1| = |\alpha|\,\|x,y\|\,.$$

4. Consider

$$\begin{aligned}\|x,y+z\| &= |x_1(y_2+z_2) - x_2(y_1+z_1)| = |x_1y_2 + x_1z_2 - x_2y_1 - x_2z_1| \\ &= |x_1y_2 - x_2y_1 + x_1z_2 - x_2z_1| \leq |x_1y_2 - x_2y_1| + |x_1z_2 - x_2z_1| \\ &= \|x,y\| + \|x,z\|\end{aligned}$$

for all $x = (x_1,x_2)$, $y = (y_1,y_2)$, $z = (z_1,z_2) \in \mathbb{R}^2$. Hence, $(X,\|\cdot,\cdot\|)$ is a linear 2-normed space. □

Example 2.3. Let $X = \mathbb{R}^3$ and consider the following 2-norm on X:

$$\|x,y\| = \left|\det \begin{bmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}\right|,$$

where $x = (x_1,x_2,x_3)$ and $y = (y_1,y_2,y_3)$. Then $(X,\|\cdot,\cdot\|)$ is a 2-normed space.

Proof. For all $x, y, z \in X$, we have.

1. Here $\|x, y\| = 0$ if and only if $\left|\det \begin{bmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}\right| = 0$ implies that x and y are linear dependent for all $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in \mathbb{R}^3$.

2. It can be easily verified that

$$\begin{aligned} \|x, y\| &= \left|\det \begin{bmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}\right| \\ &= |i(x_2y_3 - x_3y_2) - j(x_1y_3 - x_3y_1) + k(x_1y_2 - x_2y_1)| = \|y, x\| \end{aligned}$$

and therefore $\|x, y\| = \|y, x\|$.

3. For some $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} \|\alpha x, y\| &= \left|\det \begin{bmatrix} i & j & k \\ \alpha x_1 & \alpha x_2 & \alpha x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}\right| \\ &= |i(\alpha x_2y_3 - \alpha x_3y_2) - j(\alpha x_1y_3 - \alpha x_3y_1) + k(\alpha x_1y_2 - \alpha x_2y_1)| \\ &= |(\alpha(i(x_2.y_3 - x_3y_2) - j(x_1y_3 - x_3y_1) + k(x_1y_2 - x_2y_1))| \\ &= |\alpha||(i(x_2y_3 - x_3y_2) - j(x_1y_3 - x_3y_1) + k(x_1y_2 - x_2y_1)| = |\alpha|\|x, y\|. \end{aligned}$$

4. To prove $\|x, y + z\| \le \|x, y\| + \|x, z\|$, consider

$$\begin{aligned} \|x, y + z\| &= \left|\det \begin{bmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 + z_1 & y_2 + z_2 & y_3 + z_3 \end{bmatrix}\right| \\ &= |i((x_2(y_3 + z_3) - x_3(y_2 + z_2)) - j((x_1(y_3 + z_3) - x_3(y_1 + z_1)) \\ &\quad + k((x_1(y_2 + z_2) - x_2(y_1 + z_1))| \\ &= |i(x_2y_3 + x_2z_3 - x_3y_2 - x_3z_2) - j(x_1y_3 + x_1z_3 - x_3y_1 - x_3z_1) \\ &\quad + k(x_1y_2 + x_1z_2 - x_2y_1 - x_2z_1)| \\ &= |i(x_2y_3 - x_3y_2) + i(x_2z_3 - x_3z_2) - j(x_1y_3 - x_3y_1) - j(x_1z_3 - x_3z_1) \\ &\quad + k(x_1y_2 - x_2y_1) + k(x_1z_2 - x_2z_1)| \\ &= |i(x_2y_3 - x_3y_2) - j(x_1y_3 - x_3y_1) + k(x_1y_2 - x_2y_1) \\ &\quad + i(x_2z_3 - x_3z_2) - j(x_1z_3 - x_3z_1) + k(x_1z_2 - x_2z_1)| \\ &\le |i(x_2y_3 - x_3y_2) - j(x_1y_3 - x_3y_1) + k(x_1y_2 - x_2y_1)| \\ &\quad + |i(x_2z_3 - x_3z_2) - j(x_1z_3 - x_3z_1) + k(x_1z_2 - x_2z_1)| \\ &= \|x, y\| + \|x, z\| \end{aligned}$$

for all $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, $z = (z_1, z_2, z_3) \in \mathbb{R}^3$. Hence, $(X, \|\cdot, \cdot\|)$ is a linear 2-normed space. □

Definition 2.4. A sequence $\{x_n\}$ in a linear 2-normed space $(X, \|\cdot, \cdot\|)$ is called a Cauchy sequence if

$$\lim_{m,n\to\infty} \|x_m - x_n, y\| = 0 \quad \text{for every } y \in X.$$

Definition 2.5. (see [21]) A sequence $\{x_n\}$ in a linear 2-normed space $(X, \|\cdot,\cdot\|)$ is said to converge to $x \in X$ if

$$\lim_{n\to\infty} \|x_n - x, y\| = 0 \quad \text{for all } y \in X.$$

Definition 2.6. A linear 2-normed space $(X, \|\cdot,\cdot\|)$ is called a 2-Banach space if every Cauchy sequence in X is convergent.

The study of 2-normed spaces and 2-metric spaces have been developed extensively by many authors (see [7, 11, 36–38, 52, 53] and references therein).

Proposition 2.7. *Let X and Y be vector spaces. A function $\mathsf{q} : X \to Y$ satisfies*

$$\mathsf{q}(x+iy) + \mathsf{q}(x-iy) + \mathsf{q}(y+ix) + \mathsf{q}(y-ix) = 2\mathsf{q}(x) + 2\mathsf{q}(y) \tag{2.1}$$

for all $x, y \in X$. Then $\mathsf{q} : X \to Y$ is additive, i.e.,

$$\mathsf{q}(x+y) = \mathsf{q}(x) + \mathsf{q}(y) \quad \textit{for all } x, y \in X.$$

If a mapping $\mathsf{q} : X \to Y$ is additive and $\mathsf{q}(ix) = i\mathsf{q}(x)$ holds for all $x \in X$, then the mapping $\mathsf{q} : X \to Y$ satisfies (2.1).

Proof. Let $\mathsf{q} : X \to Y$ satisfy (2.1). Putting $x = 0$ in (2.1), we have

$$\begin{aligned}
&\mathsf{q}(iy) + \mathsf{q}(-iy) + \mathsf{q}(y) + \mathsf{q}(y) = 2\mathsf{q}(y),\\
&\mathsf{q}(iy) + \mathsf{q}(-iy) = 0,\\
&\mathsf{q}(iy) + \mathsf{q}(-iy) = \mathsf{q}(0),\\
&\mathsf{q}(iy) + \mathsf{q}(-iy) = \mathsf{q}(iy + (-iy)).
\end{aligned}$$

Take $iy = x$ and $-iy = y$. Then we have

$$\mathsf{q}(x+y) = \mathsf{q}(x) + \mathsf{q}(y)$$

for all $x, y \in X$.

Conversely, if a mapping $\mathsf{q} : X \to Y$ is additive and $\mathsf{q}(ix) = i\mathsf{q}(x)$ holds for all $x \in X$, then for all $x, y \in X$, we have

$$\begin{aligned}
&\mathsf{q}(x+iy) + \mathsf{q}(x-iy) + \mathsf{q}(y+ix) + \mathsf{q}(y-ix) = 2\mathsf{q}(x) + 2\mathsf{q}(y),\\
&\mathsf{q}(x) + \mathsf{q}(iy) + \mathsf{q}(x) + \mathsf{q}(-iy) + \mathsf{q}(y) + \mathsf{q}(ix) + \mathsf{q}(y) + \mathsf{q}(-ix) = 2\mathsf{q}(x) + 2\mathsf{q}(y),\\
&\mathsf{q}(x) + i\mathsf{q}(y) + \mathsf{q}(x) - i\mathsf{q}(y) + \mathsf{q}(y) + i\mathsf{q}(x) + \mathsf{q}(y) - i\mathsf{q}(x) = 2\mathsf{q}(x) + 2\mathsf{q}(y),\\
&2\mathsf{q}(x) + 2\mathsf{q}(y) = 2\mathsf{q}(x) + 2\mathsf{q}(y).
\end{aligned}$$

Therefore $\mathsf{q} : X \to Y$ is additive. □

If a mapping $\mathsf{q} : X \to Y$ satisfies Cauchy's functional equation

$$\mathsf{q}\,(x+y) = \mathsf{q}(x) + \mathsf{q}(y),$$

and $\mathsf{q}\,(ix) = i\mathsf{q}\,(x)$ for all $x, y \in X$, then we have

$$\mathsf{q}\,(x+iy) + \mathsf{q}\,(x-iy) + \mathsf{q}\,(y+ix) + \mathsf{q}\,(y-ix) = 2\mathsf{q}\,(x) + 2\mathsf{q}\,(y) \tag{2.2}$$

and $\mathsf{q}((1+i)x) = (1+i)\mathsf{q}(x)$ for all $x, y \in X$.

