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One Class of Linearly Growing C_0-Groups

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We consider the special class of C_0-groups from [12], whose generators are unbounded, have a pure point imaginary spectrum and a corresponding dense and minimal family of eigenvectors, which however does not form a Schauder basis. We obtain two-sided estimates for norms of C_0-groups from this class and thus prove that these C_0-groups have linear growth. Moreover, we show that C_0-groups from the considered class do not have any maximal asymptotics. This means that the fastest growing orbits do not exist.

Key words: C_0-group, linear growth, maximal asymptotics, XYZ theorem

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1. Introduction

In 2017, G.M. Sklyar and V. Marchenko [12] constructed classes of C_0-groups with generators possessing a pure point imaginary spectrum and a dense minimal family of eigenvectors, which is however not uniformly minimal, and hence this family does not form a Schauder basis. For definitions and various properties of Schauder bases and decompositions we refer to [6]. By the spectral XYZ theorem (see Theorem 1.1 in [19], Theorem 1.1 in [20] or XYZ Theorem in [13]) points of the spectrum of such generators must be non-separated, so they behave in [12] like

$$i \ln n, \quad n \in \mathbb{N},$$

and cluster at $i\infty$. The XYZ theorem is a spectral theorem for nonselfadjoint operators providing us with general sufficient conditions for eigenvectors (or invariant subspaces) of the generator of the C_0-group to constitute a Riesz basis in a Hilbert space, see [19, 20] for its formulations and proofs and [12, 13, 15] for discussions around it. For equivalent definitions and various properties of Riesz bases we refer to [2,6,7]. Recently in [13] G.M. Sklyar and V. Marchenko used the constructed classes of C_0-groups with non-basis family of eigenvectors from [12] to prove that XYZ Theorem is sharp in a sense that none of its conditions can be weakened or removed, see Section 2 in [13].

Throughout the paper we will use the notations from [12, 15]. Consider a separable Hilbert space H with norm $\|\cdot\|$ and fix an arbitrary Riesz basis $\{e_n\}_{n=1}^{\infty}$

in H. Then

$$H_1(\{e_n\}) = \left\{ x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n : \{c_n - c_{n-1}\}_{n=1}^{\infty} \in \ell_2,\ c_0 = 0 \right\}$$

is a Hilbert space of formal series $(\mathfrak{f}) \sum\limits_{n=1}^{\infty} c_n e_n$ with norm

$$\|x\|_1 = \left\| (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n \right\|_1 = \left\| \sum_{n=1}^{\infty} (c_n - c_{n-1}) e_n \right\|.$$

By $\mathcal{S}_1$, we denote the following class of real sequences:

$$\mathcal{S}_1 = \left\{ \{f(n)\}_{n=1}^{\infty} \subset \mathbb{R} : \lim_{n\to\infty} f(n) = +\infty; \{n(f(n) - f(n-1))\}_{n=1}^{\infty} \in \ell_\infty \right\},$$

where $f(0) = 0$. One clearly has $\{\ln n\}_{n=1}^{\infty} \in \mathcal{S}_1$ and $\{\sqrt{n}\}_{n=1}^{\infty} \notin \mathcal{S}_1$.

The construction of C_0-groups with non-basis family of eigenvectors from [12] on the space $H_1(\{e_n\})$ is given by the following theorem.

Theorem 1.1 (The case $k = 1$ in Theorem 11 from [12])**.** *Assume that $\{e_n\}_{n=1}^{\infty}$ is a Riesz basis of H. Then $\{e_n\}_{n=1}^{\infty}$ is a complete and minimal sequence in $H_1(\{e_n\})$ but does not form a Schauder basis of $H_1(\{e_n\})$, and for each $\{f(n)\}_{n=1}^{\infty} \in \mathcal{S}_1$, the operator $A_1 : H_1(\{e_n\}) \supset D(A_1) \mapsto H_1(\{e_n\})$, defined by*

$$A_1 x = A_1 \left((\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n \right) = (\mathfrak{f}) \sum_{n=1}^{\infty} i f(n) \cdot c_n e_n,$$

with domain

$$D(A_1) = \left\{ x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n \in H_1(\{e_n\}) : \{f(n)c_n - f(n-1)c_{n-1}\}_{n=1}^{\infty} \in \ell_2 \right\},$$

generates the C_0-group on $H_1(\{e_n\})$, which acts for every $t \in \mathbb{R}$ by the formula

$$e^{A_1 t} x = e^{A_1 t} (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n = (\mathfrak{f}) \sum_{n=1}^{\infty} e^{itf(n)} c_n e_n. \tag{1.1}$$

