Journal of Mathematical Physics, Analysis, Geometry
2021, Vol. 17, No. 4, pp. 463–467
doi: https://doi.org/10.15407/mag17.04.463

Stability in the Marcinkiewicz Theorem

Alexandre Eremenko and Alexander Fryntov

Dedicated to the memory of I.V. Ostrovskii

Ostrovskii's generalization of the Marcinkiewicz theorem implies that if an entire characteristic functions of a probability distribution satisfies $\log\log M(r,f) = o(r)$ and is zero-free then the distribution is normal. We show that under the same growth condition, absence of zeros in a wide vertical strip implies that the distribution is close to a normal one. This generalizes and simplifies a recent result of Michelen and Sahasrabudhe.

Key words: characteristic function, ridge function, normal distribution

Mathematical Subject Classification 2010: 60E10

Following Linnik [5], an entire function f is called a *ridge function* if $|f(z)| \leq |f(i\,\mathrm{Im}\,z)|,\ z \in \mathbf{C}$. This definition is justified by Probability theory: characteristic functions of random variables are ridge functions when they are entire. We will apply the same name to subharmonic functions u in $\mathbf{C}$ satisfying

$$u(z) \leq u(i\,\mathrm{Im}\,z), \quad z \in \mathbf{C}. \tag{1}$$

Classical theorem of Marcinkiewicz [7] says that all ridge entire functions of finite order without zeros are of the form $\exp(-az^2 + biz + c)$, where $a > 0$, b is real and c is complex. This was generalized by Ostrovskii [9] who proved a conjecture of Linnik that the condition of finite order can be relaxed to

$$\log^+ \log |f(z)| = o(|z|), \quad z \to \infty.$$

This condition was further relaxed in [11] to

$$\liminf_{z\to\infty} \frac{\log^+ \log |f(z)|}{|z|} = 0. \tag{2}$$

Paper [10] contains a survey of further generalizations of Ostrovskii's result.

We prove a "stable version" of this theorem for entire functions which are free of zeros in vertical strips. A different approach to stability in the Marcinkiewicz was proposed in [2, 3].

Theorem 1. *If u is a ridge subharmonic function in $\mathbf{C}$ satisfying*

$$\liminf_{r\to\infty} \frac{\log\max\{u(ir), u(-ir)\}}{r} = 0, \tag{3}$$

is harmonic in the strip

$$S(\Delta) = \{z : |\operatorname{Re} z| < \Delta\} \tag{4}$$

and normalized by $u(0) = u_x(0) = u_y(0) = 0$ and $u_{yy}(0) = 1$, then

$$|u(z) + \operatorname{Re}(z^2/2)| \leq c_0|z|^3/\Delta, \quad |z| \leq \Delta/3, \tag{5}$$

where c_0 is an absolute constant.

Example $u(z) = \cosh y \cos y - 1$ shows that the growth condition (3) is best possible. A new proof of Linnik's conjecture is obtained by setting $u = \log|f|$ and $\Delta = \infty$.

As a corollary we obtain a generalization of the recent theorem by Michelen and Sahasrabudhe [8, Theorem 4.1]:

Theorem 2. *Let X be a random variable with average μ and standard deviation σ. Suppose that the characteristic function f_X is entire, satisfies (2), and is free of zeros in the strip $\{z : |\operatorname{Re} z| < \delta\}$. Then the distribution function F_{X^*} of the random variable $X^* = (X - \mu)/\sigma$ satisfies $|F_{X^*} - F_N|_\infty \leq \dfrac{c_1}{\sigma\delta}$, where c_1 is an absolute constant, and N is the standard normal distribution with characteristic function $f_N(z) = \exp(-z^2/2)$.*

This theorem was proved in [8] under the additional assumption that X takes values in the set $\{0, 1, \ldots, n\}$. We generalize the result and propose a shorter proof. We will use the

Phragmén–Lindelöf Theorem. *If a subharmonic function v in a strip S satisfies*

$$\liminf_{z\to\infty} \frac{\log^+ v(z)}{|z|} = 0, \tag{6}$$

and $v(z) \leq 0$, $z \in \partial S$, then $v(z) \leq 0$ in S.