Remark 2.8. Note that the following assertions are true.

(a) Since $\mathsf{q}(ix) = i\mathsf{q}(x)$, then $\mathsf{q}(-x) = \mathsf{q}(i^2x) = i\mathsf{q}(ix) = i^2\mathsf{q}(x) = -\mathsf{q}(x)$.

(b) If $x = y$, then using (a) it is easy to see that equation (2.2) is satisfied.

(c) If $x \neq 0$ and $y = 0$, then using (a) it is easy to see that equation (2.2) is satisfied.

(d) Similarly, if $x = 0$, $y \neq 0$, then using (a) it is easy to see that equation (2.2) is satisfied.

Proposition 2.9. *Let X and Y be vector spaces. If a function $\mathsf{q} : X \to Y$ satisfies*

$$\mathsf{q}(x + iy) + \mathsf{q}(x - iy) + \mathsf{q}(y + ix) + \mathsf{q}(y - ix) = 0 \quad \textit{for all } x, y \in X, \tag{2.3}$$

then $\mathsf{q} : X \to Y$ is quadratic, i.e.,

$$\mathsf{q}(x + y) + \mathsf{q}(x - y) = 2\mathsf{q}(x) + 2\mathsf{q}(y) \quad \textit{for all } x, y \in X.$$

If a mapping $\mathsf{q} : X \to Y$ is quadratic and $\mathsf{q}(ix) = -\mathsf{q}(x)$ holds for all $x \in X$, then the mapping $\mathsf{q} : X \to Y$ satisfies (2.3).

Proof. Let $\mathsf{q} : X \to Y$ satisfy (2.3). Putting $y = ix$ in (2.3), we have

$$\mathsf{q}(x - x) + \mathsf{q}(x + x) + \mathsf{q}(2ix) + \mathsf{q}(0) = 0,$$

i.e., $\mathsf{q}(2x) + \mathsf{q}(2ix) = 0$. Hence, $\mathsf{q}(x) + \mathsf{q}(ix) = 0$, i.e., $\mathsf{q}(ix) = -\mathsf{q}(x)$. Again, taking $iy = z$, $x = 0$ in (2.3), we have

$$0 = \mathsf{q}(z) + \mathsf{q}(-z) + \mathsf{q}(-iz) + \mathsf{q}(-iz) = \mathsf{q}(z) + \mathsf{q}(-z) - \mathsf{q}(z) - \mathsf{q}(z),$$

i.e.,

$$\mathsf{q}(z) + \mathsf{q}(-z) = \mathsf{q}(z) + \mathsf{q}(z).$$

Hence,

$$\mathsf{q}(0 + z) + \mathsf{q}(0 - z) = 2\mathsf{q}(z) + 2\mathsf{q}(0).$$

Therefore,

$$\mathsf{q}(x + z) + \mathsf{q}(x - z) = 2\mathsf{q}(z) + 2\mathsf{q}(x).$$

Taking $z = y$, $x = x$, we have

$$\mathsf{q}(x + y) + \mathsf{q}(x - y) = 2\mathsf{q}(y) + 2\mathsf{q}(x) \quad \text{for all } x, y \in X.$$

Conversely, let $\mathsf{q} : X \to Y$ be quadratic and let $\mathsf{q}(ix) = -\mathsf{q}(x)$ hold for all $x \in X$. Then

$$\mathsf{q}(x + iy) + \mathsf{q}(x - iy) + \mathsf{q}(y + ix) + \mathsf{q}(y - ix) = 0 \quad \text{for all } x, y \in X.$$

Putting $iy = z$, i.e., $y = -iz$, we conclude that the relations

$$\begin{aligned}
&\mathsf{q}(x+z) + \mathsf{q}(x-z) + \mathsf{q}(-iz+ix) + \mathsf{q}(-iz-ix) = 0,\\
&\mathsf{q}(x+z) + \mathsf{q}(x-z) + \mathsf{q}(-i(z-x)) + \mathsf{q}(-i(z+x)) = 0,\\
&[\mathsf{q}(x+z) + \mathsf{q}(x-z)] - [\mathsf{q}(z-x) + \mathsf{q}(z+x)] = 0,\\
&[2\mathsf{q}(x) + 2\mathsf{q}(z)] - [2\mathsf{q}(z) + 2\mathsf{q}(x)] = 0,\\
&0 = 0
\end{aligned}$$

are equivalent. Hence, $\mathsf{q} : X \to Y$ is quadratic. □

If a mapping $\mathsf{q} : X \to Y$ satisfies the quadratic functional equation

$$\mathsf{q}\,(x+y) + \mathsf{q}\,(x-y) = 2\mathsf{q}(x) + 2\mathsf{q}(y)$$

and $\mathsf{q}\,(ix) = -\mathsf{q}\,(x)$ for all $x, y \in X$, then we have

$$\mathsf{q}\,(x+iy) + \mathsf{q}\,(x-iy) + \mathsf{q}\,(y+ix) + \mathsf{q}\,(y-ix) = 0 \tag{2.4}$$

and $\mathsf{q}((1+i)x) = 2i\mathsf{q}(x)$ for all $x, y \in X$.

Remark 2.10. (i) It is easy to see that (2.4) is satisfied by taking $x = y$.

(ii) It is easy to see that (2.4) is satisfied by taking $x = x$, $y = ix$.

(iii) Since $\mathsf{q}(ix) = -\mathsf{q}(x)$, then we have $\mathsf{q}(-ix) = -\mathsf{q}(-x) = -\mathsf{q}(i^2x) = \mathsf{q}(ix) = -\mathsf{q}(x)$.

(iv) Since $\mathsf{q}(ix) = -\mathsf{q}(x)$, then we have $\mathsf{q}(-x) = \mathsf{q}(i^2x) = -\mathsf{q}(ix) = \mathsf{q}(x)$.

3. Main results

Throughout this section, we assume that X is a complex 2-normed vector space with the 2-norm $\|\cdot,\cdot\|$ and Y is a complex 2-Banach space with the 2-norm $\|\cdot,\cdot\|$. For a given mapping $\mathsf{q} : X \to Y$, we define

$$\mathbb{C}\mathsf{q}(x,y) := \mathsf{q}(x+iy) + \mathsf{q}(x-iy) + \mathsf{q}(y+ix) + \mathsf{q}(y-ix) - 2\mathsf{q}(x) - 2\mathsf{q}(y) \quad \text{for all } x, y \in X.$$

If the mapping $\mathsf{q} : X \to Y$ satisfies the additive functional equation

$$\mathsf{q}(x+y) = \mathsf{q}(x) + \mathsf{q}(y)$$

and $\mathsf{q}(ix) = i\mathsf{q}(x)$ for all $x, y \in X$, then

$$\mathsf{q}(x+iy) + \mathsf{q}(x-iy) + \mathsf{q}(y+ix) + \mathsf{q}(y-ix) = 2\mathsf{q}(x) + 2\mathsf{q}(y) \quad \text{for all } x, y \in X.$$

In fact, $\mathsf{q} : \mathbb{C} \to \mathbb{C}$ with $\mathsf{q}(x) = x$ satisfies (1.4).

Now we prove the Hyers–Ulam stability of the additive functional equation $\mathbb{C}\mathsf{q}(x,y) = 0$.

The following result will be required in the sequel.

Lemma 3.1. *Let X be a complex 2-normed space, Y be a complex 2-Banach space and $\mathsf{q} : X \to Y$ be a mapping satisfying the mapping $\mathsf{q}((1+i)x) = (1+i)\mathsf{q}(x)$ for which there exists a function $\phi : X^2 \to [0,\infty)$ such that*

$$\bar{\phi}(x,y) := \sum_{j=0}^{\infty} \frac{1}{2^j}\phi\left(2^j x, 2^j y\right) < \infty \tag{3.1}$$

or

$$\bar{\phi}(x,y) := \sum_{j=0}^{\infty} 2^j \phi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{3.2}$$

and

$$\|\mathbb{C}\mathsf{q}(x,y), z\| \leq \phi(x,y) \quad \text{for all } x, y \in X. \tag{3.3}$$

Then for all nonnegative integers l, m with $l < m$ and $x, z \in X$, we have

$$\left\|\frac{1}{2^l}\mathsf{q}\left(2^l x\right) - \frac{1}{2^m}\mathsf{q}\left(2^m x\right), z\right\| \leq \sum_{j=l}^{m-1} \frac{1}{2^{j+1}.\sqrt{2}}\phi(2^j x, 2^j x) \tag{3.4}$$

or

$$\left\|2^l \mathsf{q}(2^{-l}x) - 2^m \mathsf{q}(2^{-m}x), z\right\| \leq \sum_{j=l}^{m-1} \frac{1}{2.\sqrt{2}} 2^{j+1}\phi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right). \tag{3.5}$$

Proof. Since $\mathsf{q}((1+i)x) = (1+i)\mathsf{q}(x)$ for all $x \in X$, we can get $\mathsf{q}(0) = 0$ and $\mathsf{q}(2x) = (1+i)\mathsf{q}((1-i)x)$ for all $x \in X$. Now, putting $x = y \neq 0$ in (3.3), we get

$$\|2\mathsf{q}((1+i)x) + 2\mathsf{q}((1-i)x) - 4\mathsf{q}(x), z\| \leq \phi(x,x).$$

Therefore, we have

$$\left\|\mathsf{q}(x) - \frac{1}{2}\mathsf{q}(2x), z\right\| \leq \frac{1}{2\sqrt{2}}\phi(x,x) \quad \text{for all } x, y \in X. \tag{3.6}$$

It is easy to see that inequality (3.4) holds for all nonnegative integers l, m with $l < m$.