It was proved in [13] that for the spectrum $\sigma(A_1)$ of the operator A_1 we have

$$\sigma(A_1) = \sigma_p(A_1) = \{if(n)\}_{n=1}^{\infty}.$$

Suppose that $f(n) = \ln n$, $n \in \mathbb{N}$. Then in [14] and [15] the authors obtained the following two-sided estimate for the norm of C_0-group $\left\{e^{A_1 t}\right\}_{t\in\mathbb{R}}$ from Theorem 1:

$$C|t| \le \left\| e^{A_1 t} \right\| \le \mathfrak{p}(|t|), \tag{1.2}$$

where $C > 0$ and $\mathfrak{p}$ is a linear function with positive coefficients, for the proof, see Theorem 6 in [15]. Thus, it was proved that for the case when $f(n) = \ln n$, $n \in \mathbb{N}$, the C_0-group $\left\{e^{A_1 t}\right\}_{t\in\mathbb{R}}$ has exactly a linear growth. Note that C_0-groups of linear growth arise naturally in the theory and applications of evolution equations, see, e.g., [1], [3], [16], [18]. A careful analysis of the scheme of the proof of Theorem 6 in [15] leads to a more general result including more general behaviour of the spectrum of the generator A_1 from Theorem 1.1. This case we discuss in details in Section 2 and thus present one class of linearly growing C_0-groups on Hilbert spaces $H_1(\{e_n\})$.

It was also proved in [15] that C_0-semigroups $\left\{e^{\pm A_1 t}\right\}_{t\geq 0}$, for the case when $f(n) = \ln n$, $n \in \mathbb{N}$, do not have maximal asymptotics. Thus, on the one hand,

$$\left\|e^{A_1 t}\right\| \sim c(A_1)\,|t|, \quad t \to \pm\infty,$$

where $c(A_1)$ is a constant depending on A_1, but, on the other hand, for all $x \in H_1(\{e_n\})$ we have that

$$\lim_{t\to\pm\infty} \frac{\left\|e^{A_1 t}x\right\|}{|t|} = 0.$$

The second aim of the paper is to show, using (1.2), that C_0-semigroups $\left\{e^{\pm A_1 t}\right\}_{t\geq 0}$ for the case of more general behaviour of the spectrum of the generator A_1 from Theorem 1.1 also do not have maximal asymptotics.

The construction of C_0-groups with non-basis family of eigenvectors was also presented in [12] on certain Banach spaces $\ell_{p,1}(\{e_n\})$, $p > 1$. The space $\ell_{p,1}(\{e_n\})$, $p > 1$, is a Banach space of formal series $(\mathfrak{f})\sum_{n=1}^{\infty} c_n e_n$,

$$\ell_{p,1}(\{e_n\}) = \left\{x = (\mathfrak{f})\sum_{n=1}^{\infty} c_n e_n : \{c_n - c_{n-1}\}_{n=1}^{\infty} \in \ell_p\right\}, \quad p > 1,$$

where $c_0 = 0$ and $\{e_n\}_{n=1}^{\infty}$ is a symmetric basis of the corresponding ℓ_p, $p > 1$, with appropriate norm, defined similarly to the case of $H_1(\{e_n\})$. The concept of a symmetric basis was first introduced and studied by I. Singer [10] in connection with one Banach problem from isomorphic theory of Banach spaces. For definition and various properties of symmetric bases see, e.g., [4,8–10]. Note that in Hilbert spaces the concepts of a Riesz basis and a symmetric basis coincide, see e.g., [6]. Since the construction of C_0-groups with non-basis family of eigenvectors on the Banach space $\ell_{p,1}(\{e_n\})$, $p > 1$, is similar to the construction on $H_1(\{e_n\})$, the third purpose of the paper is to obtain similar results for the case of corresponding C_0-groups defined on the Banach space $\ell_{p,1}(\{e_n\})$, $p > 1$.

