Journal of Mathematical Physics, Analysis, Geometry
2021, Vol. 17, No. 4, pp. 521–548
doi: https://doi.org/10.15407/mag17.04.521

Gradient Estimates and Harnack Inequalities for a Nonlinear Heat Equation with the Finsler Laplacian

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Let (M^n, F, m) be an n-dimensional compact Finsler manifold. In this paper, we study the nonlinear heat equation

$$\partial_t u = \Delta_m u \quad \text{on } M^n \times [0, T],$$

where Δ_m is the Finsler Laplacian. We derive Li–Yau type gradient estimates for positive global solutions of this equation on static Finsler manifolds, as well as under action of the Finsler–Ricci flow. As corollaries, in both cases, the corresponding Harnack inequalities are also obtained.

Key words: Li–Yau type gradient estimates, Harnack inequality, nonlinear heat equation, Finsler–Ricci flow

Mathematical Subject Classification 2010: 35K55, 53C21

1. Introduction

After Cheng–Yau's work in [6] and Li–Yau's work in [12] on gradient estimates of the heat equation

$$\partial_t u = \Delta u \tag{1.1}$$

on a complete Riemannian manifold, there have been plenty of results obtained not only for the heat equation, but more generally, for other nonlinear equations on manifolds, for example, [8–11, 14–17, 26, 30] and the references therein.

Next, we simply introduce research progress associated with this article.

Let (M^n, g) be an n-dimensional complete Riemannian manifold with Ricci curvature bounded below by $-K$, where $K \geq 0$. For the positive solution of the heat equation (1.1), Li and Yau [12] obtained the celebrated gradient estimate:

$$\frac{|\nabla u|^2}{u^2} - \vartheta \frac{\partial_t u}{u} \leq \frac{n\vartheta^2 K}{2(\vartheta - 1)} + \frac{n\vartheta^2}{2t}, \tag{1.2}$$

where $\vartheta > 1$ is a constant. In [8], Davies improved Li-Yau's estimate (1.2) to

$$\frac{|\nabla u|^2}{u^2} - \vartheta \frac{\partial_t u}{u} \leq \frac{n\vartheta^2 K}{4(\vartheta - 1)} + \frac{n\vartheta^2}{2t}. \tag{1.3}$$

Later, the Li–Yau estimate (1.3) was improved for small time by Hamilton [9], where he proved under the same assumptions as above that

$$\frac{|\nabla u|^2}{u^2} - e^{2Kt}\frac{\partial_t u}{u} \le e^{4Kt}\frac{n}{2t}. \tag{1.4}$$

But the right hand side of (1.4) will blow up as $t \to \infty$. In order to find a sharp form which works for both large and small t, Li and Xu [15] got a new gradient estimate

$$\frac{|\nabla u|^2}{u^2} - \left(1 + \frac{\sinh(Kt)\cosh(Kt) - Kt}{\sinh^2(Kt)}\right)\frac{\partial_t u}{u} \le \frac{nK}{2}(\coth(Kt) + 1) \tag{1.5}$$

and its linearized version

$$\frac{|\nabla u|^2}{u^2} - \left(1 + \frac{2}{3}Kt\right)\frac{\partial_t u}{u} \le \frac{nK}{2}\left(K + \frac{1}{t} + \frac{1}{3}K^2 t\right). \tag{1.6}$$

The estimates (1.5) and (1.6) were later generalized by Qian [23]. And more recently, Yu and Zhao [27] obtained a Li–Yau type gradient estimate for positive solutions of (1.1) which is different with the estimates by Li-Xu [15] and Qian [23] as follows:

$$\beta(t)\frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} \le \frac{n}{2t}\max_{s\in[0,t]}\left(\frac{1}{\beta(s)} + \frac{(2K\beta(s) + \beta'(s))_+ s}{4\beta(s)(1-\beta(s))}\right), \tag{1.7}$$

where $a_+ = \max\{a, 0\}$ and $\beta \in C^1([0,T])$ satisfies

(B_1) $0 < \beta(t) < 1$ for any $t \in (0,T]$;

(B_2) $(1-\beta(0))^2 + \beta'(0)^2 > 0$ and $\beta(0) > 0$.

As an application of the estimate, they obtained an improvement of Davies' Li–Yau type gradient estimate (1.3). Moreover, their results generalized (1.4).

As the most natural generalization of Riemannian geometry, Finsler geometry attracts many attentions in recent years, since it has broader applications in nature science. Simultaneously Finsler manifold is one of the most natural metric measure spaces, which plays an important role in many aspects in mathematics. There is also a hope that gradient estimates can be applied in the Finsler setting to study elliptic and parabolic operators. In the Finsler setting, there exists a natural Laplacian, which we call here Finsler Laplacian. Unlike the usual Laplacian, the Finsler Laplacian is a nonlinear operator. In [20], Ohta and Sturm have studied the associated *nonlinear heat equation*

$$\partial_t u = \Delta_m u \quad \text{on } M^n \times [0,T]. \tag{1.8}$$

The nonlinearity is inherited from the Legendre transform. The nonlinear heat equation is very recent and very little has been done about it. Some results regarding the existence, uniqueness and Sobolev regularity of a positive global solution of the nonlinear heat equation (in the sense of distributions) are obtained in [20]. In [19], Ohta and Sturm proved the Bochner–Weitzenböck formula for the Finsler Laplacian on general Finsler manifolds and derived Li–Yau type gradient estimates as well as parabolic Harnack inequalities. They proved

Theorem A (Li–Yau gradient estimate [19]). *Assume that (M^n, F, m) is an n-dimensional compact Finsler manifold and satisfies* $\mathrm{Ric}_N \geq \widetilde{K}$ *for some* $N \in [n, \infty)$ *and* $\widetilde{K} \in \mathbb{R}$, *put* $\widetilde{K}' := \min\{\widetilde{K}, 0\}$. *Let $u(x,t)$ be a positive global solution to the nonlinear heat equation* (1.8). *Then, for any $\vartheta > 1$, we have*

$$F^2(\nabla(\log u)) - \vartheta \partial_t(\log u) \leq -\frac{N\vartheta^2 \widetilde{K}'}{4(\vartheta - 1)} + \frac{N\vartheta^2}{2t} \quad \text{on } M^n \times [0, T]. \tag{1.9}$$

For their Harnack inequalities, one can refer to Theorem 4.5 in [19]. The precise definition of the Finsler measure space, weighted Ricci curvature Ric_N, gradient vector field ∇, Finsler Laplacian Δ_m and the global solution to the nonlinear heat equation will be given in Section 2 below.

Inspired by above works, we further study Li–Yau type gradient estimates for positive global solutions to the nonlinear heat equation (1.8) on compact Finsler manifolds and obtain several type estimates for the nonlinear heat equation.

Theorem 1.1. *Assume that (M^n, F, m) is an n-dimensional compact Finsler manifold and satisfies* $\mathrm{Ric}_N \geq 0$. *Let $u(x,t)$ be a nonnegative global solution of* (1.8) *on $M \times (0, \infty)$. If $\partial M \neq \emptyset$, assume that ∂M is convex, and $u(x,t)$ satisfies the Neumann boundary condition*

$$\nabla u \in T(\partial M) \quad \text{on} \quad \partial M \times (0, \infty).$$

Then we have

$$F^2(\nabla(\log u)) - \partial_t(\log u) \leq \frac{N}{2t} \quad \text{on} \quad \partial M \times (0, \infty). \tag{1.10}$$

Remark 1.2. When M^n is a compact Riemannian manifold, Ric_N becomes Ric and the Finsler Laplacian Δ_m is just the usual Laplacian Δ, then Theorem 1.1 can be reduced to the Theorem 1.1 in [12]. Hence, the above Theorem extends the corresponding result in [12].

Theorem 1.3. *Assume that (M^n, F, m) is an n-dimensional compact Finsler manifold and satisfies* $\mathrm{Ric}_N \geq -K$ *for some $N \in [n, \infty)$ and $K \in [0, \infty)$. Let $u(x,t)$ be a positive global solution to the nonlinear heat equation* (1.8) *on $M \times [0, T]$. Let*

$$\psi_1(t) = \frac{N}{2t} \max_{s \in [0,t]} \left(\frac{1}{\beta(s)} + \frac{(2K\beta(s) + \beta'(s))_+ \, s}{4\beta(s)(1 - \beta(s))} \right).$$

Then we have

$$\beta(t) F^2(\nabla(\log u)) - \partial_t(\log u) \leq \psi_1(t) \quad \text{on } M^n \times [0, T], \tag{1.11}$$

where $a_+ = \max\{a, 0\}$ and $\beta \in C^1[0, T]$ satisfies conditions (B_1) *and* (B_2).

Remark 1.4. We should note the following:

(1) When M^n is a compact Riemannian manifold, Ric_N and Δ_m become Ric and Δ respectively, then the estimate (1.11) can be reduced to the formula (1.7).

(2) For convenience of comparison, for any $\widetilde{\vartheta} \in (0,1)$, one can rewrite (1.9) as

$$\widetilde{\vartheta} F^2(\nabla(\log u)) - \partial_t(\log u) \leq -\frac{N\widetilde{K}'}{4(1-\widetilde{\vartheta})} + \frac{N}{2\widetilde{\vartheta} t}. \tag{1.12}$$

When $\beta(t)$ is a constant, the estimate (1.11) can be reduced to the formula (1.9)(or (1.12)). Hence, the above Theorem 1.3 extends the corresponding result in [19].

From Theorem 1.3, we derive Davies type estimate and Hamilton type estimate for (1.8).

Corollary 1.5. *Let the notations be the same as in Theorem 1.3. Then the following special estimates are valid.*

(1) ***Davies type:*** *For any constant $\beta \in (0,1)$, we have*

$$\beta F^2(\nabla(\log u)) - \partial_t(\log u) \leq \begin{cases} \frac{N}{2\beta t}, & t \leq \frac{1-\beta}{2K\beta} \\ \frac{3N}{8\beta t} + \frac{NK}{4(1-\beta)}, & t \geq \frac{1-\beta}{2K\beta} \end{cases} \quad \text{on } M \times (0,T] \tag{1.13}$$

(2) ***Hamilton type:*** *For any constant $\theta \in (0,1]$, we have*

$$e^{-2\theta Kt} F^2(\nabla(\log u)) - \partial_t(\log u) \leq \frac{N}{2t} e^{2\theta Kt} + \frac{NK(1-\theta)}{4(1-e^{-2\theta Kt})} \quad \text{on } M \times (0,T]. \tag{1.14}$$

Remark 1.6. We should note the following:
(1) Obviously, (1.13) improves the Li-Yau type gradient estimate (1.9) (or (1.12)). Therefore, the Corollary 1.5 improves the corresponding result in [19].
(2) When M^n is a compact Riemannian manifold and $\theta = 1$, (1.14) can be reduced to (1.4). Therefore, the Corollary 1.5 extends the corresponding result in [9].

As an application of Theorem 1.3, we derive a Harnack inequality.

Corollary 1.7. *Let (M^n, F, m) be an n-dimensional compact Finsler manifold and satisfy $\mathrm{Ric}_N \geq -K$ for some $N \in [n,\infty)$ and $K \in [0,\infty)$. Let $u : [0,T] \times M \to \mathbb{R}$ be a nonnegative global solution to the nonlinear heat equation (1.8). Then we have, for any $0 < t_1 < t_2 \leq T$ and $x_1, x_2 \in M$,*

$$u(x_2,t_2) \leq u(x_1,t_1) \exp\left\{\int_{t_2}^{t_1} \psi_1(t) + \int_{t_2}^{t_1} \frac{d(x_2,x_1)^2}{4\beta(t)(t_1-t_2)^2} dt\right\}, \tag{1.15}$$

where $\beta(t)$ and $\psi_1(t)$ are given in Theorem 1.3.