Choosing $x = \frac{x}{2}$ in equation (3.6), we have

$$\begin{aligned}
\left\|\mathsf{q}\left(\frac{x}{2}\right) - \frac{1}{2}\mathsf{q}(x), z\right\| &\leq \frac{1}{2\sqrt{2}}\phi\left(\frac{x}{2}, \frac{x}{2}\right), \\
\left\|2\mathsf{q}\left(\frac{x}{2}\right) - \mathsf{q}(x), z\right\| &\leq \frac{1}{\sqrt{2}}\phi\left(\frac{x}{2}, \frac{x}{2}\right), \\
\left\|\mathsf{q}(x) - 2\mathsf{q}\left(\frac{x}{2}\right), z\right\| &\leq \frac{1}{\sqrt{2}}\phi\left(\frac{x}{2}, \frac{x}{2}\right) \quad \text{for all } x, y \in X.
\end{aligned} \tag{3.7}$$

It is easy to verify that inequality (3.5) holds for all nonnegative integers l, m with $l < m$. □

Theorem 3.2. *Let X be a complex 2-normed space, Y be a complex 2-Banach space and $\mathsf{q} : X \to Y$ be a mapping satisfying $\mathsf{q}((1+i)x) = (1+i)\mathsf{q}(x)$ for which there exists a function $\phi : X^2 \to [0, \infty)$ such that*

$$\bar{\phi}(x, y) := \sum_{j=0}^{\infty} \frac{1}{2^j} \phi(2^j x, 2^j y) < \infty,$$
$$\|\mathbb{C}\mathsf{q}(x, y), z\| \le \phi(x, y) \quad \text{for all } x, y \in X.$$

Then there exists a unique additive mapping $g : X \to Y$ such that

$$\|\mathsf{q}(x) - g(x), z\| \le \frac{1}{2\sqrt{2}} \bar{\phi}(x, x) \quad \text{for all } x, y \in X. \tag{3.8}$$

Proof. Define the sequence of function $\{g_n\}$ by the formula

$$g_n(x) = \frac{1}{2^n} \mathsf{q}(2^n x) \quad \text{for all } x \in X, n \in \mathbb{N}. \tag{3.9}$$

Firstly we have to prove that the sequence $\{g_n\}$ is a Cauchy sequence for every $x \in X$.

For $x = 0$, it is trivial.

Taking $0 \ne x \in X$ for $n < m$ and using Lemma 3.1, we have

$$\begin{aligned} \|g_n(x) - g_m(x), z\| &= \left\| \frac{1}{2^n} \mathsf{q}(2^n x) - \frac{1}{2^m} \mathsf{q}(2^m x), z \right\| \\ &\le \sum_{j=n}^{m-1} \frac{1}{2^{j+1}.\sqrt{2}} \phi(2^j x, 2^j x) < \infty. \end{aligned}$$

Therefore, the sequence $\{g_n(x)\}$ is a Cauchy sequence. Since Y is complete, then this sequence is convergent. So, we can define a mapping $g : X \to Y$ such that

$$g(x) = \lim_{n \to \infty} g_n(x) = \lim_{n \to \infty} \left[\frac{1}{2^n} \mathsf{q}(2^n x) \right].$$

By (3.3), we get

$$\begin{aligned} \|\mathbb{C}g(x, y), z\| &= \lim_{n \to \infty} \frac{1}{2^n} \|\mathbb{C}\mathsf{q}(2^n x, 2^n y), z\| \\ &\le \lim_{n \to \infty} \frac{1}{2^n} \phi(2^n x, 2^n y) = 0 \quad \text{for all } x, y \in X. \end{aligned}$$

Thus, $\mathbb{C}g(x, y) = 0$. By Proposition 2.7, the mapping $g : X \to Y$ is additive. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in inequality (3.4), we obtain (3.8).

To prove that g is unique, assume that there exist two additive functions $g_i : X \to Y$, $i = 1, 2$, such that

$$\|\mathsf{q}(x) - g_i(x), z\| \le \frac{1}{2\sqrt{2}} \bar{\phi}(x, x). \tag{3.10}$$

Also, we have

$$g_i(x) = \frac{1}{2^n} g_i(2^n x) \quad \text{for all } x \in X, n \in \mathbb{N}. \tag{3.11}$$

Now, for every $x, z \in X$, we have $(g_1(0) = g_2(0) = 0)$. Using (3.10) and (3.11), we obtain

$$\begin{aligned} \|g_1(x) - g_2(x), z\| &= \left\|2^{-n} g_1(2^n x) - 2^{-n} g_2(2^n(x)), z\right\| \\ &= 2^{-n} \|g_1(2^n x) - g_2(2^n x), z\| \\ &= 2^{-n} \|g_1(2^n x) - \mathsf{q}(2^n x) + \mathsf{q}(2^n x) - g_2(2^n x), z\| \\ &\le 2^{-n} \left[\|\mathsf{q}(2^n x) - g_1(2^n x), z\| + \|\mathsf{q}(2^n x) - g_2(2^n x), z\|\right] \\ &\le 2^{-n} \left[2 \frac{1}{2\sqrt{2}} \bar{\phi}(2^n x, 2^n x)\right]. \end{aligned}$$

Taking the limit $n \to \infty$, we have $\|g_1(x) - g_2(x), z\| = 0$, and thus $g_1(x) = g_2(x)$. □

Theorem 3.3. *Let X be a complex 2-normed space, Y be a complex 2-Banach space and let $\mathsf{q} : X \to Y$ be a mapping satisfying $\mathsf{q}((1+i)x) = (1+i)\mathsf{q}(x)$ for which there exists a function $\phi : X^2 \to [0, \infty)$ such that*

$$\bar{\phi}(x, y) := \sum_{j=0}^{\infty} 2^j \phi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty,$$

$$\|\mathbb{C}\mathsf{q}(x, y), z\| \le \phi(x, y) \tag{3.12}$$

for all $x, y, z \in X$.

Then there exists a unique additive mapping $h : X \to Y$ such that

$$\|h(x) - \mathsf{q}(x), z\| \le \frac{1}{\sqrt{2}} \bar{\phi}\left(2^{-1}x, 2^{-1}x\right). \tag{3.13}$$

Proof. Define the sequence of functions $\{h_n(x)\}$ by

$$h_n(x) = 2^n \mathsf{q}(2^{-n}x) \quad \text{for all } x \in X,\ n \in \mathbb{N}. \tag{3.14}$$

Since $\mathsf{q}(0) = 0$, by using Lemma 3.1, for all $x, z \in X$ and $n < m$, we have

$$\begin{aligned} \|h_n(x) - h_m(x), z\| &\le \left\|2^n \mathsf{q}(2^{-n}x) - 2^m \mathsf{q}(2^{-m}x), z\right\| \\ &\le \sum_{j=n}^{m-1} \frac{1}{2.\sqrt{2}} 2^{j+1} \phi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right). \end{aligned}$$

Therefore $\{h_n(x)\}$ is a Cauchy sequence for every $x \in X$. Since Y is complete, then $\{h_n(x)\}$ is convergent. So, there exists a mapping $h : X \to Y$ such that

$$h(x) = \lim_{n \to \infty} h_n(x), \quad x \in X.$$

Then, as in Theorem 3.2, it is easy to verify that h is an additive function. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in inequality (3.5), we obtain (3.13).

To prove that h is unique, assume that there exist two additive functions $h_i : X \to Y$, $i = 1, 2$, such that

$$\|\mathsf{q}(x) - h_i(x), z\| \leq \frac{1}{\sqrt{2}}\bar{\phi}\left(2^{-1}x, 2^{-1}x\right). \tag{3.15}$$

Also, we have

$$h_i(x) = 2^n h_i\left(2^{-n}x\right). \tag{3.16}$$

Now, using (3.15), (3.16), for every $x, z \in X$ we obtain

$$\begin{aligned}
\|h_1(x) - h_2(x), z\| &= \left\|2^n h_1(2^{-n}x) - 2^n h_2(2^{-n}x), z\right\| \\
&= 2^n \left\|h_1(2^{-n}(x)) - h_2(2^{-n}(x)), z\right\| \\
&= 2^n \left\|h_1(2^{-n}x) - \mathsf{q}(2^{-n}(x)) + \mathsf{q}(2^{-n}(x)) - h_2(2^{-n}x), z\right\| \\
&\leq \left[\left\|\mathsf{q}(2^{-n}(x)) - h_1(x), z\right\| + \left\|\mathsf{q}(2^{-n}(x)) - h_2(x), z\right\|\right] \\
&\leq 2^n \left[\left\|\mathsf{q}(2^{-n}x) - h_1(2^{-n}x), z\right\| + \left\|\mathsf{q}(2^{-n}x) - h_2(2^{-n}x), z\right\|\right] \\
&\leq 2^n \left[\frac{2}{\sqrt{2}}\bar{\phi}\left(2^{-n-1}x, 2^{-n-1}x\right)\right].
\end{aligned}$$

Taking the limit $n \to \infty$, we have $h_1(x) = h_2(x)$. Hence the result follows. □

Now we need the following result to prove our next theorems.