2. A class of linearly growing C_0-groups defined on $H_1(\{e_n\})$

By Δ, we denote the backward difference operator

$$\Delta = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & \dots \\ 0 & -1 & 1 & 0 & \dots \\ 0 & 0 & -1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

by $\Delta\{\alpha_n\}_{n=1}^{\infty}$, the sequence $\{\alpha_n - \alpha_{n-1}\}_{n=1}^{\infty}$ and by $\Delta\alpha_n$, the n-th element of the sequence $\{\alpha_n - \alpha_{n-1}\}_{n=1}^{\infty}$, i.e. $\Delta\alpha_n = \alpha_n - \alpha_{n-1}$, $n \in \mathbb{N}$.

The main result of the paper is formulated as follows.

Theorem 2.1. *Let $\{e^{A_1 t}\}_{t\in\mathbb{R}}$ be the C_0-group from Theorem 1.1, defined on $H_1(\{e_n\})$, where $\{f(n)\}_{n=1}^{\infty} \in \mathcal{S}_1$. Assume that there exists a constant $K > 0$ such that for each $n \in \mathbb{N}$ we have*

$$n\,|\Delta f(n)| \geq K. \tag{2.1}$$

Then the C_0-group $\{e^{A_1 t}\}_{t\in\mathbb{R}}$ has a linear growth, i.e., there exists a linear function $\mathfrak{l}$ with positive coefficients and a constant $C > 0$ such that for all $t \in \mathbb{R}$ we have

$$C|t| \leq \left\|e^{A_1 t}\right\| \leq \mathfrak{l}(|t|). \tag{2.2}$$

Proof. The right-hand side of inequality (2.2) follows from Proposition 12 in [12].

To prove the left-hand side of inequality (2.2), we use the scheme of the proof of Theorem 6 from [15]. For the sake of completeness, we recall the full scheme of this proof. First, we consider a one-parameter family of sequences

$$a_n^{\beta} = \sum_{k=1}^{n} k^{-\beta}, \quad n \in \mathbb{N},\ \beta \in \left(\frac{1}{2}, \frac{3}{4}\right), \tag{2.3}$$

and note that for every $\beta \in \left(\frac{1}{2}, \frac{3}{4}\right)$ and each $n \in \mathbb{N}$ we have

$$\frac{2}{7}n^{1-\beta} \leq \frac{2}{7}(n+1)^{1-\beta} \leq a_n^{\beta} \leq 4n^{1-\beta}, \tag{2.4}$$

see [15] for details.

The next step of the proof is to consider a one-parameter family $x_\beta \in H_1(\{e_n\})$, $\beta \in \left(\frac{1}{2}, \frac{3}{4}\right)$, generated by sequences (2.3) and defined as follows:

$$x_\beta = (\mathfrak{f}) \sum_{n=1}^{\infty} a_n^{\beta} e_n. \tag{2.5}$$

Since the sequence $\{e_n\}_{n=1}^{\infty}$ constitutes a Riesz basis of the initial Hilbert space H, there exist constants $M \geq m > 0$ such that for each

$$y = \sum_{n=1}^{\infty} c_n e_n \in H$$

we have

$$m\sum_{n=1}^{\infty}|c_n|^2 \le \|y\|^2 \le M\sum_{n=1}^{\infty}|c_n|^2, \tag{2.6}$$

see, e.g., [2,6].

Further, by (2.6), we have that

$$\frac{m}{2\beta-1} \le \|x_\beta\|_1^2 \le \frac{3M}{2}\frac{1}{2\beta-1}, \tag{2.7}$$

see [15] for details.