Remark 1.8. Taking $\beta(t)$ is a constant in the inequality (1.15), we obtain Theorem 4.5 in [19].

Remark 1.9. In the proof of above theorems, compared with the case of compact Riemannian manifolds as in [8,9,27], in our case, we need to overcome three obstructions. First, because of the lack of higher order regularity, we need to modify the arguments in [8,9,27]. Second, in the Finsler case, $\Delta_m u$ has no definition at the maximum point of u, and thus the maximum principle can not be suitable for the Finsler Laplacian. Last but not least, in view of nonlinear property of gradient operator, it is difficult to do the calculations. The weighted linear operators play an important role in the proof. With their help, we can convert some nonlinear problems into the linear ones. Further, using the weighted Laplacian, we can obtain the gradient estimate that we need.

On the other hand, many authors used similar techniques to prove gradient estimates and Harnack inequalities for geometric flows. For instance, in [13], Liu established first order gradient estimates for positive solutions of the heat equations (1.1) on complete noncompact or closed Riemannian manifolds under Ricci flows. As applications, he derived Harnack type inequalities and second order gradient estimates for positive solutions. Generalizing Liu's work to general geometric flow, Sun [24] established first order and second order gradient estimates for positive solutions of the heat equations under general geometric flows. Bailesteanu, Cao and Pulemotov in [7] considered a series of gradient estimates for positive solutions of the heat equation under the Ricci flow. They also proved Li–Yau type gradient estimates and obtained Harnack inequalities.

Bao in [5] introduced Finsler–Ricci flow as follows,

$$\frac{\partial}{\partial t}g_{ij} = -2\,\mathrm{Ric}_{ij} = -2\frac{\partial^2\left(\frac{1}{2}F^2\,\mathrm{Ric}\right)}{\partial y^i\partial y^j} \tag{1.16}$$

with y^i and y^j gives, via Euler's theorem, $\frac{\partial F^2}{\partial t} = -2F^2\,\mathrm{Ric}$, where Ric_{ij} is Akbarzadeh's Ricci tensor and Ric is Ricci curvature. There are some results about Finsler–Ricci flow, such as the existence and uniqueness of such flow and the solitons of this flow (c.f. [1,2]). In [22], Lakzian derived differential Harnack estimates for positive solutions to (1.8) under Finsler–Ricci flow. It is worth to notice that the inequality (2) in [22] was not completely correct. In fact, due to the proof of Lemma 4.1 in [22], Lakzian thought that $\frac{\partial(\mathrm{Ric}^{ij}(\nabla f))}{\partial x^i}f_j = 0$ by Euler's theorem. This means that the parabolic differential equality (43) is lack of $\frac{\partial(\mathrm{Ric}^{ij}(\nabla f))}{\partial x^i}f_j$. Here we used the notations given in [22]. However, we compute

$$\frac{\partial(\mathrm{Ric}^{ij}(\nabla f))}{\partial x^i}f_j = \mathrm{Ric}^{ij}_{\ \ |i}\,f_j + \mathrm{Ric}^{ij}_{\ \ ;k}\,\frac{f_j}{F}(\nabla^2 f)^k_i \neq 0,$$

where $\mathrm{Ric}^{ij}_{\ \ |i}$ denotes the horizontal covariant derivative of Ric^{ij} and $\mathrm{Ric}^{ij}_{\ \ ;k}$ denotes the vertical covariant derivative of Ric^{ij}. Therefore, the gradient estimate in the inequality (2) in [22] was not completely correct. Next we follow the work of Liu [13] and Bailesteanu et al. [7], and generalize and correct the work of Lakzian in [22].

Theorem 1.10. *Let $(M^n, F(t))_{t\in[0,T]}$ be a closed solution to the Finsler–Ricci flow (1.16). Assume that there are three positive real numbers K_1, K_2 and K_3 such that for all $t \in [0,T]$, Akbarzadeh's Ricci tensor satisfies $-K_1 \le \mathfrak{R} \le K_2$ and $|\nabla\mathfrak{R}| \le K_3$ and S-curvature vanishes. Consider a positive global solution $u = u(x,t)$ of the equation (1.8). Let $f = \log u$. Let $\alpha, \lambda \in C^1(0,T]$ satisfy the following*

(C$_1$) $0 < \alpha(t) < 1$ *for any* $t \in (0,T]$;

(C$_2$) $\lim_{t\to 0^+} \lambda(t) = 0$ *and* $\lambda(t) > 0$ *for any* $t \in (0,T]$;

(C$_3$) $(\ln\lambda)' > 0$ *on* $(0,T]$.

Let

$$\psi_2(t) = \frac{n}{2(1-2\varepsilon)\lambda} \max_{s\in[0,t]} \Bigg(\frac{\lambda'(s)}{\alpha(s)} + \frac{\alpha'(s) + 2(K_1+\varepsilon)}{2\alpha(s)(1-\alpha(s))}\lambda(s) + \frac{1}{n\alpha(s)}\Big(\frac{1-2\varepsilon}{\varepsilon}\Big)^{\frac{1}{2}} \Big((n(1+\alpha(s)))^{\frac{1}{2}} K_3 + n^{\frac{3}{2}} \max\{K_1, K_2\} \Big) \lambda(s) \Bigg).$$

Then, we have

$$\alpha(t)F^2(\nabla(\log u)) - \partial_t(\log u) \le \psi_2(t) \quad \text{on } M \times (0,T]. \tag{1.17}$$

Here $\varepsilon \in \left(0, \frac{1}{2}\right)$ is an arbitrary constant.

Remark 1.11. We should note the following:

(1) Taking $\alpha = 1/\theta$ as a constant function ($\theta > 1$) and $\lambda(t) = t$ in (1.17), the estimate (1.17) is reduced to the one in [22, 31].

(2) The condition $S \equiv 0$ is often required in the study of Finsler–Ricci flow. Since the S-curvature vanishes for Berwald metrics, our results can be applied to any Finsler–Ricci flow of Berwald metrics on closed manifolds(for example, see [1, 18, 22]).

(3) An important difference in the Finsler case and Riemannian case is that the solution of the Riemannian heat equation has enough regularity to obtain $\partial_t(\Delta f) = \Delta(\partial_t f) + 2\,\mathrm{Ric}^{ij} f_{ij}$ which appeared as (2-4) in [13]. However, in the Finsler setting, the solutions of the nonlinear heat equation (1.8) are lack of higher order regularity. Therefore, we have to compute $\partial_t(\Delta_m f)$ in a weak sense, which produces $\mathrm{Ric}^{ij}_{\ |i}$ and $\mathrm{Ric}^{ij}_{\ ;k}$. In order to obtain the gradient estimate, we require $|\nabla\mathfrak{R}|$ bounded above.

Even if we assume that the solutions of the nonlinear heat equation (1.8) have enough regularity, we can't get $\partial_t(\Delta_m f) = \Delta_m^{\nabla f}(\partial_t f) + 2\,\mathrm{Ric}^{ij} f_{ij}$. Some non-Riemannian geometry quantities will appear in the RHS of the above equation, such as Cartan tensor, hv-curvature tensor P^i_{jkl}, and $\nabla\mathfrak{R}$. In order to get the gradient estimate, we have to give more assumptions. These conditions will become unnatural and too complicated.

Given the above, it is difficult to to follow the proof of Liu's paper [13] where $|\nabla\mathfrak{R}| \leq K_3$ is not assumed. Therefore, the assumption $|\nabla\mathfrak{R}| \leq K_3$ in Theorem 1.10 is necessary.

Using Theorem 1.10, we derive a Harnack inequality.

Corollary 1.12. *Let $(M^n, F(t))_{t\in[0,T]}$ be a closed solution to the Finsler–Ricci flow (1.16). Assume that there are three positive real numbers K_1, K_2 and K_3 such that for all $t \in [0,T]$, Akbarzadeh's Ricci tensor satisfies $-K_1 \leq \mathfrak{R} \leq K_2$ and $|\nabla\mathfrak{R}| \leq K_3$ and S-curvature vanishes. Consider a positive global solution $u = u(x,t)$ of the nonlinear heat equation (1.8) on $M \times [0,T]$. Let $f = \log u$. Then for $(x_1,t_1) \in M^n \times (0,T]$ and $(x_2,t_2) \in M^n \times (0,T]$ such that $t_1 < t_2$, we have*

$$u(x_1,t_1) \leq u(x_2,t_2) \exp\left\{\int_0^1 \left(\frac{1}{4\alpha}\frac{F(\dot\eta(s))^2|_\tau}{t_2-t_1} + (t_2-t_1)\psi_2(s)\right)ds\right\}. \qquad (1.18)$$

Here $\eta(s)$ be a smooth curve connecting x and y with $\eta(1) = x$ and $\eta(0) = y$, and $F(\dot\eta(s))|_\tau$ is the length of the vector $\dot\eta(s)$ at time $\tau(s) = (1-s)t_2 + st_1$.

2. Preliminaries

In this section we briefly recall the fundamentals of Finsler geometry by Bao, Chern and Shen [3], as well as some results on the analysis of Finsler geometry by Ohta–Sturm [19, 20].

2.1. Finsler metric.

We assume that M is an n-dimensional smooth connected manifold. Let TM be the tangent bundle over M with local coordinates (x,y), where $x = (x^1,\dots,x^n)$ and $y = (y^1,\dots,y^n)$. A *Finsler metric* on M is a function $F : TM \to [0,\infty)$ satisfying the following properties:

(i) F is smooth on $TM \setminus \{0\}$;

(ii) $F(x,\lambda y) = \lambda F(x,y)$ for all $\lambda > 0$;

(iii) For any nonzero tangent vector $y \in TM$, the approximated symmetric metric tensor, g_y, defined by

$$g_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial s\partial t}F^2(y+su+tv)\big|_{s=t=0},$$

is positive definite.

Such a pair (M^n, F) is called a Finsler manifold. A Finsler structure is said to be *reversible* if, in addition, F is even. Otherwise F is nonreversible. We say a Finsler manifold (M^n, F) is forward (respectively, backward) complete if every geodesic defined on $[0,a]$ (respectively, $[-a,0]$) can be extended to $[0,+\infty)$ (respectively, $(-\infty,0]$). Compact Finsler manifolds are both forward and backward complete. By a *Finsler measure space* we mean a triple (M^n, F, m) constituted with a smooth, connected n-dimensional manifold M, a Finsler structure F on M and a measure m on M.

2.2. Geodesic spray and Chern connection. It is straightforward to observe that the *geodesic spray* in the Finsler setting is of the form, $G = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$, where

$$G^i(x,y) = \frac{1}{4} g_y^{ik} \left\{ 2\frac{\partial (g_y)_{jk}}{\partial x^l} - \frac{\partial (g_y)_{jl}}{\partial x^k} \right\} y^j y^l. \tag{2.1}$$

For every nonvanishing vector field V, $g_{ij}(V)$ induces a Riemannian structure g_V of T_xM via

$$g_V(X,Y) = \sum_{i,j}^{n} g_{ij}(V) X^i Y^j \quad \text{for } X, Y \in T_xM.$$

In particular, $g_V(V,V) = F^2(V)$.