Lemma 3.4. *Let X be a complex 2-normed space, Y be a complex 2-Banach space and let $r - \{1\} \in \mathbb{R}$ and θ be a positive real number. Let $\mathsf{q} : X \to Y$ be a mapping satisfying the mapping $\mathsf{q}((1+i)x) = (1+i)\mathsf{q}(x)$ and the inequality*

$$\|\mathbb{C}\mathsf{q}(x, y), z\| \leq \theta\left(\|x, z\|^r + \|y, z\|^r\right) \quad \text{for all } x, y, z \in X. \tag{3.17}$$

Then, for all $x, z \in X$ and for all nonnegative integers l, m with $l < m$, we have

$$\left\|\frac{1}{2^l}\mathsf{q}(2^l x) - \frac{1}{2^m}\mathsf{q}(2^m x), z\right\| \leq \sum_{j=l}^{m-1} \frac{2^{rj}}{2^j}\left[\frac{1}{\sqrt{2}}\theta\,\|x, z\|^r\right] \tag{3.18}$$

or

$$\left\|2^l\mathsf{q}(2^{-l}x) - 2^m\mathsf{q}(2^{-m}x), z\right\| \leq \sum_{j=l}^{m-1} \frac{2^{j+1}}{2^{rj+r}}\left[\frac{\sqrt{2}}{2}\theta\|x, z\|^r\right]. \tag{3.19}$$

Proof. Since $\mathsf{q}((1+i)x) = (1+i)\mathsf{q}(x)$ for all $x \in X$, we can get $\mathsf{q}(0) = 0$ and $\mathsf{q}(2x) = (1+i)\mathsf{q}((1-i)x)$ for all $x, y \in X$.

Now, putting $x = y \neq 0$ in (3.17), we get

$$\|2\mathsf{q}((1+i)x) + 2\mathsf{q}((1-i)x) - 4\mathsf{q}(x), z\| \leq 2\theta \|x, z\|^r .$$

Therefore, we have

$$\left\|\mathsf{q}(x) - \frac{1}{2}\mathsf{q}(2x), z\right\| \leq \frac{1}{\sqrt{2}}\theta \|x, z\|^r \quad \text{for all } x, z \in X. \tag{3.20}$$

It is easy to verify that inequality (3.18) holds for all nonnegative integers l, m with $l < m$.

Choosing $x = \frac{x}{2}$ in equation (3.20), we have

$$\left\|\mathsf{q}\left(\frac{x}{2}\right) - \frac{1}{2}\mathsf{q}(x), z\right\| \leq \frac{1}{\sqrt{2}}\theta \left\|\frac{x}{2}, z\right\|^r$$

$$\left\|\mathsf{q}(x) - 2\mathsf{q}\left(\frac{x}{2}\right), z\right\| \leq \sqrt{2}\theta \left\|\frac{x}{2}, z\right\|^r . \tag{3.21}$$

Now it is easy to verify that inequality (3.19) holds for all nonnegative integers l, m with $l < m$. □

Theorem 3.5. *Let X be a complex 2-normed space, Y be a complex 2-Banach space and let $r < 1$, θ be a positive real number. Let $\mathsf{q} : X \to Y$ be a mapping satisfying $\mathsf{q}((1+i)x) = (1+i)\mathsf{q}(x)$ and the inequality*

$$\|\mathbb{C}\mathsf{q}(x, y), z\| \leq \theta \left(\|x, z\|^r + \|y, z\|^r\right) \quad \textit{for all } x, y, z \in X. \tag{3.22}$$

Then there exists a unique additive mapping $g : X \to Y$ such that

$$\|\mathsf{q}(x) - g(x), z\| \leq \frac{\sqrt{2}}{2 - 2^r}\theta \|x, z\|^r \quad \textit{for all } x, z \in X. \tag{3.23}$$

Proof. Define the sequence of function $\{g_n\}$ by the formula

$$g_n(x) = \frac{1}{2^n}\mathsf{q}(2^n x) \quad \text{for all } x \in X,\ n \in \mathbb{N}. \tag{3.24}$$

Firstly we have to prove that the sequence $\{g_n\}$ is a Cauchy sequence for every $x \in X$. For $x = 0$, it is trivial. Taking $0 \neq x \in X$ for $n < m$ and using Lemma 3.4, we have

$$\begin{aligned} \|g_n(x) - g_m(x), z\| &= \left\|\frac{1}{2^n}\mathsf{q}(2^n x) - \frac{1}{2^m}\mathsf{q}(2^m x), z\right\| \\ &\leq \sum_{j=n}^{m-1} \frac{2^{rj}}{2^j}\left[\frac{1}{\sqrt{2}}\theta\|x, z\|^r\right] < \infty, \quad r < 1. \end{aligned}$$

Therefore the sequence $\{g_n(x)\}$ is a Cauchy sequence. Since Y is complete, then this sequence is convergent. Thus, we can define a mapping $g : X \to Y$ such that

$$g(x) = \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty}\left[\frac{1}{2^n}\mathsf{q}(2^n x)\right].$$

By (3.22), we get

$$\begin{aligned}\|\mathbb{C}g(x,y),z\| &= \lim_{n\to\infty}\frac{1}{2^n}\|\mathbb{C}\mathsf{q}(2^n x,2^n y),z\|\\ &\le \lim_{n\to\infty}\frac{2^{nr}}{2^n}\theta(\|x,z\|^r+\|y,z\|^r)=0 \quad \text{for all } x,y,z\in X.\end{aligned}$$

Hence $\mathbb{C}g(x,y)=0$. By Proposition 2.7, the mapping $g : X \to Y$ is additive. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (3.18), we obtain (3.23).

To prove that g is unique, assume that there exist two additive functions $g_i : X \to Y$, $i=1,2$ such that

$$\|\mathsf{q}(x)-g_i(x),z\| \le \frac{\sqrt{2}}{2-2^r}\theta\,\|x,z\|^r. \tag{3.25}$$

Also, we have

$$g_i(x)=2^{-n}g_i\left(2^n x\right). \tag{3.26}$$

Now, for every $x,z\in X$, we have $g_1(0)=g_2(0)=0$. Using (3.25), (3.26), we obtain

$$\begin{aligned}\|g_1(x)-g_2(x),z\| &= \left\|2^{-n}g_1(2^n x)-2^{-n}g_2(2^n(x)),z\right\|\\ &= 2^{-n}\|g_1(2^n x)-g_2(2^n x),z\|\\ &= 2^{-n}\|g_1(2^n x)-\mathsf{q}(2^n x)+\mathsf{q}(2^n x)-g_2(2^n x),z\|\\ &\le 2^{-n}\left[\|\mathsf{q}(2^n x)-g_1(2^n x),z\|+\|\mathsf{q}(2^n x)-g_2(2^n x),z\|\right]\\ &\le 2\frac{2^{nr}}{2^n}\left[\frac{\sqrt{2}}{2-2^r}\theta\,\|x,z\|^r\right], \quad r<1.\end{aligned}$$

As $n\to\infty$, we get $\|g_1(x)-g_2(x),z\|=0$. Therefore, $g_1(x)=g_2(x)$. □

Theorem 3.6. *Let $r>1$, θ be a positive real number, X be a complex 2-normed space and Y be a 2-Banach space. Assume that $\mathsf{q}: X\to Y$ is a mapping satisfying $\mathsf{q}((1+i)x)=(1+i)\mathsf{q}(x)$ and the inequality*

$$\|\mathbb{C}\mathsf{q}(x,y),z\| \le \theta\left[\|x,z\|^r+\|y,z\|^r\right] \quad \text{for all } x,y,z\in X. \tag{3.27}$$

Then there exists a unique additive mapping $h: X\to Y$ such that

$$\|\mathsf{q}(x)-h(x),z\| \le \frac{\sqrt{2}}{2^r-2}\theta\,\|x,z\|^r \quad \text{for all } x,z\in X. \tag{3.28}$$

Proof. Define the sequence of functions $\{h_n\}$ by the formula

$$h_n(x)=2^n\mathsf{q}(2^{-n}x) \tag{3.29}$$

for all $x\in X$ and $n\in\mathbb{N}$. Since $\mathsf{q}(0)=0$, by using Lemma 3.4, we have for all $x\in X$ and $n>m$,

$$\|h_n(x)-h_m(x),z\| \le \left\|2^n\mathsf{q}(2^{-n}x)-2^m\mathsf{q}(2^{-m}x),z\right\| \le \sum_{j=l}^{m-1}\frac{2^{j+1}}{2^{rj+r}}\left[\frac{\sqrt{2}}{2}\theta\|x,z\|^r\right].$$

Therefore, $\{h_n(x)\}$ is a Cauchy sequence for every $x \in X$. Since Y is complete, then $\{h_n(x)\}$ is convergent. Thus, there exists a mapping $h : X \to Y$ such that

$$h(x) = \lim_{n\to\infty} h_n(x), \quad x \in X.$$

Then, in the same way as in Theorem 3.5, it easy to verify that h is an additive function. Using inequality (3.19), we obtain (3.28).