The next step of the proof is to estimate from below the norm of the C_0-group, $\|e^{A_1 t}\|$. To this end, we use a one-parameter family $x_\beta \in H_1(\{e_n\})$, defined by (2.3), (2.5), fix arbitrary $t \in \mathbb{R}$ and, using (2.6), note that

$$\begin{aligned}
\left\|e^{A_1 t}x_\beta\right\|_1^2 &= \left\|(\mathfrak{f})\sum_{n=1}^{\infty} e^{itf(n)}a_n^\beta e_n\right\|_1^2 \\
&\ge m\left(1+\sum_{n=1}^{\infty}\left|a_{n+1}^\beta e^{itf(n+1)} - a_n^\beta e^{itf(n)}\right|^2\right) \\
&= m\left(1+\sum_{n=1}^{\infty}\left|a_{n+1}^\beta\left(e^{itf(n+1)}-e^{itf(n)}\right)+e^{itf(n)}\left(a_{n+1}^\beta - a_n^\beta\right)\right|^2\right) \\
&= m\left(1+\sum_{n=1}^{\infty}\left|a_{n+1}^\beta\left(e^{it(f(n+1)-f(n))}-1\right)+\frac{1}{(n+1)^\beta}\right|^2\right) \\
&\ge m\left(1+\sum_{n=1}^{\infty}\left|a_{n+1}^\beta\right|^2 \sin^2\left(t\left(f(n+1)-f(n)\right)\right)\right).
\end{aligned}$$

Since $\{f(n)\}_{n=1}^\infty \in \mathcal{S}_1$, there exists a constant $L>0$ such that for all $n \in \mathbb{N}$,

$$n\left|\Delta f(n)\right| \le L.$$

Hence, for all $t \in \mathbb{R}$ and each $n \in \mathbb{N}$,

$$\left|t\left(f(n+1)-f(n)\right)\right| \le \frac{L|t|}{n},$$

and thus for all $n \ge L|t|$ we obtain

$$\left|t\left(f(n+1)-f(n)\right)\right| \le 1. \tag{2.8}$$

Since for all $s \in [0,1]$ we have

$$\sin s \ge \frac{s}{2}, \tag{2.9}$$

we infer, applying (2.8), (2.9) and (2.1), that for arbitrary $t \in \mathbb{R}$ and for all $n \ge L|t|$,

$$\sin^2\left(t\left(f(n+1)-f(n)\right)\right) \ge \frac{t^2}{4}\left(f(n+1)-f(n)\right)^2 = \frac{t^2}{4}\left(\Delta f(n)\right)^2 \ge \frac{K^2t^2}{4n^2}.$$

Thus we can continue the estimation for $\left\|e^{A_1 t}x_\beta\right\|_1$ and, using (2.4), obtain that

$$\begin{aligned}\left\|e^{A_1 t}x_\beta\right\|_1^2 - m &\geq m \sum_{n\geq L|t|} \left|a_{n+1}^\beta\right|^2 \sin^2\left(t\left(f(n+1)-f(n)\right)\right) \\ &\geq m \sum_{n\geq L|t|} \frac{K^2\left|a_{n+1}^\beta\right|^2 t^2}{4n^2} \geq \frac{4m}{49}\cdot\frac{K^2t^2}{4}\sum_{n\geq L|t|}\frac{(n+1)^{2-2\beta}}{n^2} \\ &\geq \frac{mK^2t^2}{49}\sum_{n\geq L|t|}(n+1)^{-2\beta} \geq \frac{mK^2t^2}{49}\int_{L|t|+1}^{\infty}(s+1)^{-2\beta}ds \\ &= \frac{mK^2t^2}{49}\frac{1}{2\beta-1}\frac{1}{(L|t|+2)^{2\beta-1}}.\end{aligned}$$

By applying of (2.7), we arrive at

$$\frac{\left\|e^{A_1 t}x_\beta\right\|_1^2}{\left\|x_\beta\right\|_1^2} \geq \frac{2mK^2t^2}{147M(L|t|+2)^{2\beta-1}} + \frac{m}{\left\|x_\beta\right\|_1^2} \geq \frac{2mK^2t^2}{147M(L|t|+2)^{2\beta-1}}.$$