The projection $\pi : TM \to M$ gives rise to the pull-back bundle π^*TM over $TM\setminus\{0\}$. As is well known, on π^*TM there exists uniquely the *Chern connection* D. The Chern connection is determined by the following structure equations, which characterize "torsion freeness":

$$D_X^V Y - D_Y^V X = [X,Y]$$

and "almost g-compatibility"

$$Z(g_V(X,Y)) = g_V(D_Z^V X, Y) + g_V(X, D_Z^V Y) + C_V(D_Z^V V, X, Y) \tag{2.2}$$

for $V \in TM \setminus \{0\}$, $X, Y, Z \in TM$. Here

$$C_V(X,Y,Z) = \frac{1}{4}\frac{\partial^3 F^2}{\partial V^i \partial V^j \partial V^k}(V) X^i Y^j Z^k$$

denotes the Cartan tensor and $D_X^V Y$ the covariant derivative with respect to reference vector $V \in TM \setminus \{0\}$. We mention here that $C_V(V,X,Y) = 0$ due to the homogeneity of F. The Chern connection coefficients are given by

$$\Gamma^i_{jk} := \frac{1}{2} g^{il} \left\{ \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{lj}}{\partial y^r} N^r_k + \frac{\partial g_{jk}}{\partial y^r} N^r_l - \frac{\partial g_{kl}}{\partial y^r} N^r_j \right\},$$

where $N^i_j = \frac{\partial G^i}{\partial y^j}$ and g is in fact g_y.

2.3. Covariant derivative of tensor field. Given the coordinates $\{x^i, y^i\}$ on TM, one can observe that the pair $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ forms a horizontal and vertical frames for TTM, where $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^k_i \frac{\partial}{\partial y^k}$. Let $\{dx^i, \delta y^i\}$ denote the local frame dual to $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$, where $\delta y^i = dy^i + N^i_j dx^j$. Then we obtain a decomposition for $T(TM \setminus \{0\})$ and $T^*(TM \setminus \{0\})$,

$$T(TM \setminus \{0\}) = \mathcal{H}TM \oplus \mathcal{V}TM, \quad T^*(TM \setminus \{0\}) = \mathcal{H}^*TM \oplus \mathcal{V}^*TM,$$

where

$$\mathcal{H}TM = \mathrm{span}\{\frac{\delta}{\delta x^i}\}, \qquad \mathcal{V}TM = \mathrm{span}\{\frac{\partial}{\partial y^i}\},$$
$$\mathcal{H}^*TM = \mathrm{span}\{dx^i\}, \qquad \mathcal{V}^*TM = \mathrm{span}\{\delta y^i\}.$$

Let $T = T^{ij}\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial x^j}$ be an arbitrary smooth local section of $\pi^*TM\otimes\pi^*T^*M$. They can therefore be expanded in terms of the natural basis $\{dx^s, \frac{\delta y^s}{F}\}$. The covariant derivative of T^{ij} denotes

$$(\nabla T)^{ij} = T^{ij}_{\ |s}dx^s + T^{ij}_{\ ;s}\frac{\delta y^s}{F}. \tag{2.3}$$

The horizontal covariant derivative $T^{ij}_{\ |s}$ denotes

$$T^{ij}_{\ |s} = \frac{\delta T^{ij}}{\delta x^s} + T^{ki}\Gamma^j_{ks} + T^{kj}\Gamma^i_{ks}. \tag{2.4}$$

The vertical covariant derivative $T^{ij}_{\ ;s}$ denotes

$$T^{ij}_{\ ;s} = F\frac{\partial T^{ij}}{\partial y^s}. \tag{2.5}$$

2.4. Distance function. For $x_1, x_2 \in M$, the *distance function* from x_1 to x_2 is defined by

$$d(x_1,x_2) = \inf_\gamma \int_0^1 F(\dot\gamma(t))\,dt,$$

where the infimum is taken over all C^1-curves $\gamma : [0,1] \to M$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$. Note that the distance function may not be symmetric unless F is reversible. A C^∞-curve $\gamma : [0,1] \to M$ is called a *geodesic* if $F(\dot\gamma)$ is constant and it is locally minimizing. In terms of the Chern connection, a geodesic γ satisfies $D^{\dot\gamma}_{\dot\gamma}\dot\gamma = 0$.

2.5. S-curvature. Associated to any Finsler structure, there is one canonical measure, called the Busemann–Hausdorff measure, given by

$$dV_F := \sigma_F(x)dx^1\wedge\cdots\wedge dx^n,$$

where $\sigma_F(x)$ is the volume ratio

$$\sigma_F(x) = \frac{\mathrm{vol}(B_{\mathbb{R}^n}(1))}{\mathrm{vol}((a_i)\in\mathbb{R}^n|F(\sum a_i\frac{\partial}{\partial x^i})<1)}.$$

The S-curvature is then defined as

$$S(y) := \frac{\partial G^i}{\partial y^i}(x,y) - y^i\frac{\partial}{\partial x^i}(\ln\sigma_F(x)). \tag{2.6}$$

2.6. Legendre transform, gradient, Hessian and Finsler Laplacian. In order to define the *gradient* of a function, we define the *Legendre transform* $\mathcal{L} : TM \to T^*M$, as $\mathcal{L}(y) = FF_{y^i}dx^i$, which satisfies $\mathcal{L}(0) = 0$ and $\mathcal{L}(\lambda y) = \lambda\mathcal{L}(y)$ for all $\lambda > 0$ and $y \in TM \setminus \{0\}$. Then $\mathcal{L} : TM \setminus \{0\} \to T^*M \setminus \{0\}$ is a norm-preserving C^∞ diffeomorphism. For a smooth function $u : M \to \mathbb{R}$, the *gradient vector* of u at $x \in M$ is defined as $\nabla u(x) := \mathcal{L}^{-1}(du(x)) \in T_xM$, which can be written as

$$\nabla u(x) := \begin{cases} g^{ij}(x, \nabla u)\frac{\partial u}{\partial x^j}\frac{\partial}{\partial x^i}, & \mathrm{d}u(x) \neq 0 \\ 0, & \mathrm{d}u(x) = 0 \end{cases}.$$

Set $M_u := \{x \in M \mid du(x) \neq 0\}$. We define $\nabla^2 u(x) \in T_x^*M \otimes T_xM$ for $x \in M_u$ by using the following covariant derivative [28, 29]:

$$\nabla^2 u(v) := D_v^{\nabla u}\nabla u(x) \in T_xM, \quad v \in T_xM.$$

Set

$$D^2u(X, Y) := g_{\nabla u}(\nabla^2 u(X), Y) = g_{\nabla u}(D_X^{\nabla u}(\nabla u), Y).$$

Then we have

$$g_{\nabla u}(D_X^{\nabla u}(\nabla u), Y) = D^2u(X, Y) = D^2u(Y, X) = g_{\nabla u}(D_Y^{\nabla u}(\nabla u), X)$$

for any $X, Y \in T_xM$.

In order to define a Laplacian on Finsler manifolds, we need a measure m (or a volume form dm) on M. From now on, we consider the Finsler measure space (M, F, m) equipped with a fixed smooth measure m. Let $V \in TM$ be a smooth vector field on M. In a local coordinate (x^i), expressing $dm = e^\Phi dx^1dx^2 \cdots dx^n$, we can write $\mathrm{div}_m V$ as

$$\mathrm{div}_m V = \sum_{i=1}^n \left(\frac{\partial V^i}{\partial x^i} + V^i\frac{\partial \Phi}{\partial x^i}\right).$$

A Laplacian, which is called the *Finsler Laplacian*, can now be defined by

$$\Delta_m u = \mathrm{div}_m(\nabla u).$$

We remark that the Finsler Laplacian is better to be viewed in a weak sense that for $u \in W^{1,2}(M)$,

$$\int_M \phi\Delta_m u dm = -\int_M D\phi(\nabla u)\, dm \quad \text{for } \phi \in C_c^\infty(M),$$

where $D\phi$ is the differential 1-form of ϕ.

The relation between $\Delta_m u$ and $\nabla^2 u$ is that

$$\Delta_m u = tr_{g_{\nabla u}}(\nabla^2 u) - S(\nabla u) = \sum_{i=1}^n \nabla^2 u(e_i, e_i) - S(\nabla u),$$

where $\{e_i\}$ is an orthonormal basis of T_xM with respect to $g_{\nabla u}$.

Given a vector field V, the *weighted Laplacian* is defined on the weighted Riemannian manifold (M, g_V, m) by

$$\Delta_m^V u = \mathrm{div}_m(\nabla^V u),$$

where

$$\nabla^V u(x) := \begin{cases} g^{ij}(x, V)\frac{\partial u}{\partial x^j}\frac{\partial}{\partial x^i}, & du(x) \neq 0 \\ 0, & du(x) = 0 \end{cases}.$$

Similarly, the weighted Laplacian can be viewed in a weak sense for $u \in W^{1,2}(M)$. We note that $\Delta_m^{\nabla u} u = \Delta_m u$.

2.7. Weighted Ricci curvature. The Ricci curvature of Finsler manifolds is defined as the trace of the flag curvature. Explicitly, given two linearly independent vectors $V, W \in TM \setminus \{0\}$, the *flag curvature* is defined by

$$K^V(V, W) = \frac{g_V(R^V(V, W)W, V)}{g_V(V, V)g_V(W, W) - g_V(V, W)^2},$$

where R^V is the *Chern curvature* (or Riemannian curvature):

$$R^V(X, Y)Z = D_X^V D_Y^V Z - D_Y^V D_X^V Z - D_{[X,Y]}^V Z.$$

Then the *Ricci curvature* is defined by

$$\mathrm{Ric}(V) = \sum_{i=1}^{n-1} K^V(V, e_i),$$

where $e_1, \ldots, e_{n-1}, \frac{V}{F(V)}$ form an orthonormal basis of T_xM with respect to g_V.

We recall the definition of the *weighted Ricci curvature* on Finsler manifolds, which was introduced by Ohta in [21].

Given a vector $V \in T_xM$, let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a geodesic with $\gamma(0) = x$, $\dot\gamma(0) = V$. Define

$$\dot S(V) := F^{-2}(V)\frac{d}{dt}[S(\gamma(t), \dot\gamma(t))]_{t=0},$$

where $S(V)$ denotes the S-curvature at (x, V). The *weighted Ricci curvature* of (M, F, m) is defined by

$$\begin{aligned} \mathrm{Ric}_n(V) &:= \begin{cases} \mathrm{Ric}(V) + \dot S(V) & \text{for } S(V) = 0 \\ -\infty & \text{otherwise} \end{cases}, \\ \mathrm{Ric}_N(V) &:= \mathrm{Ric}(V) + \dot S(V) - \frac{S(V)^2}{(N-n)F(V)^2}, \qquad N \in (n, \infty), \\ \mathrm{Ric}_\infty(V) &:= \mathrm{Ric}(V) + \dot S(V). \end{aligned}$$

We note that the curvature Ric_N is 0-homogeneous.