Lemma 1. *If a harmonic function in a strip $S(\Delta)$ satisfies (3) and (1), then for all real y, the function $x \mapsto u(x + iy)$ is decreasing for $x \in [0, \Delta/2]$.*

Proof. Let us fix $s \in (0, \Delta/2)$ and let $z \mapsto z^*$ be the reflection with respect to the line $\operatorname{Re} z = s$, that is $z^* = 2s - \overline{z}$. We define $u^*(z) = u(z^*)$, and

$$v(z) = \max\{u(z), u^*(z)\}, \quad 0 < \operatorname{Re} z < 2s.$$

On the lines $\operatorname{Re} z = 0$ and $\operatorname{Re} z = s$ we have $v(z) \leq u(z)$. For a ridge function u, condition (3) implies (6) so u and v satisfy (6), and by the Phragmén–Lindelöf theorem we conclude that $v(z) \leq u(z)$ in the strip $\{z : 0 < \operatorname{Re} z < s\}$. On the other hand $v(z) \geq u(z)$ by definition, so

$$v(z) = u(z), \quad 0 < \operatorname{Re} z < s. \tag{7}$$

On the lines $\operatorname{Re} z = s$ and $\operatorname{Re} z = 2s$ we have $v(z) \leq u^*(z)$, so by a similar application of the the Phragmén–Lindelöf theorem we conclude that $v(z) \leq u^*(z)$ in the strip $\{z : s < \operatorname{Re} z < 2s\}$. On the other hand, $v(z) \geq u^*(z)$ by definition, so

$$v(z) = u^*(z), \quad s < \operatorname{Re} z < 2s. \tag{8}$$

Since $v(z)$ is subharmonic, we have $v_x(s-0) \leq v_x(s+0)$, and in view of (7), (8) we have $v_x(s-0) = u_x(s)$ and $v_x(s+0) = u_x^*(s) = -u_x(s)$, and so we obtain that $u_x(s) \leq -u_x(s)$ that is $u_x(s) \leq 0$, which proves the lemma. □

Lemma 2. *Let Q be the square,*

$$Q = \{x + iy : 0 < x < 2, \ |y| < 1\}, \tag{9}$$

and let $P(z, \zeta)$ be the Poisson kernel of Q, where $z = x + iy \in Q$, and $\zeta \in \partial Q$. Then for $\zeta \in \partial Q \backslash (-i, i)$ we have $P_x(0, \zeta) \geq c_2$, where c_2 is an absolute constant.

Lemma 3. *The family of harmonic functions in a vertical strip $S(\Delta)$ as in (4) satisfying (3), (1) and normalized both conditions $u(0) = u_y(0) = 0$ and $u_{yy}(0) = 1$, is uniformly bounded from above on every compact set $K \subset S(\Delta/2)$ by a constant depending only on K and Δ.*

Proof. By Lemma 1, harmonic functions $-u_x$ are positive in the right half of the strip, and $u_x(0, y) = 0$ in view of (1). Applying to them the Poisson representation in rectangles cQ where Q is defined in (9) and using Lemma 2, we obtain that the total measure in this representation is bounded. So u_x are uniformly bounded on compacts. We conclude that the analytic functions $u_x - iu_y$, are uniformly bounded on compacts. Since $u_x(0) = 0$ by the ridge property and $u_y(0) = 0$ by assumption, we conclude that functions u are uniformly bounded on compacts in $S(\Delta/2)$. This proves Lemma 3. □

Proof of Theorem 1. We may assume without loss of generality that $\Delta \geq 1$. Consider the expansion at 0:

$$u(z) = \operatorname{Re}\left(-z^2/2 + \sum_{n=3}^{\infty} a_n z^n\right).$$

Let

$$u_\Delta = \Delta^{-2} u(z\Delta) = \operatorname{Re}\left(-z^2 + \sum_{n=3}^{\infty} a_n \Delta^{n-2} z^n\right), \quad z \in S(1).$$

By Lemma 3, its coefficients are uniformly bounded, therefore $|a_n| \leq c_3 \Delta^{2-n}$, and

$$\sum_{n=3}^{\infty} |a_n||z^n| \leq c_3 \Delta^{-1} \frac{|z|^3}{1 - |z|/\Delta} \leq c_0 |z|^3/\Delta, \quad \text{when} \quad |z| \leq \Delta/3.$$

This proves Theorem 1. □

Derivation of Theorem 2 from Theorem 1. Following [8] and [4], we use the Berry–Esseen inequality

$$\sup_{t\in\mathbf{R}} |F_{X^*}(t) - F_Z(t)| \leq \frac{1}{\pi}\int_{-T}^{T}\left|\frac{f_{X*}(x) - e^{-x^2/2}}{x}\right| dx + \frac{c}{T}, \qquad (10)$$

where c is an absolute constant. This estimate can be found in [1, Ch. XVI, 3, Lemma 2] and in [6, Lemma 8.2.2].

We set $\Delta = \delta\sigma$. The statement of Theorem 2 is meaningful only when Δ is large, so we assume that $\Delta > c_0$, where c_0 is the constant in Theorem 1.