To prove that h is unique, assume that there exist two additive functions $h_i : X \to Y$, $i = 1, 2$, such that

$$\|\mathsf{q}(x) - h_i(x), z\| \le \frac{\sqrt{2}}{2^r - 2}\theta \|x, z\|^r . \tag{3.30}$$

Also, we have

$$h_i(x) = 2^n h_i\left(2^{-n}x\right). \tag{3.31}$$

Now, for every $x, z \in X$, using (3.30), (3.31), we get

$$\begin{aligned}
\|h_1(x) - h_2(x), z\| &= \left\|2^n h_1(2^{-n}x) - 2^n h_2(2^{-n}x), z\right\| \\
&= 2^n \left\|h_1(2^{-n}(x)) - h_2(2^{-n}(x)), z\right\| \\
&= 2^n \left\|h_1(2^{-n}x) - \mathsf{q}(2^{-n}(x)) + \mathsf{q}(2^{-n}(x)) - h_2(2^{-n}x), z\right\| \\
&\le \left[\left\|\mathsf{q}(2^{-n}(x)) - h_1(x), z\right\| + \left\|\mathsf{q}(2^{-n}(x)) - h_2(x), z\right\|\right] \\
&\le 2^n \left[\left\|\mathsf{q}(2^{-n}x) - h_1(2^{-n}x), z\right\| + \left\|\mathsf{q}(2^{-n}x) - h_2(2^{-n}x), z\right\|\right] \\
&\le \frac{2^{n+1}}{2^{nr}}\left[\frac{\sqrt{2}}{2^r - 2}\theta \|x, z\|^r\right], \quad r > 1.
\end{aligned}$$

Taking the limit $n \to \infty$, we have $h_1(x) = h_2(x)$. Hence the result follows. □

Lemma 3.7. *Let X be a complex 2-normed space, Y be a complex 2-Banach space and let $r - \{1/2\} \in \mathbb{R}$ and θ be a positive real number. Let $\mathsf{q} : X \to Y$ be a mapping satisfying the mapping $\mathsf{q}((1 + i)x) = (1 + i)\mathsf{q}(x)$ and the inequality*

$$\|\mathbb{C}\mathsf{q}(x, y), z\| \le \theta \|x, z\|^r \|y, z\|^r \quad \text{for all } x, y, z \in X. \tag{3.32}$$

Then, for all nonnegative integers l, m with $l < m$ and $x, z \in X$, we have

$$\left\|\frac{1}{2^l}\mathsf{q}(2^l x) - \frac{1}{2^m}\mathsf{q}(2^m x), z\right\| \le \sum_{j=l}^{m-1} \frac{4^{rj}}{2^{j+1}}\left[\frac{1}{\sqrt{2}}\theta\|x, z\|^{2r}\right] \tag{3.33}$$

or

$$\left\|2^l\mathsf{q}(2^{-l}x) - 2^m\mathsf{q}(2^{-m}x), z\right\| \le \sum_{j=l}^{m-1} \frac{2^j}{4^{rj+r}}\left[\frac{1}{\sqrt{2}}\theta\|x, z\|^{2r}\right]. \tag{3.34}$$

Proof. Since $\mathsf{q}((1+i)x) = (1+i)\mathsf{q}(x)$ for all $x \in X$, we can get $\mathsf{q}(0) = 0$ and $\mathsf{q}(2x) = (1+i)\mathsf{q}((1-i)x)$ for all $x \in X$.

Now, putting $x = y \neq 0$ in (3.32), we get

$$\|2\mathsf{q}((1+i)x) + 2\mathsf{q}((1-i)x) - 4\mathsf{q}(x), z\| \leq \theta \|x, z\|^{2r}.$$

Therefore, we have

$$\left\|\mathsf{q}(x) - \frac{1}{2}\mathsf{q}(2x), z\right\| \leq \frac{1}{2\sqrt{2}}\theta \|x, z\|^{2r} \tag{3.35}$$

for all $x, z \in X$. It is easy to verify that inequality (3.33) holds for all nonnegative integers l, m with $l < m$. Choosing $x = \frac{x}{2}$ in equation (3.35), we have

$$\begin{aligned} \left\|\mathsf{q}\left(\frac{x}{2}\right) - \frac{1}{2}\mathsf{q}(x), z\right\| &\leq \frac{1}{2\sqrt{2}}\theta \left\|\frac{x}{2}, z\right\|^{2r} \\ \left\|\mathsf{q}(x) - 2\mathsf{q}\left(\frac{x}{2}\right), z\right\| &\leq \frac{1}{\sqrt{2}}\theta \left\|\frac{x}{2}, z\right\|^{2r}. \end{aligned} \tag{3.36}$$

Finally, it is easy to verify that inequality (3.34) holds for all nonnegative integers l, m with $l < m$. □

Theorem 3.8. *Let X be a complex 2-normed space, Y be a complex 2-Banach space and let $r < 1/2$, θ be a positive real number. Let $\mathsf{q} : X \to Y$ be a mapping satisfying $\mathsf{q}((1+i)x) = (1+i)\mathsf{q}(x)$ and the inequality*

$$\|\mathbb{C}\mathsf{q}(x, y), z\| \leq \theta \|x, z\|^r . \|y, z\|^r \quad \textit{for all } x, y, z \in X. \tag{3.37}$$

Then there exists a unique additive mapping $g : X \to Y$ such that

$$\|\mathsf{q}(x) - g(x), z\| \leq \frac{1}{(2-4^r)\sqrt{2}}\theta \|x, z\|^{2r} \quad \textit{for all } x, z \in X. \tag{3.38}$$

Proof. Define the sequence of functions $\{g_n\}$ by the formula

$$g_n(x) = \frac{1}{2^n}\mathsf{q}(2^n x) \quad \text{for all } x \in X,\ n \in \mathbb{N}. \tag{3.39}$$

Firstly we have to prove that the sequence $\{g_n\}$ is a Cauchy sequence for every $x \in X$. For $x = 0$, it is trivial. Take $0 \neq x \in X$ for $n < m$. By using Lemma 3.7, we have

$$\begin{aligned} \|g_n(x) - g_m(x), z\| &= \left\|\frac{1}{2^n}\mathsf{q}(2^n x) - \frac{1}{2^m}\mathsf{q}(2^m x), z\right\| \\ &\leq \sum_{j=n}^{m-1} \frac{4^{rj}}{2^{j+1}} \left[\frac{1}{\sqrt{2}}\theta\|x, z\|^{2r}\right] < \infty, \quad r < \frac{1}{2}. \end{aligned}$$

Therefore the sequence $\{g_n(x)\}$ is a Cauchy sequence. Since Y is complete, then this sequence is convergent. Hence, we can define a mapping $g : X \to Y$ such that

$$g(x) = \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \left[\frac{1}{2^n}\mathsf{q}(2^n x)\right].$$

By (3.37), we get

$$\begin{aligned}\|\mathbb{C}g(x,y), z\| &= \lim_{n\to\infty} \frac{1}{2^n}\|\mathbb{C}\mathsf{q}(2^n x, 2^n y), z\| \\ &\le \lim_{n\to\infty} \frac{4^{nr}}{2^n}\theta\|x,z\|^r\|y,z\|^r = 0 \quad \text{for all } x,y,z\in X.\end{aligned}$$

Thus $\mathbb{C}g(x,y) = 0$. By Proposition(2.7), the mapping $g : X \to Y$ is additive. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in inequality (3.33), we obtain (3.38).