2.8. Akbarzadeh's Ricci tensor Ric_{ij}**.** Akbarzadeh's Ricci tensor Ric_{ij} is defined as follows

$$\mathrm{Ric}_{ij} := \frac{\partial^2}{\partial y^i \partial y^j}\left(\frac{F^2\,\mathrm{Ric}}{2}\right). \tag{2.7}$$

We denote second order contravariant tensor of Akbarzadeh's Ricci tensor by $\mathfrak{R}$, that is

$$\mathfrak{R} := \mathrm{Ric}^{ij}\,\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}, \tag{2.8}$$

where $\mathrm{Ric}^{ij} = g^{ik}g^{jl}\,\mathrm{Ric}_{kl}$. For further details regarding Akbarzadeh's Ricci tensor, see [4].

2.9. Bochner–Weitzenböck formula. The following Bochner–Weitzenböck type formula, established by Ohta-Sturm in [19], plays an important role in this paper.

Theorem B (Bochner–Weitzenböck formula [19])**.** *Given* $u \in W^{2,2}_{loc}(M) \cap C^1(M)$ *with* $\Delta_m u \in W^{1,2}_{loc}(M)$*, we have*

$$\begin{aligned} -\int_M D\psi\left(\nabla^{\nabla u}\left(\frac{F^2(x,\nabla u)}{2}\right)\right) dm \\ = \int_M \psi\{D(\Delta_m u)(\nabla u) + \mathrm{Ric}_\infty(\nabla u) + |\nabla^2 u|^2_{HS(\nabla u)}\}\, dm \end{aligned} \tag{2.9}$$

for all nonnegative functions $\psi \in W^{1,2}_c(M) \cap L^\infty(M)$*. Given* $u \in C^\infty(M)$*, the pointwise version of the identity is*

$$\Delta_m^{\nabla u}\left(\frac{F^2(\nabla u)}{2}\right) = D(\Delta_m u)(\nabla u) + \mathrm{Ric}_\infty(\nabla u) + |\nabla^2 u|^2_{HS(\nabla u)}. \tag{2.10}$$

Here $|\nabla^2 u|^2_{HS(\nabla u)}$ *denotes the Hilbert-Schmidt norm with respect to* $g_{\nabla u}$*.*

2.10. Global solutions to $\partial_t u = \Delta_m u$**.** We say that a function u on $[0,T] \times M$, $T > 0$, is a global solution to the nonlinear heat equation $\partial_t u = \Delta_m u$ if it satisfies the following:

(i) $u(x,t) \in L^2([0,T], H^1(M)) \cap H^1([0,T], H^{-1}(M))$;

(ii) For any test function $\phi \in C^\infty_c(M)$ and for all $t \in [0,T]$,

$$\int_M \phi \partial_t u\, dm = -\int_M D\phi(\nabla u)\, dm. \tag{2.11}$$

2.11. Finsler manifolds with boundary. Let $\Omega \subset M$ be a domain of M. Then $\partial\Omega$ can be viewed as a hypersurface of (M, F, m). For any $x \in \partial\Omega$, there exist exactly two unit normal vectors ν_i, $i = 1, 2$ such that

$$T_x(\partial\Omega) = \{V \in T_x(M) | g_{\nu_i}(\nu_i, V) = 0, g_{\nu_i}(\nu_i,\nu_i) = 1\}.$$

If F is reversible, then $\nu_1 = -\nu_2$.

Now we briefly illustrate the convexity defined in [25, 28], which is adopted in Theorem 1.1. Let (M, F, m) be a Finsler manifold with boundary ∂M and ν be the normal vector that points outward M. The normal curvature $\Lambda_\nu(V)$ at $x \in \partial M$ in the direction $V \in T_x(\partial M)$ is defined by

$$\Lambda_\nu(V) := g_\nu(\nu, D_{\dot\gamma}^{\dot\gamma}\dot\gamma|_x),$$

where γ is the unique local geodesic for the Finsler structure $F_{\partial M}$ on ∂M induced by F with the initial data $\gamma(0) = x$ and $\dot\gamma(0) = V$. M is said to have *convex* boundary if for any $x \in \partial M$, the normal curvature Λ at x is non-positive in any direction $V \in T_x(\partial M)$.

3. Gradients estimates on static Finsler manifolds

3.1. Some lemmas. In this section, we consider a positive global solution of the heat equation (1.8) on static Finsler manifolds (M^n, F, m). The Laplacian, gradient and Legendre transform are all with respect to $V := \nabla u$ and are valid on $M_u := \{x \in M \mid \nabla u(x) \neq 0\}$.

We consider the function $f := \log u$ which is H^2 in space and $C^{1,\alpha}$ in both space and time, then $u = e^f$. We have

$$\partial_t f = e^{-f}\partial_t u$$

and

$$\nabla f = e^{-f}\nabla u, \qquad \Delta_m f = e^{-f}\Delta_m u - F^2(\nabla f). \tag{3.1}$$

Hence f satisfies the following equation

$$\partial_t f = \Delta_m f + F^2(\nabla f) \tag{3.2}$$

for every t in the weak sense that

$$\int_M \{-D\phi(\nabla f) + \phi F^2(\nabla f)\}\, dm = \int_M \phi\partial_t f\, dm$$

for each $\phi \in H^1(M)$. From (3.1), we have $g_{\nabla f} = g_{\nabla u}$ a.e. on M_u and $\Delta_m f \in H^1(M)$ for each t.

Now let us consider the function

$$G(x,t) := \mu\{\beta F^2(\nabla f) - \alpha\partial_t f - \varphi(t)\} = \mu\beta F^2(\nabla f) - \sigma - \mu\varphi, \tag{3.3}$$

where $\mu(t)$, $\beta(t)$, $\alpha(t)$ and $\varphi(t)$ are four functions depending on t. In addition, $\sigma(x,t) = \mu(t)\alpha(t)\partial_t f$ lies in $H^1(M)$ and $G(x,t)$ lies in $H^1(M)$ for each t and is Hölder continuous in both space and time.

Lemma 3.1 ([19]). *Let (M^n, F, m) be an n-dimensional compact Finsler manifold, then*

$$\partial_t[F^2(\nabla f)] = 2D(\partial_t f)(\nabla f). \tag{3.4}$$

Lemma 3.2. *In the sense of distributions, $\sigma(x,t)$ satisfies the parabolic differential equality*

$$\Delta_m^{\nabla f}\sigma - \partial_t\sigma + 2D\sigma(\nabla f) = -\mu\alpha'\partial_t f - \alpha\mu'\partial_t f. \tag{3.5}$$

Proof. For each $\phi \in H_0^1(M\times(0,T))$, we have

$$\begin{aligned}
&\int_0^T\int_M \Big\{-D\phi(\nabla^{\nabla f}\sigma) + \sigma\partial_t\phi + 2\phi D\sigma(\nabla f)\Big\}\,dm\,dt\\
&= \int_0^T\int_M \Big\{-D(\alpha\mu\phi)(\nabla^{\nabla f}(\partial_t f)) + \alpha\mu\partial_t f\partial_t\phi + 2\alpha\mu\phi D(\partial_t f)(\nabla f)\Big\}\,dm\,dt\\
&= \int_0^T\int_M \{-D(\alpha\mu\phi)(\nabla^{\nabla f}(\partial_t f)) + \partial_t(\phi\alpha\mu)\partial_t f - \phi\mu\alpha'\partial_t f - \phi\alpha\mu'\partial_t f\\
&\quad + 2\alpha\mu\phi D(\partial_t f)(\nabla f)\}\,dm\,dt\\
&= \int_0^T\int_M \{D(\partial_t(\alpha\mu\phi))(\nabla f) - \phi\mu\alpha'\partial_t f + \partial_t(\phi\alpha\mu)(\Delta_m f + F^2(\nabla f))\\
&\quad - \phi\alpha\mu'\partial_t f + \alpha\phi\mu\partial_t(F^2(\nabla f))\}\,dm\,dt\\
&= \int_0^T\int_M \{-\mu\alpha'\partial_t f - \alpha\mu'\partial_t f\}\,\phi\,dm\,dt,
\end{aligned}$$

where the third equality used (3.2) and (3.4). □

Now we can compute a parabolic partial differential inequality for $G(x,t)$.

Lemma 3.3. *In the sense of distributions, $G(x,t)$ satisfies*

$$\Delta_m^{\nabla f}G - \partial_t G + 2DG(\nabla f) \geq D, \tag{3.6}$$

where

$$D(x,t) = \frac{2\mu\beta(\Delta_m f)^2}{N} - (2\mu\beta K + (\mu\beta)')F^2(\nabla f) + \mu\alpha'\partial_t f + \alpha\mu'\partial_t f + (\mu\varphi)'.$$

Proof. For each $\phi \in H_0^1(M\times(0,T))$,

$$\begin{aligned}
&\int_0^T\int_M \Big\{-D\phi(\nabla^{\nabla f}G) + G\partial_t\phi + 2\phi DG(\nabla f)\Big\}\,dm\,dt\\
&= \int_0^T\int_M \{-D(\mu\beta\phi)(\nabla^{\nabla f}(F^2(\nabla f))) + \mu\beta F^2(\nabla f)\partial_t\phi\\
&\quad + 2\mu\beta\phi D(F^2(\nabla f))(\nabla f) + \phi\mu\alpha'\partial_t f + \phi\alpha\mu'\partial_t f + \phi(\mu\varphi)'\}\,dm\,dt\\
&= \int_0^T\int_M \{-D(\mu\beta\phi)(\nabla^{\nabla f}(F^2(\nabla f))) - 2\mu\beta\phi D(\Delta_m f + F^2(\nabla f))(\nabla f)\\
&\quad - \phi F^2(\nabla f)(\mu\beta)' + 2\phi\mu\beta D(F^2(\nabla f))(\nabla f)\\
&\quad + \phi\mu\alpha'\partial_t f + \phi\alpha\mu'\partial_t f + \phi(\mu\varphi)'\}\,dm\,dt\\
&= \int_0^T\int_M \{-D(\mu\beta\phi)(\nabla^{\nabla f}(F^2(\nabla f))) - 2\mu\beta\phi D(\Delta_m f)(\nabla f)
\end{aligned}$$

$$
\begin{aligned}
&- \phi F^2(\nabla f)(\mu\beta)' + \phi\mu\alpha'\partial_t f + \phi\alpha\mu'\partial_t f + \phi(\mu\varphi)'\Big\}\, dm\, dt \\
\geq &\int_0^T \int_M \phi \left\{ \frac{2\mu\beta(\Delta_m f)^2}{N} - (2\mu\beta K + (\mu\beta)')F^2(\nabla f) \right. \\
&\left. + \mu\alpha'\partial_t f + \alpha\mu'\partial_t f + (\mu\varphi)' \right\} dm\, dt,
\end{aligned}
$$

where the second equality used (3.2) and (3.4) and the last one used (2.10). □

3.2. Proof of Theorem 1.1. To prove Theorem 1.1, let us first give some auxiliary lemmas.

Lemma 3.4. *Let $G_1(x,t) := t\{F^2(\nabla f(x,t)) - \partial_t f(x,t)\}$, then $G_1(x,t)$ satisfies the parabolic differential inequality*

$$
\Delta_m^{\nabla f} G_1 - \partial_t G_1 + 2DG_1(\nabla f) \geq D_1(x,t) \tag{3.7}
$$

in the distributional sense on $M \times (0,T)$, where

$$
D_1(x,t) = \frac{2}{Nt} G_1 \left(G_1 - \frac{N}{2} \right).
$$

Proof. By Lemma 3.3, we obtain

$$
\begin{aligned}
\Delta_m^{\nabla f} G_1 - \partial_t G_1 + 2DG_1(\nabla f) \geq &\frac{2t(\Delta_m f)^2}{N} - F^2(\nabla f) + \partial_t f \\
= &- \frac{G_1}{t} + \frac{2t}{N}(F^2(\nabla f) - \partial_t f)^2 \frac{2}{Nt} G_1 \left(G_1 - \frac{N}{2} \right)
\end{aligned}
$$

in the distributional on $M \times (0,T)$. □

We also need to use a new normal vector field on ∂M defined in [25], that is normal with respect to the Riemannian metric $g_{\nabla u}$. To be more general, for every $X \in TM$, there is a unique normal vector field ν_X such that

$$
g_X(\nu_X, Y) = 0 \quad \text{for any } Y \in T(\partial M),\ g_X(\nu_X, \nu_X) = 1,\ g_\nu(\nu, \nu_X) > 0. \tag{3.8}
$$