Finally, the latter estimate leads for all $t\in\mathbb{R}$ to the following:

$$\begin{aligned}\left\|e^{A_1 t}\right\|^2 &= \sup_{x\in H_1(\{e_n\})}\frac{\left\|e^{A_1 t}x\right\|_1^2}{\left\|x\right\|_1^2} \geq \sup_{\beta\in\left(\frac12,\frac34\right)}\frac{\left\|e^{A_1 t}x_\beta\right\|_1^2}{\left\|x_\beta\right\|_1^2} \\ &\geq \lim_{\beta\to+\frac12}\left(\frac{2mK^2t^2}{147M(L|t|+2)^{2\beta-1}}\right) = \frac{2mK^2t^2}{147M},\end{aligned}$$

and thus (2.2) is proved with $C=\sqrt{\frac{2m}{147M}}K$. □

Remark 2.2. Note that for the case when $f(n)=\ln n$, $n\in\mathbb{N}$, condition (2.1) obviously holds. For this case, Theorem 2.1 was first obtained in [14] and proved in [15].

Remark 2.3. As it was shown in [3], C_0-groups corresponding to abstract wave equations also have linear growth. Let $n\in\mathbb{N}$. The partial case of abstract wave equations is the classical d'Alembert wave equation

$$\frac{\partial^2 u(t,x)}{\partial t^2} = \Delta u(t,x), \quad t\in\mathbb{R},\ x\in\mathbb{R}^n,$$

on the space $L_2(\mathbb{R}^n)$, where Δ is the usual Laplacian, see [3] for details.

3. A class of linearly growing C_0-groups defined on Banach spaces $\ell_{p,1}(\{e_n\})$, $p > 1$

3.1. Preliminary constructions. Let $\{e_n\}_{n=1}^{\infty}$ be an arbitrary symmetric basis of ℓ_p, $p > 1$. Then $\ell_{p,1}(\{e_n\})$, $p > 1$, is a Banach space of formal series $(\mathfrak{f}) \sum\limits_{n=1}^{\infty} c_n e_n$,

$$\ell_{p,1}(\{e_n\}) = \left\{ x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n : \ \{c_n\}_{n=1}^{\infty} \in \ell_p(\Delta) \right\},$$

where

$$\ell_p(\Delta) = \{x = \{\alpha_n\}_{n=1}^{\infty} \subset \mathbb{C} : \ \Delta x \in \ell_p\}, \quad p > 1.$$

By Proposition 5 in [12], we have that $\overline{\mathrm{Lin}}\{e_n\}_{n=1}^{\infty} = \ell_{p,1}(\{e_n\})$, the sequence $\{e_n\}_{n=1}^{\infty}$ is minimal but $\{e_n\}_{n=1}^{\infty}$ is not uniformly minimal in $\ell_{p,1}(\{e_n\})$, hence it does not form a Schauder basis of $\ell_{p,1}(\{e_n\})$. We refer to Section 2.2 in [12] for more details.

Consider the operator

$$\widetilde{A_1} : \ell_{p,1}(\{e_n\}) \supset D\left(\widetilde{A_1}\right) \mapsto \ell_{p,1}(\{e_n\}),$$

defined on a Banach space $\ell_{p,1}(\{e_n\})$, $p > 1$, as follows:

$$\widetilde{A_1} x = \widetilde{A_k} (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n = (\mathfrak{f}) \sum_{n=1}^{\infty} i f(n)\, c_n e_n, \tag{3.1}$$

where $x \in D\left(\widetilde{A_1}\right)$, $\{f(n)\}_{n=1}^{\infty} \in \mathcal{S}_1$ and $\{e_n\}_{n=1}^{\infty}$ is a symmetric basis of the initial Banach space ℓ_p, $p > 1$, with domain

$$D\left(\widetilde{A_1}\right) = \left\{ x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n \in \ell_{p,1}(\{e_n\}) : \ \{f(n) \cdot c_n\}_{n=1}^{\infty} \in \ell_p(\Delta) \right\}. \tag{3.2}$$