To prove that g is unique, assume that there exist two additive functions $g_i : X \to Y$, $i = 1, 2$, such that

$$\|\mathsf{q}(x) - g_i(x), z\| \le \frac{1}{(2-4^r)\sqrt{2}}\theta\,\|x,z\|^{2r}. \tag{3.40}$$

Also, we have

$$g_i(x) = 2^{-n} g_i\,(2^n x)\,. \tag{3.41}$$

Now we obtain for every $x, z \in X$ that $g_1(0) = g_2(0) = 0$ and using (3.40), (3.41), we have

$$\begin{aligned}\|g_1(x) - g_2(x), z\| &= \left\|2^{-n}g_1(2^n x) - 2^{-n}g_2(2^n(x)), z\right\| \\ &= 2^{-n}\,\|g_1(2^n x) - g_2(2^n x), z\| \\ &= 2^{-n}\,[\|g_1(2^n x) - \mathsf{q}(2^n x) + \mathsf{q}(2^n x) - g_2(2^n x), z\|] \\ &\le 2^{-n}\,[\|\mathsf{q}(2^n x) - g_1(2^n x), z\| + \|\mathsf{q}(2^n x) - g_2(2^n x), z\|] \\ &\le 2\frac{2^{2nr}}{2^n}\left[\frac{1}{(2-4^r)\sqrt{2}}\theta\,\|x,z\|^{2r}\right], \quad r < \frac{1}{2}.\end{aligned}$$

Taking $n \to \infty$, we get $\|g_1(x) - g_2(x), z\| = 0$. Therefore $g_1(x) = g_2(x)$. □

Theorem 3.9. *Let X be a complex 2-normed space, Y be a complex 2-Banach space and let $r > 1/2$, θ be a positive real number. Let $\mathsf{q} : X \to Y$ be a mapping satisfying $\mathsf{q}((1+i)x) = (1+i)\mathsf{q}(x)$ and the inequality*

$$\|\mathbb{C}\mathsf{q}(x,y), z\| \le \theta\,\|x,z\|^r\,\|y,z\|^r \quad \textit{for all } x,y,z\in X. \tag{3.42}$$

Then there exists a unique additive mapping $h : X \to Y$ such that

$$\|\mathsf{q}(x) - h(x), z\| \le \frac{1}{(4^r-2)\sqrt{2}}\theta\,\|x,z\|^{2r} \quad \textit{for all } x,z\in X. \tag{3.43}$$

Proof. Define the sequence of functions $\{h_n\}$ by the formula

$$h_n(x) = 2^n \mathsf{q}(2^{-n}x) \quad \text{for all } x, z \in X, \ n \in \mathbb{N}. \tag{3.44}$$

Since $\mathsf{q}(0) = 0$, by using Lemma 3.7, we have for all $x, z \in X$ and $n > m$,

$$\begin{aligned} \|h_n(x) - h_m(x), z\| &\leq \left\|2^n \mathsf{q}(2^{-n}x) - 2^m \mathsf{q}(2^{-m}x), z\right\| \\ &\leq \sum_{j=l}^{m-1} \frac{2^j}{4^{rj+r}} \left[\frac{1}{\sqrt{2}}\theta \|x, z\|^{2r}\right]. \end{aligned}$$

Therefore $\{h_n(x)\}$ is a Cauchy sequence for every $x \in X$. Since Y is complete, then $\{h_n(x)\}$ is convergent. Hence, there exists a mapping $h : X \to Y$ such that

$$h(x) = \lim_{n\to\infty} h_n(x), \quad x \in X.$$

Then, in the same way as in Theorem 3.8, it easy to verify that h is an additive function. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.34), we obtain (3.43).

To prove that h is unique, assume that there exist two additive functions $h_i : X \to Y$, $i = 1, 2$, such that

$$\|\mathsf{q}(x) - h_i(x), z\| \leq \frac{1}{(4^r - 2)\sqrt{2}}\theta \|x, z\|^{2r}. \tag{3.45}$$

Also, we have

$$h_i(x) = 2^n h_i\left(2^{-n}x\right). \tag{3.46}$$

Now, for every $x, z \in X$, by using (3.45), (3.46), we get

$$\begin{aligned} \|h_1(x) - h_2(x), z\| &= \left\|2^n h_1(2^{-n}x) - 2^n h_2(2^{-n}x), z\right\| \\ &= 2^n \left\|h_1(2^{-n}(x)) - h_2(2^{-n}(x)), z\right\| \\ &= 2^n \left\|h_1(2^{-n}x) - \mathsf{q}(2^{-n}(x)) + \mathsf{q}(2^{-n}(x)) - h_2(2^{-n}x), z\right\| \\ &\leq \left[\left\|\mathsf{q}(2^{-n}(x)) - h_1(x), z\right\| + \left\|\mathsf{q}(2^{-n}(x)) - h_2(x), z\right\|\right] \\ &\leq 2^n \left[\left\|\mathsf{q}(2^{-n}x) - h_1(2^{-n}x), z\right\| + \left\|\mathsf{q}(2^{-n}x) - h_2(2^{-n}x), z\right\|\right] \\ &\leq \frac{2^{n+1}}{2^{2nr}} \left[\frac{1}{(4^r - 2)\sqrt{2}}\theta \|x, z\|^{2r}\right], \quad r > \frac{1}{2}. \end{aligned}$$

Taking the limit $n \to \infty$, we have $h_1(x) = h_2(x)$. Hence the result follows. □

4. Hyers–Ulam stability of quadratic functional equations

For a given mapping $\mathsf{q} : X \to Y$, we define

$$\mathbb{C}\mathsf{q}(x, y) := \mathsf{q}(x + iy) + \mathsf{q}(x - iy) + \mathsf{q}(y + ix) + \mathsf{q}(y - ix) \quad \text{for all } x, z \in X.$$

If a mapping $\mathsf{q} : X \to Y$ satisfies the quadratic functional equation

$$\mathsf{q}(x+y) + \mathsf{q}(x-y) = 2\mathsf{q}(x) + 2\mathsf{q}(y),$$

and $\mathsf{q}(ix) = -\mathsf{q}(x)$ for all $x, y \in X$, then

$$\mathsf{q}(x+iy) + \mathsf{q}(x-iy) + \mathsf{q}(y+ix) + \mathsf{q}(y-ix) = 0 \quad \text{for all } x, z \in X.$$

In fact, $\mathsf{q} : \mathbb{C} \to \mathbb{C}$ with $\mathsf{q}(x) = x^2$ satisfies (1.5).

Now we prove the Hyers–Ulam stability of the quadratic functional equation $\mathbb{C}\mathsf{q}(x, y) = 0$.

Lemma 4.1. *Let X be a complex 2-normed space, Y be a complex 2-Banach space and let $r - \{2\} \in \mathbb{R}$ and θ be a positive real number. Let $\mathsf{q} : X \to Y$ be a mapping satisfying the mapping $\mathsf{q}((1+i)x) = 2i\mathsf{q}(x)$ and the inequality*

$$\|\mathbb{C}\mathsf{q}(x, y), z\| \le \theta \left(\|x, z\|^r + \|y, z\|^r\right) \quad \text{for all } x, z \in X. \tag{4.1}$$

Then, for all $x \in X$ and for all nonnegative integers l, m with $l < m$, we have

$$\left\| \frac{1}{4^l}\mathsf{q}(2^l x) - \frac{1}{4^m}\mathsf{q}(2^m x), z \right\| \le \sum_{j=l}^{m-1} \frac{2^{rj}}{4^j} \left[\frac{1}{2}\theta \|x, z\|^r\right] \tag{4.2}$$

or

$$\left\| 4^l \mathsf{q}(2^{-l}x) - 4^m \mathsf{q}(2^{-m}x), z \right\| \le \sum_{j=l}^{m-1} \frac{4^j}{2^{rj+r}} \left[2\theta \|x, z\|^r\right]. \tag{4.3}$$

Proof. Since $\mathsf{q}((1+i)x) = 2i\mathsf{q}(x)$ for all $x \in X$, we can get $\mathsf{q}(0) = 0$ and $\mathsf{q}(2x) = 2i\mathsf{q}((1-i)x)$ for all $x \in X$.

Now, putting $x = y \ne 0$ in (4.1), we get

$$\|2\mathsf{q}((1+i)x) + 2\mathsf{q}((1-i)x), z\| \le 2\theta \|x, z\|^r.$$

Therefore we have

$$\left\| \mathsf{q}(x) - \frac{1}{4}\mathsf{q}(2x), z \right\| \le \frac{1}{2}\theta \|x, z\|^r \quad \text{for all } x, z \in X. \tag{4.4}$$

It is easy to see that inequality (4.2) holds for all nonnegative integers l, m with $l < m$.

Choosing $x = 1/2$ in equation (4.4), we have

$$\left\| \mathsf{q}\left(\frac{x}{2}\right) - \frac{1}{4}\mathsf{q}(x), z \right\| \le \frac{1}{2}\theta \left\| \frac{x}{2}, z \right\|^r$$

$$\left\| \mathsf{q}(x) - 4\mathsf{q}\left(\frac{x}{2}\right), z \right\| \le 2\theta \left\| \frac{x}{2}, z \right\|^r. \tag{4.5}$$

Now it is easy to see that inequality (4.3) holds for all nonnegative integers l, m with $l < m$. □

Theorem 4.2. *Let X be a complex 2-normed space, Y be a complex 2-Banach space and let $r < 2$, θ be a positive real number. Let $\mathsf{q} : X \to Y$ be a mapping satisfying $\mathsf{q}((1+i)x) = 2i\mathsf{q}(x)$ and the inequality*

$$\|\mathbb{C}\mathsf{q}(x,y), z\| \leq \theta\left(\|x,z\|^r + \|y,z\|^r\right) \quad \text{for all } x, y, z \in X. \tag{4.6}$$

Then there exists a unique quadratic mapping $f : X \to Y$ such that

$$\|\mathsf{q}(x) - f(x), z\| \leq \frac{2}{4 - 2^r}\theta\,\|x,z\|^r \quad \text{for all } x, z \in X. \tag{4.7}$$