A simple calculation shows that

$$
g_X(\nu_X, \nu) > 0. \tag{3.9}
$$

Lemma 3.5 ([25]). *Let $X, Y \in T(M)$. Then*

$$
g_\nu(\nu, Y) = 0 \ \Leftrightarrow \ Y \in T(\partial M) \ \Leftrightarrow \ g_X(\nu_X, Y) = 0. \tag{3.10}
$$

Lemma 3.6 ([25]). *Let*

$$
T_\pm^\nu M = \{ Y \in TM \mid g_\nu(\nu, Y) > 0 (< 0) \}
$$

and

$$
T_\pm^{\nu_X} M = \{ Y \in TM \mid g_X(\nu_X, Y) > 0 (< 0) \}.
$$

Then

$$
T_+^\nu M = T_+^{\nu_X} M, \qquad T_-^\nu M = T_-^{\nu_X} M.
$$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. By setting $f = \log(u+\varepsilon)$ for $\varepsilon > 0$, one verifies that f satisfies (3.2). The theorem claims that G_1 is at most $\frac{N}{2}$. If not, at the maximum point (x_0, t_0) of G_1 on $M \times (0, T)$ for some $T > 0$,

$$G_1(x_0, t_0) > \frac{N}{2} > 0.$$

Clearly, $t_0 > 0$, because $G_1(x, 0) = 0$. If x_0 is an interior point of M, then by the fact that (x_0, t_0) is a maximum point of G_1 on $M \times (0, T)$, we have $D_1(x_0, t_0) \leq 0$. Assume the contrary, $D_1(x_0, t_0) > 0$. It would imply $D_1 > 0$ on a neighborhood of (x_0, t_0). Hence, according to (3.7) on such a neighborhood, the function G_1 would be strict to the linear parabolic operator

$$\Delta_m^{\nabla f} G_1 - \partial_t G_1 + 2DG_1(\nabla f).$$

Therefore, $G_1(x_0, t_0)$ would be strictly less than the supremum of G_1 on the boundary of any small parabolic cylinder $[t_0 - \delta, t_0] \times B_\delta(x_0)$, where $B_\delta(x_0) := \{y \in M | d(x_0, y) < \delta\}$. In particular, G_1 could not be maximal at (x_0, t_0), which is a contradiction. Hence, $D_1(x_0, t_0) \leq 0$, that is

$$\frac{2}{Nt_0} G_1(x_0, t_0) \left(G_1(x_0, t_0) - \frac{N}{2} \right) \leq 0,$$

which is a contradiction. Hence x_0 must be on ∂M.

Now we consider the case when G_1 attains its maximum at $x_0 \in \partial M$. Recall that $G_1 \in C^1(M_u)$. Since $\nu_{\nabla u}$ points outward due to its definition, by (3.7), the strong maximum principle yields

$$DG_1(\nu_{\nabla u})(x_0, t_0) > 0.$$

On one hand, the Neumann boundary condition $\nabla u \in T(\partial M)$ implies that

$$Du(\nu_{\nabla u})(x, t) = 0 \quad \text{for } (x, t) \in \partial M \times (0, \infty).$$

Thus we have

$$DG_1(\nu_{\nabla u})(x_0, t_0) = (tD(F^2(\nabla f))(\nu_{\nabla u}))(x_0, t_0). \tag{3.11}$$

On the other hand, using (2.2) and the symmetry of $\nabla^2 u$, we have

$$\begin{aligned} D(F^2(\nabla f))(\nu_{\nabla u}) &= D(g_{\nabla u}(\nabla u, \nabla u))(\nu_{\nabla u}) && (3.12) \\ &= 2g_{\nabla u}(D^{\nabla u}_{\nu_{\nabla u}} \nabla u, \nabla u) = 2g_{\nabla u}(D^{\nabla u}_{\nabla u} \nabla u, \nu_{\nabla u}). && (3.13) \end{aligned}$$

By the convexity of ∂M, for any $X \in T(\partial M)$, $g_\nu(D^X_X X, \nu) \leq 0$. In particular, set $X = \nabla u$, we know that

$$g_\nu(D^{\nabla u}_{\nabla u} \nabla u, \nu) \leq 0. \tag{3.14}$$

It follows from Lemmas 3.5 and 3.6 that (3.14) is equivalent to

$$g_{\nabla u}(D^{\nabla u}_{\nabla u}\nabla u, \nu_{\nabla u}) \leq 0. \tag{3.15}$$

Combining (3.11), (3.12) and (3.15), we conclude that

$$DG_1(\nu_{\nabla u})(x_0, t_0) \leq 0.$$

It yields a contradiction. Hence

$$G_1 \leq \frac{N}{2}$$

and the theorem follows by letting $\varepsilon \to 0$. □

Remark 3.7. In the proof of Theorem 1.1, compared with the Riemannian case as in [12], in our case, we need to overcome two obstructions, the one is how to prove $D_1(x_0, t_0) \leq 0$. In the Finsler case, $\Delta_m u$ has no definition at the maximum point of u, and thus we cannot use Finsler Laplacian to adopt maximum principle. To overcome it, we use the methods as in the proof of Theorem 4.4 in [19]. The other one is how to apply Neumann boundary condition and convexity to find contradictions. In the Finsler case, we use the methods as in the proof of Theorem 3.1 in [25].

3.3. Proof of Theorem 1.3. In this section we will complete the proof of Theorem 1.3. We first give an auxiliary lemma.

Lemma 3.8. *Let* $G_2(x,t) := t\{\beta(t)F^2(\nabla f(x,t)) - \partial_t f(x,t)\}$*, then* $G_2(x,t)$ *satisfies the parabolic differential inequality*

$$\Delta^{\nabla f}_m G_2 - \partial_t G_2 + 2DG_2(\nabla f) \geq D_2(x,t) \tag{3.16}$$

in the distributional sense on $M \times (0,T)$*, where*

$$D_2(x,t) = -\frac{G_2}{t} + \frac{2t\beta(t)}{N}\left[\frac{G_2}{t} + (1-\beta(t))F^2(\nabla f)\right]^2 - [2K\beta(t) + \beta'(t)]tF^2(\nabla f).$$

Proof. By Lemma 3.3, we obtain

$$\begin{aligned}
\Delta^{\nabla f}_m G_2 - \partial_t G_2 + 2DG_2(\nabla f) \geq& \frac{2t\beta(\Delta_m f)^2}{N} - (2t\beta K + (t\beta)')F^2(\nabla f) + \partial_t f \\
=&\frac{2t\beta(\Delta_m f)^2}{N} - (2t\beta K + t\beta')F^2(\nabla f) - \beta F^2(\nabla f) + \partial_t f \\
=& -\frac{G_2}{t} + \frac{2t\beta(\Delta_m f)^2}{N} - (2t\beta K + t\beta')F^2(\nabla f) \\
=& -\frac{G_2}{t} + \frac{2t\beta}{N}\left[\frac{G_2}{t} + (1-\beta)F^2(\nabla f)\right]^2 - [2K\beta + \beta']tF^2(\nabla f)
\end{aligned}$$

in the distributional on $M \times (0,T)$, where the last equality used $\Delta_m f = \frac{G_2}{t} + (1-\beta)F^2(\nabla f)$. □

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Fix arbitrary $t \in (0, T]$ and assume that G_2 achieves its maximum at the point $(x_0, t_0) \in M \times [0, t]$ and $G_2(x_0, t_0) > 0$ (otherwise the proof is trivial), which implies $t_0 > 0$. By an argument analogue to the proof of Theorem 1.1, one can show that $D_2(x_0, t_0) \leq 0$, that is at (x_0, t_0),

$$\begin{aligned} 0 &\geq -\frac{G_2}{t_0} + \frac{2t_0\beta(t_0)}{N}\left[\frac{G_2}{t_0} + (1-\beta(t_0))F^2(\nabla f)\right]^2 - [2K\beta(t_0) + \beta'(t_0)]t_0F^2(\nabla f) \\ &\geq \frac{2t_0\beta(t_0)}{N}\left(\frac{1}{t_0} + (1-\beta(t_0))Q\right)^2 G_2^2 - \left[(2K\beta(t_0) + \beta'(t_0))_+ t_0 Q + \frac{1}{t_0}\right] G_2, \end{aligned}$$

where $Q = G_2^{-1}F^2(\nabla f)(x_0, t_0)$. Multiplying t_0 to the last inequality, we have, at the point (x_0, t_0),

$$0 \geq \frac{2\beta(t_0)}{N}\left(1 + (1-\beta(t_0))Qt_0\right)^2 G_2^2 - \left[(2K\beta(t_0) + \beta'(t_0))_+ t_0^2 Q + 1\right] G_2. \quad (3.17)$$

By (B_2), we know that $\beta(0) > 0$, so $\min_{[0,T]} \beta > 0$. Hence,

$$\frac{2\beta(t_0)}{N}\left(1 + (1-\beta(t_0))Qt_0\right)^2 > 0.$$

Then, by (3.17),

$$G_2(x_0, t_0) \leq \frac{N}{2\beta(t_0)} \frac{(2K\beta(t_0) + \beta'(t_0))_+ t_0^2 Q + 1}{(1 + (1-\beta(t_0))t_0 Q)^2}. \quad (3.18)$$

Moreover, note that

$$\frac{aQ + c}{(1 + bQ)^2} \leq \frac{a}{4b} + c, \quad (3.19)$$

where $a, b, c, Q > 0$. Since $t \geq t_0$, applying (3.19) to (3.18), we have

$$\begin{aligned} G_2(x, t) \leq G_2(x_0, t_0) &\leq \frac{N}{2\beta(t_0)}\left(\frac{(2K\beta(t_0) + \beta'(t_0))_+ t_0}{4(1-\beta(t_0))} + 1\right) \\ &= \frac{N}{2}\left(\frac{(2K\beta(t_0) + \beta'(t_0))_+ t_0}{4(1-\beta(t_0))\beta(t_0)} + \frac{1}{\beta(t_0)}\right) \leq t\psi_1(t). \quad (3.20) \end{aligned}$$

Since t is arbitrary in $0 < t \leq T$, we have (1.11). Theorem 1.3 is proved. □

Next, we will prove Corollary 1.5 and 1.7 from Theorem 1.3.