By virtue of Theorem 16 in [12], the operator $\widetilde{A_1}$ generates the C_0-group $\left\{ \widetilde{e^{A_1 t}} \right\}_{t \in \mathbb{R}}$ on $\ell_{p,1}(\{e_n\})$, $p > 1$, which acts on $\ell_{p,1}(\{e_n\})$ for every $t \in \mathbb{R}$ by the formula

$$\widetilde{e^{A_1 t}} x = \widetilde{e^{A_1 t}} (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n = (\mathfrak{f}) \sum_{n=1}^{\infty} e^{itf(n)} c_n e_n. \tag{3.3}$$

3.2. A class of linearly growing C_0-groups on $\ell_{p,1}(\{e_n\})$, $p > 1$. For the case of the construction of C_0-groups with non-basis family of eigenvectors from [12] on a Banach space $\ell_{p,1}(\{e_n\})$, $p > 1$, we obtain the following theorem on their linear growth, similar to Theorem 2.1.

Theorem 3.1. *Let* $\left\{\widetilde{e^{A_1 t}}\right\}_{t\in\mathbb{R}}$ *be the* C_0*-group given by* (3.3) *, defined on a Banach space* $\ell_{p,1}(\{e_n\})$ *,* $p>1$*, where* $\{f(n)\}_{n=1}^{\infty}\in\mathcal{S}_1$*. Assume that there exists* $K>0$ *such that for each* $n\in\mathbb{N}$ *we have* (2.1)*, i.e.,*

$$n\left|\Delta f(n)\right|\geq K.$$

Then the C_0*-group* $\left\{\widetilde{e^{A_1 t}}\right\}_{t\in\mathbb{R}}$ *has a linear growth, i.e., there exists a linear function* $\widetilde{\mathfrak{l}}$ *with positive coefficients and a constant* $C>0$ *such that for all* $t\in\mathbb{R}$ *we have*

$$C|t|\leq\left\|\widetilde{e^{A_1 t}}\right\|\leq\widetilde{\mathfrak{l}}(|t|). \tag{3.4}$$

Proof. The right-hand side of inequality (3.4) follows from Proposition 17 in [12].

To prove the left-hand side of inequality (3.4), we first note that if $\{e_n\}_{n=1}^{\infty}$ is a symmetric basis of the Banach space ℓ_p, $p\geq 1$, then there exist constants $M\geq m>0$ such that for each

$$z=\sum_{n=1}^{\infty}c_n e_n\in\ell_p$$

we have

$$m\sum_{n=1}^{\infty}|c_n|^p\leq\|z\|^p\leq M\sum_{n=1}^{\infty}|c_n|^p, \tag{3.5}$$

i.e., a two-sided estimate similar to (2.6), see Proposition 4 in [12] and [4] for more details. Thus the proof of the left-hand side of inequality (3.4) repeats the lines of the proof of the left-hand side of inequality (2.2) in Theorem 2.1. For $p\geq 2$, one just needs to consider a one-parameter family

$$a_n^{\beta},\quad n\in\mathbb{N},\ \beta\in\left(\frac{1}{p},1-\frac{1}{p^2}\right)$$

instead of (2.3), for $p\in(1,2]$, a family

$$a_n^{\beta},\quad n\in\mathbb{N},\ \beta\in\left(\frac{1}{p},1-\frac{1}{q^2}\right),$$

where q is a number satisfying $\frac{1}{p}+\frac{1}{q}=1$. Then one needs to put into play a one-parameter family $x_\beta\in\ell_{p,1}(\{e_n\})$, defined as in (2.5), with the corresponding interval for β depending on p, and to estimate the norm of the C_0-group from below. The necessity to control the convergence of the series

$$\sum_{n=1}^{\infty}n^{-p\beta}$$

at the second step of the proof together with the positivity of the power $p\beta-1$ for all β from the interval at the end of the proof leads to the need of distinction of intervals for $p\geq 2$ and $p\in(1,2]$, for details, see the proof of Theorem 2.1. □

Remark 3.2. Note that C_0-groups generated by certain perturbations of generators of uniformly bounded C_0-groups on Banach spaces grow at most linearly in t, for details, see Corollary 2 in [5].