Proof. Define the sequence of functions $\{g_n\}$ by the formula

$$f_n(x) = \frac{1}{4^n}\mathsf{q}(2^n x) \quad \text{for all } x \in X,\ n \in \mathbb{N}. \tag{4.8}$$

Firstly we have to prove that the sequence $\{f_n\}$ is a Cauchy sequence for every $x \in X$. For $x = 0$, it is trivial. Take $0 \neq x \in X$ for $n < m$. By using Lemma 4.1, we have

$$\begin{aligned}\|f_n(x) - f_m(x), z\| &= \left\|\frac{1}{4^n}\mathsf{q}(2^n x) - \frac{1}{4^m}\mathsf{q}(2^m x), z\right\| \\ &\leq \sum_{j=n}^{m-1}\frac{2^{rj}}{4^j}\left[\frac{1}{2}\theta\|x,z\|^r\right] < \infty, \quad r < 2.\end{aligned}$$

Therefore the sequence $\{f_n(x)\}$ is a Cauchy sequence. Since Y is complete, then this sequence is convergent. Thus, we can define a mapping $f : X \to Y$ such that

$$\begin{aligned}f(x) &= \lim_{n\to\infty} f_n(x), \\ f(x) &= \lim_{n\to\infty}\left[\frac{1}{4^n}\mathsf{q}(2^n x)\right].\end{aligned}$$

By (4.6),

$$\begin{aligned}\|\mathbb{C}f(x,y), z\| &= \lim_{n\to\infty}\frac{1}{4^n}\|\mathbb{C}\mathsf{q}(2^n x, 2^n y), z\| \\ &\leq \lim_{n\to\infty}\frac{2^{nr}}{4^n}\theta(\|x,z\|^r + \|y,z\|^r) = 0 \quad \text{for all } x, y, z \in X.\end{aligned}$$

Hence $\mathbb{C}f(x,y) = 0$. By Proposition (2.9), the mapping $f : X \to Y$ is quadratic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in inequality (4.2), we obtain (4.7).

To prove that f is unique, assume that there exist two quadratic functions $g_i : X \to Y$, $i = 1, 2$, such that

$$\|\mathsf{q}(x) - f_i(x), z\| \leq \frac{2}{4 - 2^r}\theta\,\|x,z\|^r. \tag{4.9}$$

Also, we have

$$f_i(x) = 4^{-n} f_i(2^n(x)) \tag{4.10}$$

For every $x, z \in X$, we obtain that $f_1(0) = f_2(0) = 0$. By using (4.9),(4.10), we have

$$\begin{aligned}
\|f_1(x) - f_2(x), z\| &= \left\|4^{-n} f_1(2^n x) - 4^{-n} f_2(2^n(x)), z\right\| \\
&= 4^{-n} \|f_1(2^n x) - f_2(2^n x), z\| \\
&= 4^{-n} \|f_1(2^n x) - \mathsf{q}(2^n x) + \mathsf{q}(2^n x) - f_2(2^n x), z\| \\
&\le 4^{-n} [\|\mathsf{q}(2^n x) - f_1(2^n x), z\| + \|\mathsf{q}(2^n x) - f_2(2^n x), z\|] \\
&\le 2\frac{2^{nr}}{4^n}\left[\frac{2}{4-2^r}\theta \|x, z\|^r\right], \quad r < 2.
\end{aligned}$$

Taking $n \to \infty$, we get $\|f_1(x) - f_2(x), z\| = 0$, and thus $f_1(x) = f_2(x)$. □

Theorem 4.3. *Let $r > 2$, θ be a positive real number, X be a complex 2-normed space and Y be a 2-Banach space. Let $\mathsf{q} : X \to Y$ be a mapping satisfying $\mathsf{q}((1+i)x) = 2i\mathsf{q}(x)$ and the inequality*

$$\|\mathbb{C}\mathsf{q}(x, y), z\| \le \theta [\|x, z\|^r + \|y, z\|^r] \quad \text{for all } x, y, z \in X. \tag{4.11}$$

Then there exists a unique quadratic mapping $k : X \to Y$ such that

$$\|\mathsf{q}(x) - k(x), z\| \le \frac{2}{2^r - 4}\theta \|x, z\|^r \quad \text{for all } x, z \in X. \tag{4.12}$$

Proof. Define the sequence of functions $\{k_n\}$ by the formula

$$k_n(x) = 4^n \mathsf{q}(2^{-n} x) \quad \text{for all } x \in X,\ n \in \mathbb{N}. \tag{4.13}$$

Since $k(0) = 0$, by using Lemma 4.1, we have for all $x \in X$ and $n > m$,

$$\|k_n(x) - k_m(x), z\| \le \left\|4^n \mathsf{q}(2^{-n}x) - 4^m \mathsf{q}(2^{-m}x), z\right\| \le \sum_{j=l}^{m-1} \frac{4^j}{2^{rj+r}} [2\theta \|x, z\|^r].$$

Therefore $\{k_n(x)\}$ is a Cauchy sequence for every $x \in X$. Since Y is complete, $\{k_n(x)\}$ is convergent. Hence, there exists a mapping $k : X \to Y$ such that

$$k(x) = \lim_{n\to\infty} k_n(x), \quad x \in X.$$

Then, in the same way as in Theorem 4.2, it easy to verify that k is a quadratic function. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in inequality (4.3), we obtain (4.12).

To prove that k is unique, assume that there exist two quadratic functions $k_i : X \to Y$, $i = 1, 2$, such that

$$\|\mathsf{q}(x) - k_i(x), z\| \le \frac{2}{2^r - 4}\theta \|x, z\|^r. \tag{4.14}$$

Also, we have

$$k_i(x) = 4^n k_i\left(2^{-n} x\right). \tag{4.15}$$

Now, for every $x, z \in X$ by using (4.14), (4.15), we get

$$\begin{aligned}\|k_1(x) - k_2(x), z\| &= \left\|4^n k_1(2^{-n}x) - 4^n k_2(2^{-n}x), z\right\| \\ &= 4^n \left\|k_1(2^{-n}(x)) - k_2(2^{-n}(x)), z\right\| \\ &= 4^n \left\|k_1(2^{-n}x) - \mathsf{q}(2^{-n}(x)) + \mathsf{q}(2^{-n}(x)) - k_2(2^{-n}x), z\right\| \\ &\leq 4^n \left[\left\|\mathsf{q}(2^{-n}x) - k_1(2^{-n}x), z\right\| + \left\|\mathsf{q}(2^{-n}x) - k_2(2^{-n}x), z\right\|\right] \\ &\leq 2\frac{4^n}{2^{nr}} \left[\frac{2}{2^r - 4}\theta \|x, z\|^r\right], \quad ; r > 2.\end{aligned}$$

Taking the limit $n \to \infty$, we have $k_1(x) = k_2(x)$. Hence the result follows. □

Lemma 4.4. *Let X be a complex 2-normed space, Y be a complex 2-Banach space and let $r - \{1\} \in \mathbb{R}$ and θ be a positive real number. Let $\mathsf{q} : X \to Y$ be a mapping satisfying the mapping $\mathsf{q}((1+i)x) = 2i\mathsf{q}(x)$ and the inequality*

$$\|\mathbb{C}\mathsf{q}(x, y), z\| \leq \theta \|x, z\|^r . \|y, z\|^r \quad \textit{for all } x, y, z \in X. \tag{4.16}$$

Then, for all nonnegative integers l, m with $l < m$ and $x, z \in X$, we have

$$\left\|\frac{1}{4^l}\mathsf{q}(2^l x) - \frac{1}{4^m}\mathsf{q}(2^m x), z\right\| \leq \sum_{j=l}^{m-1} \frac{4^{rj}}{4^{j+1}} \left[\theta \|x, z\|^{2r}\right] \tag{4.17}$$

or

$$\left\|4^l \mathsf{q}(2^{-l}x) - 4^m \mathsf{q}(2^{-m}x), z\right\| \leq \sum_{j=l}^{m-1} \frac{4^j}{4^{rj+r}} \left[\theta \|x, z\|^{2r}\right]. \tag{4.18}$$

Proof. Since $\mathsf{q}((1+i)x) = 2i\mathsf{q}(x)$ for all $x \in X$, we can get $\mathsf{q}(0) = 0$ and $\mathsf{q}(2x) = 2i\mathsf{q}((1-i)x)$ for all $x \in X$.