Proof of Corollary 1.5. This can be proved using arguments similar to those for Corollary 2.1 and Corollary 2.3 in [27]. We omit the proofs here. □

Proof of Corollary 1.7. Replacing u by $u + \varepsilon$ if necessary, we may assume without restriction that u is positive. Along the lines of Li–Yau, let the reverse curve $\gamma(\tau) = \exp_{x_1}((t_1 - \tau)v)$ for $\tau \in [t_2, t_1]$ be a shortest geodesic joining $x_1 =$

$\gamma(t_1)$ and $x_2 = \gamma(t_2)$ with suitable $v \in T_{x_1}M$. Then obviously $F(-\dot\gamma(\tau)) = \frac{d(x_2,x_1)}{(t_1-t_2)}$ for all τ. From (1.11), we have

$$-\partial_t(\log u) \le \psi_1(t) - \beta(t)F(\nabla(\log u))^2.$$

We also put $f := \log u$ and have

$$\begin{aligned} f(x_2,t_2) - f(x_1,t_1) &= \int_{t_1}^{t_2} \frac{d}{dt}\Big(f(\gamma(t),t)\Big)dt = \int_{t_1}^{t_2} \Big\{Df(\dot\gamma) + \partial_t f\Big\}dt \\ &= \int_{t_2}^{t_1} \Big\{-Df(\dot\gamma) - \partial_t f\Big\}dt \le \int_{t_2}^{t_1} \Big\{F(-\dot\gamma(t))F(\nabla f) - \partial_t f\Big\}dt \\ &\le \int_{t_2}^{t_1} \left\{-\beta(t)F^2(\nabla f) + F(-\dot\gamma(t))F(\nabla f) + \psi_1(t)\right\} dt \\ &\le \int_{t_2}^{t_1} \left\{\frac{1}{4\beta(t)}\frac{d(x_1,x_2)^2}{(t_1-t_2)^2} + \psi_1(t)\right\} dt, \end{aligned} \tag{3.21}$$

where the last inequality used $-Ax^2 + Cx \le \frac{C^2}{4A}$. From (3.21), we have

$$\log\left(\frac{u(x_2,t_2)}{u(x_1,t_1)}\right) = f(x_2,t_2) - f(x_1,t_1) \le \int_{t_2}^{t_1}\left\{\frac{1}{4\beta(t)}\frac{d(x_1,x_2)^2}{(t_1-t_2)^2} + \psi_1(t)\right\} dt.$$

Therefore, we arrive at

$$u(x_2,t_2) \le u(x_1,t_1)\exp\left\{\int_{t_2}^{t_1}\psi_1(t)dt + \int_{t_2}^{t_1}\frac{d(x_1,x_2)^2}{4\beta(t)(t_1-t_2)^2}dt\right\}.$$

It ends the proof of Corollary 1.7. □

4. Gradient estimates under Finsler–Ricci flows

4.1. Some lemmas. In this section, we consider a positive global solution of the nonlinear heat equation (1.8) under Finsler–Ricci flows $(M^n, F(t), m)$. The Laplacian and gradient are both with respect to $V := \nabla u$ and are valid on $M_u := \{x \in M \mid \nabla u(x) \neq 0\}$. Let $f = \log u$. Although Chern connection coefficient $\Gamma^i_{jk}(\nabla f)$ is not compatible with respect to $g_{\nabla f}$, it is torsion free. Hence, similar to Riemannian case, for a given time t, we can choose a normal coordinate system at a fixed point of M_u. We will compute at a fixed point and at this point we have

$$\begin{aligned} \mathrm{Ric}^{ij}(\nabla f) = \mathrm{Ric}_{ij}(\nabla f), \quad |\nabla^2 f|^2_{HS(\nabla f)} = \sum_{i,j} f^2_{ij}, \\ \sum_{i=1}^n f_{ii} = \Delta_m f, \quad \Gamma^i_{jk}(\nabla f) = 0. \end{aligned} \tag{4.1}$$

First, we will use the following obvious lemma:

Lemma 4.1 ([3]). *Let* Ric_{ij} *be a component of Akbarzadeh's Ricci tensor and* Ric *be the Ricci curvature on* (M^n, F, m). *We have*

$$\mathrm{Ric}^{ij}(\nabla f) f_i f_j = \mathrm{Ric}(\nabla f). \tag{4.2}$$

Proceeding, we have the following lemma:

Lemma 4.2 ([22]). *Let* $(M^n, F(t), m)$ *be a closed solution to the Finsler–Ricci flow* (1.16). *Then we have*

$$\partial_t(F^2(\nabla f)) = 2\,\mathrm{Ric}(\nabla f) + 2D(\partial_t f)(\nabla f). \tag{4.3}$$

Now let us consider the function

$$\mathcal{H} = \lambda(\alpha F^2(\nabla f) - \partial_t f) = \lambda\alpha F^2(\nabla f) - \mathcal{L}, \tag{4.4}$$

where $\lambda(t)$ and $\alpha(t)$ are two functions depending on t. $\mathcal{L}(x,t) = \lambda(t)\partial_t f$ lies in $H^1(M)$ and $\mathcal{H}(x,t)$ lies in $H^1(M)$ for each t and is Hölder continuous in both space and time.

Proceeding, we have the following lemma:

Lemma 4.3. *In the sense of distributions,* $\mathcal{L}(x,t)$ *satisfies the parabolic differential equality*

$$\begin{aligned}\Delta_m^{\nabla f}\mathcal{L} - \partial_t\mathcal{L} + 2D\mathcal{L}(\nabla f) = {}& -2\lambda\,\mathrm{Ric}^{ij}(\nabla f) f_i f_j - 2\lambda\,\mathrm{Ric}^{ij}(\nabla f) f_{ij} - 2\lambda\,\mathrm{Ric}^{ij}_{\ |i} f_j \\ & - 2\lambda\,\mathrm{Ric}^{ij}_{\ ;k}\frac{f_j}{F}(\nabla^2 f)_i^k - \lambda'\partial_t f,\end{aligned} \tag{4.5}$$

where $\mathrm{Ric}^{ij}_{\ |i}$ *denotes the horizontal covariant derivative of* Ric^{ij} *and* $\mathrm{Ric}^{ij}_{\ ;k}$ *denotes the vertical covariant derivative of* Ric^{ij}.

Proof. For any non-negative test function $\phi \in H_0^1(M\times(0,T))$ whose support is included in the domain of the local coordinate, we have

$$\begin{aligned}\partial_t(D(\lambda\phi)(\nabla f)) &= \partial_t(g^{ij}(\nabla f)(\lambda\phi)_i f_j) \\ &= \partial_t(g^{ij}(\nabla f))(\lambda\phi)_i f_j + g^{ij}(\nabla f)(\partial_t(\lambda\phi))_i f_j + g^{ij}(\nabla f)(\lambda\phi)_i(\partial_t f)_j \\ &= \partial_t(g^{ij})(\nabla f)(\lambda\phi)_i f_j + \frac{\partial g^{ij}}{\partial y^k}\partial_t(f^k)(\lambda\phi)_i f_j + g^{ij}(\nabla f)(\partial_t(\lambda\phi))_i f_j \\ &\quad + g^{ij}(\nabla f)(\lambda\phi)_i(\partial_t f)_j \\ &= 2\,\mathrm{Ric}^{ij}(\nabla f)(\lambda\phi)_i f_j + D(\partial_t(\lambda\phi))(\nabla f) + D(\lambda\phi)(\nabla^{\nabla f}(\partial_t f)),\end{aligned}$$

where the last equality used $\partial_t g^{ij} = 2\,\mathrm{Ric}^{ij}$ and $C_V(V,X,Y)=0$. That is,

$$\begin{aligned}-D(\lambda\phi)(\nabla^{\nabla f}(\partial_t f)) = {}& -\partial_t(D(\lambda\phi)(\nabla f)) + D(\partial_t(\lambda\phi))(\nabla f) \\ & + 2\,\mathrm{Ric}^{ij}(\nabla f)\frac{\partial(\lambda\phi)}{\partial x^i}\frac{\partial f}{\partial x^j}.\end{aligned} \tag{4.6}$$

Multiplying the left-hand side of (4.5) by ϕ, integrating and then substituting (4.6), we get

$$\begin{aligned}
\mathcal{A} &= \int_0^T \int_M \left\{ -D\phi(\nabla^{\nabla f}\mathcal{L}) + \mathcal{L}\partial_t\phi + 2\phi D\mathcal{L}(\nabla f) \right\} dm\, dt \\
&= \int_0^T \int_M \left\{ -D(\lambda\phi)(\nabla^{\nabla f}(\partial_t f)) + \lambda\partial_t f\partial_t\phi + 2\lambda\phi D(\partial_t f)(\nabla f) \right\} dm\, dt \\
&= \int_0^T \int_M \{ -D(\lambda\phi)(\nabla^{\nabla f}(\partial_t f)) + \partial_t(\phi\lambda)\partial_t f \\
&\quad - \lambda'\phi\partial_t f + 2\lambda\phi D(\partial_t f)(\nabla f) \} \, dm\, dt \\
&= \int_0^T \int_M \{ -\partial_t(D(\lambda\phi)(\nabla f)) + D(\partial_t(\lambda\phi))(\nabla f) + 2\operatorname{Ric}^{ij}(\nabla f)\frac{\partial(\lambda\phi)}{\partial x^i}\frac{\partial f}{\partial x^j} \\
&\quad - \lambda'\phi\partial_t f + \partial_t(\phi\lambda)(\Delta_m f + F^2(\nabla f)) + 2\lambda\phi D(\partial_t f)(\nabla f) \} \, dm\, dt.
\end{aligned}$$

Using Lemma 4.2 and the fact that

$$\int_0^T \int_M \partial_t(D(\lambda\phi)(\nabla f))\, dm\, dt = 0,$$

we arrive at

$$\begin{aligned}
\mathcal{A} &= \int_0^T \int_M \{ -\partial_t(D(\lambda\phi)(\nabla f)) + D(\partial_t(\lambda\phi))(\nabla f) + 2\operatorname{Ric}^{ij}(\nabla f)\frac{\partial(\lambda\phi)}{\partial x^i}\frac{\partial f}{\partial x^j} \\
&\quad - \phi\partial_t f\partial_t\lambda + \partial_t(\phi\lambda)(\Delta_m f + F^2(\nabla f)) + 2\lambda\phi D(\partial_t f)(\nabla f) \} \, dm\, dt \\
&= \int_0^T \int_M \left\{ 2\operatorname{Ric}^{ij}(\nabla f)\frac{\partial(\lambda\phi)}{\partial x^i}\frac{\partial f}{\partial x^j} - \lambda'\phi\partial_t f + \partial_t(\phi\lambda)(F^2(\nabla f)) \right. \\
&\quad \left. + 2\lambda\phi D(\partial_t f)(\nabla f) \right\} dm\, dt \\
&= \int_0^T \int_M \left\{ 2\operatorname{Ric}^{ij}(\nabla f)\frac{\partial(\lambda\phi)}{\partial x^i}\frac{\partial f}{\partial x^j} - \lambda'\phi\partial_t f + \partial_t(\phi\lambda)(F^2(\nabla f)) \right. \\
&\quad \left. + \lambda\phi\partial_t(F^2(\nabla f)) - 2\lambda\phi\operatorname{Ric}^{ij}(\nabla f) f_i f_j \right\} dm\, dt \\
&= \int_0^T \int_M \phi \left\{ -2\lambda\operatorname{Ric}^{ij}(\nabla f) f_i f_j - 2\lambda\operatorname{Ric}^{ij}(\nabla f) f_{ij} \right. \\
&\quad \left. - 2\lambda\frac{\partial(\operatorname{Ric}^{ij}(\nabla f))}{\partial x^i} f_j - \lambda'\partial_t f \right\} dm\, dt. \qquad (4.7)
\end{aligned}$$

We note that

$$\frac{\partial(\operatorname{Ric}^{ij}(\nabla f))}{\partial x^i} f_j = \left(\frac{\partial\operatorname{Ric}^{ij}}{\partial x^i}(\nabla f) + \frac{\partial\operatorname{Ric}^{ij}}{\partial y^k}\frac{\partial(\nabla f)^k}{\partial x^i} \right) f_j$$

$$= \mathrm{Ric}^{ij}_{\ |i} f_j + \mathrm{Ric}^{ij}_{\ ;k} \frac{f_j}{F} (\nabla^2 f)^k_i. \tag{4.8}$$

Hence we have

$$\mathcal{A} = \int_0^T \int_M \phi \Big\{ -2\lambda \mathrm{Ric}^{ij}(\nabla f) f_i f_j - 2\lambda \mathrm{Ric}^{ij}(\nabla f) f_{ij} - 2\lambda \mathrm{Ric}^{ij}_{\ |i} f_j - 2\lambda \mathrm{Ric}^{ij}_{\ ;k} \frac{f_j}{F} (\nabla^2 f)^k_i - \lambda' \partial_t f \Big\} dm dt. \tag{4.9}$$

The lemma is proved. □

Now we can compute a parabolic partial differential equality for $\mathcal{H}(x,t)$ which will be the key to the proof of our Theorem 1.10.