4. The lack of maximal asymptotics for linearly growing C_0-groups on spaces $H_1(\{e_n\})$ and $\ell_{p,1}(\{e_n\})$, $p>1$

The question on the existence of maximal asymptotics for a C_0-semigroup $\{e^{At}\}_{t\geq 0}$, or for the corresponding abstract linear differential equation

$$\begin{cases} \dot{x}(t) = Ax(t), & t\geq 0, \\ x(0) = x_0, \end{cases} \tag{4.1}$$

on a Banach space X, as the existence of its fastest growing in time t weak solution $e^{At}x_0$, $t\geq 0$, $x_0\in X$, was first formulated by G. Sklyar in 2010, see [11]. The definition of a maximal asymptotics for a C_0-semigroup is the following.

Definition 4.1 ([11])**.** The C_0-semigroup $\{e^{At}\}_{t\geq 0}$ (or the corresponding abstract linear differential equation (4.1)) on a Banach space X has a maximal asymptotics (a real and positive function $f(t)$, $t\geq 0$) provided that

(1) for some $a\geq 0$ and for each $x\in X$, the function

$$\frac{\|e^{At}x\|}{f(t)}$$

is bounded for all $t\in[a,+\infty)$;

(2) there exists at least one $x_0\in X$ such that

$$\lim_{t\to+\infty}\frac{\|e^{At}x_0\|}{f(t)} = 1.$$

Clearly, if $f(t)$ is a maximal asymptotics of a C_0-semigroup $\{e^{At}\}_{t\geq 0}$, then $cf(t)$ for any $c>0$ also is a maximal asymptotics of a C_0-semigroup $\{e^{At}\}_{t\geq 0}$.

If $A:\mathbb{R}^n\to\mathbb{R}^n$ is a finite dimensional linear operator, then the associated C_0-semigroup $\{e^{At}\}_{t\geq 0}$ always has the maximal asymptotics

$$f(t) = e^{\mu t}t^{N-1},$$

where $\mu = \max\limits_{\lambda\in\sigma(A)}\Re\lambda$, $\sigma(A)$ is the spectrum of A, and $N\leq n$ is the maximal geometric multiplicity of an eigenvalue of A with a real part μ. For infinite dimensional case, even in the class of bounded operators A, there may exist corresponding C_0-semigroups $\{e^{At}\}_{t\geq 0}$ without any maximal asymptotics, for details, see [11].

We recall that the growth bound ω_0 of the C_0-semigroup $\{e^{At}\}_{t\geq 0}$ on a Banach space can be defined as the following limit:

$$\omega_0 = \lim_{t\to+\infty}\frac{\ln\|e^{At}\|}{t}.$$

The main result on the lack of maximal asymptotics for linearly growing C_0-groups on the spaces $H_1(\{e_n\})$ and $\ell_{p,1}(\{e_n\})$, $p>1$, from Theorem 2.1 and Theorem 3.1 is formulated as follows.

Theorem 4.2. *Let* $\left\{e^{A_1 t}\right\}_{t\in\mathbb{R}}$ *be the* C_0*-group from Theorem 2.1, defined on* $H_1(\{e_n\})$, *and* $\left\{\widetilde{e^{A_1 t}}\right\}_{t\in\mathbb{R}}$ *be the* C_0*-group (3.3) from Theorem 3.1, defined on a Banach space* $\ell_{p,1}(\{e_n\})$, *where* $p>1$. *Then the* C_0*-semigroups* $\left\{e^{\pm A_1 t}\right\}_{t\geq 0}$ *and* $\left\{\widetilde{e^{\pm A_1 t}}\right\}_{t\geq 0}$ *do not have a maximal asymptotics.*

Proof. To prove this theorem, we use Theorem 12 from [15], a new theorem on the lack of maximal asymptotics for C_0-semigroups on Banach spaces, see also [17] for its proof.