Now, putting $x = y \neq 0$ in (4.16), we get

$$\|2\mathsf{q}((1+i)x) + 2\mathsf{q}((1-i)x), z\| \leq \theta \|x, z\|^{2r} .$$

Therefore, we have

$$\left\|\mathsf{q}(x) - \frac{1}{4}\mathsf{q}(2x), z\right\| \leq \frac{1}{4}\theta \|x, z\|^{2r} \quad \text{for all } x, z \in X. \tag{4.19}$$

It is easy to see that inequality (4.17) holds for all nonnegative integers l, m with $l < m$. Choosng $x = x/2$ in equation (4.19), we have

$$\left\|\mathsf{q}\left(\frac{x}{2}\right) - \frac{1}{4}\mathsf{q}(x), z\right\| \leq \frac{1}{4}\theta \left\|\frac{x}{2}, z\right\|^{2r}$$

$$\left\|\mathsf{q}(x) - 4\mathsf{q}\left(\frac{x}{2}\right), z\right\| \leq \theta \left\|\frac{x}{2}, z\right\|^{2r}. \tag{4.20}$$

It is easy to see that inequality (4.18) holds for all nonnegative integers l, m with $l < m$. □

Theorem 4.5. *Let X be a complex 2-normed space, Y be a complex 2-Banach space and let $r < 1$, θ be a positive real number. Let $\mathsf{q} : X \to Y$ be a mapping satisfying $\mathsf{q}((1+i)x) = 2i\mathsf{q}(x)$ and the inequality*

$$\|\mathbb{C}\mathsf{q}(x,y), z\| \leq \theta \|x,z\|^r \|y,z\|^r \quad \text{for all } x, y, z \in X. \tag{4.21}$$

Then there exists a unique quadratic mapping $f : X \to Y$ such that

$$\|\mathsf{q}(x) - f(x), z\| \leq \frac{\theta}{4-4^r} \|x,z\|^{2r} \quad \text{for all } x, z \in X. \tag{4.22}$$

Proof. Define the sequence of functions $\{g_n\}$ by the formula

$$f_n(x) = \frac{1}{4^n}\mathsf{q}(2^n x) \quad \text{for all } x \in X,\ n \in \mathbb{N}. \tag{4.23}$$

Firstly we have to prove that the sequence $\{f_n\}$ is a Cauchy sequence for every $x \in X$. For $x = 0$, it is trivial. Taking $0 \neq x \in X$ for $n < m$ and using Lemma 4.4, we have

$$\begin{aligned}\|f_n(x) - f_m(x), z\| &= \left\| \frac{1}{4^n}\mathsf{q}(2^n x) - \frac{1}{4^m}\mathsf{q}(2^m x), z \right\| \\ &\leq \sum_{j=n}^{m-1} \frac{4^{rj}}{4^{j+1}} \left[\theta \|x,z\|^{2r}\right] < \infty, \quad r < 1.\end{aligned}$$

Therefore the sequence $\{f_n(x)\}$ is a Cauchy sequence. Since Y is complete, then this sequence is convergent. Hence, we can define a mapping $f : X \to Y$ such that

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \left[\frac{1}{4^n}\mathsf{q}(2^n x)\right].$$

By (4.21), we get

$$\begin{aligned}\|\mathbb{C}f(x,y), z\| &= \lim_{n\to\infty} \frac{1}{2^n}\|\mathbb{C}\mathsf{q}(2^n x, 2^n y), z\| \\ &\leq \lim_{n\to\infty} \frac{4^{nr}}{4^n}\theta\|x,z\|^r\|y,z\|^r = 0 \quad \text{for all } x, y, z \in X.\end{aligned}$$

Thus $\mathbb{C}f(x,y) = 0$. By Proposition 2.9, the mapping $f : X \to Y$ is quadratic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in inequality (4.17), we obtain (4.22).

To prove that f is unique, assume that there exist two quadratic functions $f_i : X \to Y$, $i = 1, 2$, such that

$$\|\mathsf{q}(x) - f_i(x), z\| \leq \frac{\theta}{4-4^r} \|x,z\|^{2r}. \tag{4.24}$$

Also, we have

$$f_i(x) = 4^{-n} f_i(2^n x). \tag{4.25}$$

For every $x, z \in X$, we obtain that $f_1(0) = f_2(0) = 0$. Using (4.24),(4.25), we have

$$\begin{aligned}\|f_1(x) - f_2(x), z\| &= \left\|4^{-n} f_1(2^n x) - 2^{-n} f_2(2^n(x)), z\right\| \\ &= 4^{-n}\|f_1(2^n x) - f_2(2^n x), z\| \\ &= 4^{-n}\|f_1(2^n x) - \mathsf{q}(2^n x) + \mathsf{q}(2^n x) - f_2(2^n x), z\| \\ &\leq 4^{-n}\left[\|\mathsf{q}(2^n x) - f_1(2^n x), z\| + \|\mathsf{q}(2^n x) - f_2(2^n x), z\|\right] \\ &\leq 2\frac{2^{2nr}}{4^n}\left[\frac{\theta}{4-4^r}\|x, z\|^{2r}\right], \quad r < 1.\end{aligned}$$

Taking $n \to \infty$, we get $\|f_1(x) - f_2(x), z\| = 0$, and thus $f_1(x) = f_2(x)$. □

Theorem 4.6. *Let X be a complex 2-normed space, Y be a complex 2-Banach space and let $r > 1$, θ be a positive real number. Let $\mathsf{q} : X \to Y$ be a mapping satisfying $\mathsf{q}((1+i)x) = 2i\mathsf{q}(x)$ and*

$$\|\mathbb{C}\mathsf{q}(x, y), z\| \leq \theta\,\|x, z\|^r\,\|y, z\|^r \quad \text{for all } x, y, z \in X. \tag{4.26}$$

Then there exists a unique quadratic mapping $k : X \to Y$ such that

$$\|\mathsf{q}(x) - k(x), z\| \leq \frac{\theta}{4^r - 4}\,\|x, z\|^{2r} \quad \text{for all } x, z \in X. \tag{4.27}$$

Proof. Define the sequence of functions $\{k_n\}$ by the formula

$$k_n(x) = 4^n \mathsf{q}(2^{-n}x) \quad \text{for all } x \in X,\ n \in \mathbb{N}. \tag{4.28}$$

Since $k(0) = 0$, by using Lemma 4.4, we have for all $x, z \in X$ and $n > m$,

$$\|k_n(x) - k_m(x), z\| \leq \left\|4^n \mathsf{q}(2^{-n}x) - 4^m \mathsf{q}(2^{-m}x), z\right\| \leq \sum_{j=l}^{m-1} \frac{4^j}{4^{rj+r}}\left[\theta\|x, z\|^{2r}\right].$$

Therefore, $\{k_n(x)\}$ is a Cauchy sequence for every $x \in X$. Since Y is complete, then $\{k_n(x)\}$ is convergent. Hence, there exists a mapping $k : X \to Y$ such that

$$k(x) = \lim_{n\to\infty} k_n(x), \quad x \in X.$$

Then, in the same way as in Theorem 4.5, it easy to verify that k is a quadratic function. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in inequality (4.18), we obtain (4.27).

To prove that k is unique, assume that there exist two linear functions $k_i : X \to Y$, $i = 1, 2$, such that

$$\|\mathsf{q}(x) - k_i(x), z\| \leq \frac{\theta}{4^r - 4}\,\|x, z\|^{2r}. \tag{4.29}$$

Also, we have

$$k_i(x) = 4^n k_i\left(2^{-n}x\right). \tag{4.30}$$

Now, for every $x, z \in X$ by using (4.29), (4.30), we get

$$\begin{aligned}\|k_1(x) - k_2(x), z\| &= \left\|4^n k_1(2^{-n}x) - 4^n k_2(2^{-n}x), z\right\| \\ &= 4^n \left\|k_1(2^{-n}(x)) - k_2(2^{-n}(x)), z\right\| \\ &= 4^n \left\|k_1(2^{-n}x) - \mathsf{q}(2^{-n}(x)) + \mathsf{q}(2^{-n}(x)) - k_2(2^{-n}x), z\right\| \\ &\leq 4^n \left[\left\|\mathsf{q}(2^{-n}x) - k_1(2^{-n}x), z\right\| + \left\|\mathsf{q}(2^{-n}x) - k_2(2^{-n}x), z\right\|\right] \\ &\leq 2\frac{4^n}{2^{2nr}}\left[\frac{1}{(4^r - 4)}\theta \left\|x, z\right\|^{2r}\right], \quad r > 1.\end{aligned}$$

Taking the limit $n \to \infty$, we have $k_1(x) = k_2(x)$. Hence the result follows. □

Remark 4.7. In this paper, we have extended the main results of Cao et al. [9] (Theorem II.1. and Theorem II.3) and of Kwon et al. [33] (Theorems 2.1–2.4 , 3.1–3.4) in the framework of a complex 2-normed space (Theorems 3.2–3.3, 3.5–3.9 and 4.2–4.6). Also, we obtained the Hyers–Ulam stability of the additive and quadratic functional equations.

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Стійкість комплексних функціональних рівнянь у 2-банаховому просторі

Anshul Rana, Ravinder Kumar Sharma, and Sumit Chandok

У роботі ми одержуємо деякі результати для стійкості Хайерса–Улама наступних рівнянь

$$\mathsf{q}(x+iy)+\mathsf{q}(x-iy)+\mathsf{q}(y+ix)+\mathsf{q}(y-ix)=2\mathsf{q}(x)+2\mathsf{q}(y)$$

і

$$\mathsf{q}(x+iy)+\mathsf{q}(x-iy)+\mathsf{q}(y+ix)+\mathsf{q}(y-ix)=0$$

за у 2-банахових просторах.

Ключові слова: 2-нормовані простори, 2-банахові простори, стійкість Хайерса–Улама–Рассіаса, адитивне відображення, квадратичне рівняння