Lemma 4.4. *In the sense of distributions, $\mathcal{H}(x,t)$ satisfies the parabolic differential equality*

$$\Delta_m^{\nabla f} \mathcal{H} - \partial_t \mathcal{H} + 2D\mathcal{H}(\nabla f) = \mathcal{B}, \tag{4.10}$$

where

$$\mathcal{B}(x,t) = \left(\frac{\alpha'}{1-\alpha} - (\ln \lambda)' \right) \mathcal{H} + \lambda \bigg(\frac{\alpha'}{1-\alpha} \Delta_m f + 2 \mathrm{Ric}^{ij}(\nabla f) f_i f_j + 2 \mathrm{Ric}^{ij}(\nabla f) f_{ij} + 2 \mathrm{Ric}^{ij}_{\ |i} f_j + 2 \mathrm{Ric}^{ij}_{\ ;k} \frac{f_j}{F} (\nabla^2 f)^k_i + 2\alpha |\nabla^2 f|^2_{HS(\nabla f)} \bigg).$$

Proof. For any non-negative test function $\phi \in H^1_0(M \times (0,T))$, we compute

$$\begin{aligned}
&\int_0^T \int_M \Big\{ -D\phi(\nabla^{\nabla f} \mathcal{H}) + \mathcal{H} \partial_t \phi + 2\phi D\mathcal{H}(\nabla f) \Big\} dm\, dt \\
&= -\mathcal{A} + \int_0^T \int_M \big\{ -D(\lambda\alpha\phi)(\nabla^{\nabla f}(F^2(\nabla f))) + \lambda\alpha F^2(\nabla f) \partial_t \phi \\
&\quad + 2\lambda\alpha\phi D(F^2(\nabla f))(\nabla f) \big\} dm\, dt \\
&= -\mathcal{A} + \int_0^T \int_M \big\{ -D(\lambda\alpha\phi)(\nabla^{\nabla f}(F^2(\nabla f))) - \phi((\lambda\alpha)' F^2(\nabla f) \\
&\quad + \lambda\alpha \partial_t (F^2(\nabla f))) + 2\lambda\alpha\phi D(F^2(\nabla f))(\nabla f) \big\} dm\, dt \\
&= -\mathcal{A} + \int_0^T \int_M \big\{ -D(\lambda\alpha\phi)(\nabla^{\nabla f}(F^2(\nabla f))) - \phi(\lambda\alpha)' F^2(\nabla f) \\
&\quad - \phi\lambda\alpha(2 \mathrm{Ric}(\nabla f) + 2D(\partial_t f)(\nabla f)) + 2\lambda\alpha\phi D(F^2(\nabla f))(\nabla f) \big\} dm\, dt \\
&= -\mathcal{A} + \int_0^T \int_M \big\{ -D(\lambda\alpha\phi)(\nabla^{\nabla f}(F^2(\nabla f))) - \phi(\lambda\alpha)' F^2(\nabla f) \\
&\quad - 2\phi\lambda\alpha \mathrm{Ric}(\nabla f) - 2\phi\lambda\alpha D(\Delta_m f + F^2(\nabla f))(\nabla f)) \\
&\qquad + 2\lambda\alpha\phi D(F^2(\nabla f))(\nabla f) \big\} dm\, dt \\
&= -\mathcal{A} + \int_0^T \int_M \big\{ -D(\lambda\alpha\phi)(\nabla^{\nabla f}(F^2(\nabla f))) - \phi(\lambda\alpha)' F^2(\nabla f) \\
&\quad - 2\phi\lambda\alpha \mathrm{Ric}(\nabla f) - 2\phi\lambda\alpha D(\Delta_m f)(\nabla f) \big\} dm\, dt.
\end{aligned}$$

By applying the Bochner–Weitzenböck formula (2.9) and noticing that $S = 0$ implies $\mathrm{Ric}_\infty(V) = \mathrm{Ric}(V)$, we have

$$\begin{aligned} -\mathcal{A} + \int_0^T \int_M \{ &- D(\lambda\alpha\phi)(\nabla^{\nabla f}(F^2(\nabla f))) - \phi(\lambda\alpha)' F^2(\nabla f) \\ &- 2\phi\lambda\alpha \,\mathrm{Ric}(\nabla f) - 2\phi\lambda\alpha D(\Delta_m f)(\nabla f)\} \, dm\, dt \\ &= -\mathcal{A} + \int_0^T \int_M \{2\phi\lambda\alpha|\nabla^2 f|^2_{HS(\nabla f)} - \phi(\lambda\alpha)'F^2(\nabla f)\}\, dm\, dt. \end{aligned}$$

Now, substituting $\mathcal{A}$ from (4.9), we have

$$\begin{aligned} \mathcal{B}(x,t) = {} & 2\lambda \,\mathrm{Ric}^{ij}(\nabla f) f_i f_j + 2\lambda\,\mathrm{Ric}^{ij}(\nabla f) f_{ij} + 2\lambda\,\mathrm{Ric}^{ij}_{\ |i} f_j \\ & + 2\lambda\alpha|\nabla^2 f|^2_{HS(\nabla f)} + 2\lambda\,\mathrm{Ric}^{ij}_{\ ;k}\frac{f_j}{F}(\nabla^2 f)^k_i + \lambda'\partial_t f - (\lambda\alpha)'F^2(\nabla f). \end{aligned}$$

Using the fact that

$$\begin{aligned} \lambda'\partial_t f - (\lambda\alpha)'F^2(\nabla f) &= \left(\frac{\alpha'}{1-\alpha} - (\ln\lambda)'\right)\lambda(\alpha F^2(\nabla f) - \partial_t f) + \lambda\frac{\alpha'}{1-\alpha}\Delta_m f \\ &= \left(\frac{\alpha'}{1-\alpha} - (\ln\lambda)'\right)\mathcal{H} + \lambda\frac{\alpha'}{1-\alpha}\Delta_m f, \end{aligned}$$

we arrive at

$$\begin{aligned} \mathcal{B}(x,t) = {} & \left(\frac{\alpha'}{1-\alpha} - (\ln\lambda)'\right)\mathcal{H} + \lambda\bigg(\frac{\alpha'}{1-\alpha}\Delta_m f + 2\,\mathrm{Ric}^{ij}(\nabla f) f_i f_j \\ & + 2\,\mathrm{Ric}^{ij}(\nabla f) f_{ij} + 2\,\mathrm{Ric}^{ij}_{\ |i} f_j + 2\,\mathrm{Ric}^{ij}_{\ ;k}\frac{f_j}{F}(\nabla^2 f)^k_i + 2\alpha|\nabla^2 f|^2_{HS(\nabla f)}\bigg). \end{aligned}$$

The lemma is proved. □

4.2. Proof of Theorem 1.10. By Lemma 4.4, we have

$$\begin{aligned} & \Delta_m^{\nabla f}\mathcal{H} - \partial_t\mathcal{H} + 2D\mathcal{H}(\nabla f) \\ & = \left(\frac{\alpha'}{1-\alpha} - (\ln\lambda)'\right)\mathcal{H} + \lambda\bigg(\frac{\alpha'}{1-\alpha}\Delta_m f + 2\,\mathrm{Ric}^{ij}(\nabla f) f_i f_j \\ & \quad + 2\,\mathrm{Ric}^{ij}(\nabla f) f_{ij} + 2\,\mathrm{Ric}^{ij}_{\ |i} f_j + 2\,\mathrm{Ric}^{ij}_{\ ;k}\frac{f_j}{F}(\nabla^2 f)^k_i + 2\alpha|\nabla^2 f|^2_{HS(\nabla f)}\bigg) \\ & \geq \left(\frac{\alpha'}{1-\alpha} - (\ln\lambda)'\right)\mathcal{H} + \lambda\bigg(\frac{\alpha'}{1-\alpha}\Delta_m f - 2K_1F^2(\nabla f) - 2\varepsilon\alpha f_{ij}^2 \\ & \quad - \frac{n^2}{2\varepsilon\alpha}\max\{K_1^2, K_2^2\} \\ & \quad - 2\varepsilon F^2(\nabla f) - \frac{1}{2\varepsilon}K_3^2 - 2\varepsilon\alpha f_{ik}^2 - \frac{1}{2\varepsilon\alpha}K_3^2 + 2\alpha|\nabla^2 f|^2_{HS(\nabla f)}\bigg) \\ & \geq \left(\frac{\alpha'}{1-\alpha} - (\ln\lambda)'\right)\mathcal{H} + \lambda\bigg(\frac{2\alpha(1-2\varepsilon)}{n}(\Delta_m f)^2 + \frac{\alpha'}{1-\alpha}\Delta_m f \end{aligned}$$

$$- 2(K_1+\varepsilon)F^2(\nabla f) - \frac{1+\alpha}{2\varepsilon\alpha}K_3^2 - \frac{n^2}{2\varepsilon\alpha}\max\{K_1^2, K_2^2\}\Big), \tag{4.11}$$

where we used

$$2\,\mathrm{Ric}_{ij} f_{ij} \geq -2\varepsilon\alpha\sum_{i,j} f_{ij}^2 - \frac{1}{2\varepsilon\alpha}\sum_{i,j}\mathrm{Ric}_{ij}^2 \geq -2\varepsilon\alpha\sum_{i,j} f_{ij}^2 - \frac{1}{2\varepsilon\alpha}\max\{K_1^2, K_2^2\},$$

$$2\,\mathrm{Ric}^{ij}_{\ |i} f_j \geq -2\varepsilon F^2(\nabla f) - \frac{1}{2\varepsilon}\sum_{i,j}(\mathrm{Ric}^{ij}_{\ |i})^2 \geq -2\varepsilon F^2(\nabla f) - \frac{1}{2\varepsilon}K_3^2,$$

$$\begin{aligned}2\,\mathrm{Ric}^{ij}_{\ ;k}\frac{f_j}{F}(\nabla^2 f)^k_i &\geq -2\varepsilon\alpha\sum_{i,k} f_{ik}^2 - \frac{1}{2\varepsilon\alpha}\sum_{i,k}\left(\sum_j \mathrm{Ric}^{ij}_{\ ;k}\frac{f_j}{F}\right)^2 \\ &\geq -2\varepsilon\alpha\sum_{i,k} f_{ik}^2 - \frac{1}{2\varepsilon\alpha}\sum_{i,j,k}\left(\mathrm{Ric}^{ij}_{\ ;k}\right)^2 \geq -2\varepsilon\alpha\sum_{i,k} f_{ik}^2 - \frac{1}{2\varepsilon\alpha}K_3^2\end{aligned}$$

and the Cauchy inequality

$$\sum_{i,j} f_{ij}^2 \geq \frac{1}{n}(\Delta_m f)^2.$$

Noticing that

$$\Delta_m f = -(F^2(\nabla f) - \partial_t f),$$

hence (4.11) shows

$$\begin{aligned}\Delta_m^{\nabla f}\mathcal{H} - \partial_t\mathcal{H} + 2D\mathcal{H}(\nabla f) &\geq \left(\frac{\alpha'}{1-\alpha} - (\ln\lambda)'\right)\mathcal{H} \\ &+ \lambda\Big(\frac{2\alpha(1-2\varepsilon)}{n}(F^2(\nabla f) - \partial_t f)^2 - \frac{\alpha'}{1-\alpha}(F^2(\nabla f) - \partial_t f) \\ &- 2(K_1+\varepsilon)F^2(\nabla f) - \frac{1+\alpha}{2\varepsilon\alpha}K_3^2 - \frac{n^2}{2\varepsilon\alpha}\max\{K_1^2, K_2^2\}\Big) := \mathcal{D}(x,t).\end{aligned} \tag{4.12}$$