By virtue of Theorem 2.1 and Theorem 3.1, we obtain that

$$\omega_0=\lim_{t\to+\infty}\frac{\ln\left\|e^{A_1 t}\right\|}{t}=0=\lim_{t\to+\infty}\frac{\ln\left\|\widetilde{e^{A_1 t}}\right\|}{t}=\widetilde{\omega_0},$$

where ω_0 is the growth bound of the C_0-semigroup $\left\{e^{A_1 t}\right\}_{t\geq 0}$ and $\widetilde{\omega_0}$ is the growth bound of the C_0-semigroup $\left\{\widetilde{e^{A_1 t}}\right\}_{t\geq 0}$. Since by Theorem 3.1 in [13],

$$\sigma(A_1)=\sigma_p(A_1)=\{if(n)\}_{n=1}^{\infty}\subset i\mathbb{R},$$

Condition 1 of Theorem 12 from [15] is satisfied for the C_0-semigroup $\left\{e^{A_1 t}\right\}_{t\geq 0}$. Analogously, by Theorem 3.2 in [13], this condition holds for $\left\{\widetilde{e^{A_1 t}}\right\}_{t\geq 0}$.

Further we note that for any $n\in\mathbb{N}$ the eigenspace, corresponding to the point $if(n)\in\sigma_p(A_1)$, is $Lin\{e_n\}$. Then, for any $x=c_n e_n\in Lin\{e_n\}$, we clearly have

$$\left\|e^{A_1 t}x\right\|_1=\left\|e^{itf(n)}c_n e_n\right\|_1=|c_n|\left\|e_n\right\|_1.$$

Therefore, by virtue of Theorem 2.1, we obtain that

$$\lim_{t\to+\infty}\frac{\left\|e^{A_1 t}x\right\|_1}{\left\|e^{A_1 t}\right\|}\leq\lim_{t\to+\infty}\frac{|c_n|\left\|e_n\right\|_1}{C|t|}=0,$$

and hence Condition 2 of Theorem 12 from [15] is satisfied for the C_0-semigroup $\left\{e^{A_1 t}\right\}_{t\geq 0}$. By similar arguments and application of Theorem 3.1, we obtain that that Condition 2 of Theorem 12 from [15] holds also for the C_0-semigroup $\left\{\widetilde{e^{A_1 t}}\right\}_{t\geq 0}$. Thus, by virtue of Theorem 12 from [15], we infer that the C_0-semigroups $\left\{e^{A_1 t}\right\}_{t\geq 0}$ and $\left\{\widetilde{e^{A_1 t}}\right\}_{t\geq 0}$ do not have any maximal asymptotics.

The operator $-A_1$ generates the C_0-semigroup $\left\{e^{-A_1 t}\right\}_{t\geq 0}$ with $D(-A_1)=D(A_1)$, and for its spectrum we have that

$$\sigma(-A_1)=\sigma_p(-A_1)=\{-if(n)\}_{n=1}^{\infty}\subset i\mathbb{R}.$$

Therefore, by virtue of Theorem 2.1, Theorem 3.1 and Theorem 12 from [15], we conclude that the C_0-semigroups $\left\{e^{-A_1 t}\right\}_{t\geq 0}$ and $\left\{\widetilde{e^{-A_1 t}}\right\}_{t\geq 0}$ also do not have any maximal asymptotics. □

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Один клас лінійно зростальних C_0-груп

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Ми розглядаємо спеціальний клас C_0-груп з [12], генератори яких є необмеженими, мають чисто точковий уявний спектр та відповідну щільну і мінімальну сім'ю власних векторів, яка, проте, не утворює базис Шаудера. Ми одержуємо двосторонні оцінки норм C_0-груп з цього класу і таким чином доводимо, що ці C_0-групи зростають лінійно. Крім того, ми доводимо, що C_0-групи з класу, що розглядається, не мають жодної максимальної асимптотики. Це означає, що не існує орбіти, що зростає найшвидше.

Ключові слова: C_0-група, лінійне зростання, максимальна асимптотика, XYZ теорема