We are going to apply Theorem 1 to $u = \log|f_{X^*}|$, where f_{X*} is the characteristic function of X^*. Since X^* is normalized, u is normalized as required in Theorem 1. Since by assumption the characteristic function f_X has no zeros in the strip $S(\delta)$, the function f_{X^*} has no zeros in the strip $S(\Delta)$. Then Theorem 1 implies that

$$f_{X^*}(x) = \exp(-x^2/2 + R(x)), \quad \text{where} \quad |R(x)| \leq c_0|x|^3/\Delta,\ |x| < \Delta/2.$$

Set $T = \Delta/(4c_0)$ in (10). To estimate the integral in (10) we break it into two parts:

Let $a := (\Delta/c_0)^{1/3} \geq 1$. When $|x| < a$, we have $|R(x)| \leq 1$, so

$$|e^{R(x)} - 1| \leq 2|R(x)| \leq 2c_0x^3/\Delta,$$

so

$$\int_{-a}^{a}\left|\frac{f_{X^*}(x) - e^{-x^2/2}}{x}\right| dx = \frac{2c_0}{\Delta}\int_{-\infty}^{\infty} e^{-x^2/2}x^2 dx \leq c_5/\Delta.$$

When $|x| \in [a, T]$ we use $f_{X^*}(x) = \exp(-x^2/2 + R(x))$ and

$$x^2(-1/2 + |x|c_0/\Delta) \leq x^2(-1/2 + 1/4) = -x^2/4.$$

So

$$\int_{|x|\in[a,T]}\left|\frac{f_{X*}(x) - e^{-x^2/2}}{x}\right| dx \leq 4\int_{a}^{\infty} e^{-x^2/4}dx \leq c_6/\Delta.$$

This completes the proof. □

Acknowledgments. The authors thank L. Golinskii, F. Nazarov and M. Sodin for useful discussions.

References

[1] W. Feller, *An introduction to probability theory and its applications*, Vol. II, J. Wiley NY, 1971.

[2] L. Golinskii, *Stability estimates in a theorem of J. Marcinkiewicz*, J. Soviet Math. **57** (1991), No. 4, 3193–3209.

[3] L. Golinskii and G. Chistyakov, *On stability estimates in the Marcinkiewicz theorem and its generalization*. Entire and subharmonic functions, Amer. Math. Soc., Providence, RI, 1992, 243–250.

[4] J. Lebowitz, B. Pittel, D. Ruelle and E. Speer, *Central limit theorems, Lee–Young zeros, and graph-counting polynomials*, J. Comb. Theory, **A 141** (2016) 147–183.

[5] Yu. Linnik, *Decomposition of probability distributions*, Dover Publ., New York, 1964.

[6] Yu. Linnik and I. Ostrovskii, *Decompositions of random variables and vectors*, AMS, Providence, RI, 1977.

[7] J. Marcinkiewicz, *Sur une propriété de la loi de Gauss*, Math. Z. **4** (1939), No. 1, 612–618.

[8] M. Michelen and J. Sahasrabudhe, *Central limit theorems and the geometry of polynomials*, https://arxiv.org/abs/2106.14078.

[9] I. V. Ostrovskii, *On the application of a law established by Wiman and Valiron to the investigation of characteristic functions of probability laws*, Sov. Math., Dokl. 3, 433–437 (1962).

[10] A. Vishnyakova, I. Ostrovskii and A. Ulanovskii, *On a conjecture of YU. V. Linnik*, Leningrad Math. J., **2** (1991), No. 4, 765–773.

[11] V. Zimogljad, *The growth of entire functions that satisfy special inequalities*, Teor. Funkcii Funkcional. Anal. i Prilozen. **6** (1968), 30–41.

Received June 26, 2021.

Alexandre Eremenko,
Purdue University, West Lafayette IN 47907, USA,
E-mail: eremenko@purdue.edu

Alexander Fryntov,
6198 Townswood ct., Mississauga ON, L5N2L4, Canada,

Стійкість у теоремі Марцинкевича

Alexandre Eremenko and Alexander Fryntov

Узагальнення Островського теореми Марцинкевича стверджує, що якщо ціла характеристична функція ймовірнісного розподілу задовольняє умову $\log^+ \log|f(z)| = o(|z|),\ r \to \infty$, і не має нулів, то розподіл є нормальним. Ми доводимо, що при тому самому обмеженні на зростання з відсутності нулів у широкій вертикальній смузі випливає, що розподіл є близьким до нормального. Це узагальнює один результат Мішелена і Сахастрабудхе та спрощує його доведення.

Ключові слова: характеристична функція, хребтова функція, нормальний розподіл