Let $w = F^2(\nabla f)$ and $z = \partial_t f$. We have

$$\begin{aligned}(w-z)^2 &= (\alpha w - z)^2 + 2(1-\alpha)w(\alpha w - z) + (1-\alpha)^2 w^2 \\ &= \frac{\mathcal{H}^2}{\lambda^2} + 2(1-\alpha)w\frac{\mathcal{H}}{\lambda} + (1-\alpha)^2 w^2.\end{aligned}$$

Then,

$$\begin{aligned}\mathcal{D}(x,t) &= \left(\frac{\alpha'}{1-\alpha} - (\ln\lambda)'\right)\mathcal{H} + \lambda\Big(\frac{2\alpha(1-2\varepsilon)}{n}(w-z)^2 - \frac{\alpha'}{1-\alpha}(w-z) \\ &\quad - 2(K_1+\varepsilon)w - \frac{1+\alpha}{2\varepsilon\alpha}K_3^2 - \frac{n^2}{2\varepsilon\alpha}\max\{K_1^2, K_2^2\}\Big) \\ &= \left(\frac{\alpha'}{1-\alpha} - (\ln\lambda)'\right)\mathcal{H} + \lambda\Big(\frac{2\alpha(1-2\varepsilon)}{n}\left(\frac{\mathcal{H}^2}{\lambda^2} + 2(1-\alpha)w\frac{\mathcal{H}}{\lambda} + (1-\alpha)^2 w^2\right)\end{aligned}$$

$$
\begin{aligned}
&- \frac{\alpha'}{1-\alpha}\frac{\mathcal{H}}{\lambda} - \alpha' w - 2(K_1+\varepsilon)w - \frac{1+\alpha}{2\varepsilon\alpha}K_3^2 - \frac{n^2}{2\varepsilon\alpha}\max\{K_1^2, K_2^2\}\Big) \\
&= -(\ln\lambda)'\mathcal{H} + \lambda\Big(\frac{2\alpha(1-2\varepsilon)}{n}\frac{\mathcal{H}^2}{\lambda^2} + \frac{4(1-2\varepsilon)}{n}\alpha(1-\alpha)w\frac{\mathcal{H}}{\lambda} \\
&\quad + \frac{2(1-2\varepsilon)}{n}\alpha(1-\alpha)^2 w^2 - (\alpha' + 2(K_1+\varepsilon))w \\
&\quad - \frac{1+\alpha}{2\varepsilon\alpha}K_3^2 - \frac{n^2}{2\varepsilon\alpha}\max\{K_1^2, K_2^2\}\Big) \\
&\ge \frac{2\alpha(1-2\varepsilon)}{n}\frac{\mathcal{H}^2}{\lambda} - (\ln\lambda)'\mathcal{H} - \frac{n(\alpha'+2(K_1+\varepsilon))^2}{8(1-2\varepsilon)\alpha(1-\alpha)^2}\lambda \\
&\quad - \frac{1+\alpha}{2\varepsilon\alpha}K_3^2\lambda - \frac{n^2}{2\varepsilon\alpha}\max\{K_1^2, K_2^2\}\lambda,
\end{aligned}
$$

where the last inequality used $-Ax^2 + Bx \le \frac{B^2}{4A}$.

Fix arbitrary $t \in (0, T]$ and assume that $\mathcal{H}$ achieves its maximum at the point $(x_0, t_0) \in M \times [0, t]$ and $\mathcal{H}(x_0, t_0) > 0$ (otherwise the proof is trivial), which implies $t_0 > 0$. By an argument analogue to the proof of Theorem 1.1 and 1.3, one can show that $\mathcal{D}(x_0, t_0) \le 0$, that is at (x_0, t_0),

$$
\begin{aligned}
0 \ge \frac{2\alpha(1-2\varepsilon)}{n}\mathcal{H}^2 - \lambda'\mathcal{H} - \frac{n(\alpha'+2(K_1+\varepsilon))^2}{8(1-2\varepsilon)\alpha(1-\alpha)^2}\lambda^2 \\
- \frac{1+\alpha}{2\varepsilon\alpha}K_3^2\lambda^2 - \frac{n^2}{2\varepsilon\alpha}\max\{K_1^2, K_2^2\}\lambda^2. \qquad (4.13)
\end{aligned}
$$

For a positive number a and two nonnegative numbers b, c, from the inequality $ax^2 - bx - c \le 0$ we have $x \le \frac{b}{a} + \sqrt{\frac{c}{a}}$. Hence, solving the quadratic inequality of $\mathcal{H}$ in (4.13) yields

$$
\begin{aligned}
\mathcal{H} \le \frac{n}{2(1-2\varepsilon)\alpha}\lambda' + \frac{n(\alpha'+2(K_1+\varepsilon))}{4(1-2\varepsilon)\alpha(1-\alpha)}\lambda \\
+ \frac{1}{2\alpha(\varepsilon(1-2\varepsilon))^{\frac{1}{2}}}\left((n(1+\alpha))^{\frac{1}{2}}K_3 + n^{\frac{3}{2}}\max\{K_1, K_2\}\right)\lambda. \qquad (4.14)
\end{aligned}
$$

Since $t \ge t_0$, we have

$$
\begin{aligned}
\mathcal{H}(x,t) \le \mathcal{H}(x_0,t_0) \le \frac{n}{2(1-2\varepsilon)\alpha}\lambda' + \frac{n(\alpha'+2(K_1+\varepsilon))}{4(1-2\varepsilon)\alpha(1-\alpha)}\lambda \\
+ \frac{1}{2\alpha(\varepsilon(1-2\varepsilon))^{\frac{1}{2}}}\left((n(1+\alpha))^{\frac{1}{2}}K_3 + n^{\frac{3}{2}}\max\{K_1, K_2\}\right)\lambda \le \lambda\psi_2(t)
\end{aligned}
$$

and for all $x \in M$, it holds that

$$
\alpha F^2(\nabla f) - \partial_t f \le \psi_2(t). \qquad (4.15)
$$

Since t is arbitrary in $[0, T]$, we obtain (1.17). Hence, we complete the proof.

4.3. Proof of Corollary 1.12. Let $\eta(s)$ be a smooth curve connecting x and y with $\eta(1) = x$ and $\eta(0) = y$, and $F(\dot{\eta}(s))|_\tau$ is the length of the vector $\dot{\eta}(s)$ at time $\tau(s) = (1-s)t_2 + st_1$. From (1.17) we have

$$-\partial_t f \leq -\alpha F^2(\nabla(\log u)) + \psi_2(t).$$

Let $l(s) = \log u(\eta(s), \tau(s)) = f(\eta(s), \tau(s))$. Then

$$\begin{aligned} f(x_1,t_1) - f(x_2,t_2) &= \int_0^1 \frac{d}{ds}\Big(f(\eta(s),\tau(s))\Big)\, ds \\ &= \int_0^1 (t_2-t_1)\Big(\frac{Df(\dot{\eta}(s))}{t_2-t_1} - \partial_t f\Big)\, ds \\ &\leq \int_0^1 (t_2-t_1)\Big\{\frac{F(\dot{\eta}(s))F(\nabla f)}{t_2-t_1} - \partial_t f\Big\}\, ds \\ &\leq \int_0^1 (t_2-t_1)\left\{\frac{F(\dot{\eta}(s))F(\nabla f)}{t_2-t_1} - \alpha F^2(\nabla(\log u)) + \psi_2(t)\right\} ds \\ &\leq \int_0^1 \left\{\frac{1}{4\alpha}\frac{F(\dot{\eta}(s))^2|_\tau}{t_2-t_1} + (t_2-t_1)\psi_2(t)\right\} ds, \end{aligned} \tag{4.16}$$

where the last inequality used $-Ax^2 + Bx \leq \frac{B^2}{4A}$. Using (4.16), we derive

$$\log\left(\frac{u(x_1,t_1)}{u(x_2,t_2)}\right) = f(x_1,t_1) - f(x_2,t_2) \leq \int_0^1 \left\{\frac{1}{4\alpha}\frac{F(\dot{\eta}(s))^2|_\tau}{t_2-t_1} + (t_2-t_1)\psi_2(t)\right\} ds.$$

Therefore, we arrive at

$$u(x_1,t_1) \leq u(x_2,t_2)\exp\left\{\int_0^1 \left(\frac{1}{4\alpha}\frac{F(\dot{\eta}(s))^2|_\tau}{t_2-t_1} + (t_2-t_1)\psi_2(t)\right) ds\right\}.$$

Hence, we complete the proof.

Acknowledgments. The author would like to thank the referee for valuable comments. The author is also grateful to his advisor professor Qun He for her constant encouragement.

The author is supported by NSFC 11971415, Henan Province Science Foundation for Youths (No.212300410235), and the Key Scientific Research Program in Universities of Henan Province (No.21A110021) and Nanhu Scholars Program for Young Scholars of XYNU (No. 2019).

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Received October 7, 2020, revised April 30, 2021.

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Градієнтні оцінки та нерівності Гарнака для нелінійного рівняння теплопровідності з фінслеровим лапласіаном

Fanqi Zeng

Нехай (M^n, F, m) є n-вимірним компактним фінслеровим многовидом. У цій роботі ми вивчаємо нелінійне рівняння теплопровідності

$$\partial_t u = \Delta_m u \quad \text{на } M^n \times [0, T],$$

де Δ_m є фінслеровим лапласіаном. Одержано градієнтні оцінки типу Лі–Яу для позитивних глобальних розв'язків цього рівняння на статичних фінслерових многовидах, а також під дією потоку Фінслера–Річчі, Як наслідок, в обох випадках також одержано відповідні нерівності Гарнака.

Ключові слова: градієнтні оцінки Лі–Яу, нерівність Гарнака, нелінійне рівняння теплопровідності, потік Фінслера–Річчі