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Implicit Linear Nonhomogeneous Difference Equation over $\mathbb{Z}$ with a Random Right-Hand Side

S.L. Gefter and A.L. Piven'

Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of independent identically distributed integer-valued random variables which are defined on a probability space $(\Omega, \mathcal{F}, P)$. We assume that these variables have a non-degenerate distribution. Let a and b be integers, $b \neq 0, \pm 1$, and let a be not divisible by b. For every $\omega \in \Omega$, we consider the implicit first-order linear nonhomogeneous difference equation $bx_{n+1} + ax_n = f_n(\omega)$, $n = 0, 1, 2, \ldots$. It is proved that the probability that there exists an integer solution of this implicit difference equation is equal to zero. Hence, under the random choice of integers $f_0, f_1, f_2, \ldots$, the implicit linear difference equation $bx_{n+1} + ax_n = f_n$, $n = 0, 1, 2, \ldots$, has no solutions in integers. We also prove that if a and b are co-prime integers, then the solvability set for this difference equation is an uncountable dense meagre set in the space of all sequences of integers.

Key words: difference equation, independent random variables, solvability set

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1. Introduction

Let a and b be integers, $\mathbb{N}_0$ be the set of nonnegative integers, $\{f_n\}_{n=0}^{\infty}$ be a given sequence of integers. Consider the problem of solving in integers for the following first order difference equation:

$$bx_{n+1} + ax_n = f_n, \quad n = 0, 1, 2, \ldots \tag{1.1}$$

If $b = \pm 1$, then, obviously, this equation has infinitely many solutions in integers. If $b \neq \pm 1$, then equation (1.1) is said to be *implicit over the ring* $\mathbb{Z}$ (see [7, 10]).

We note that an implicit equation may not have integer solutions. For example, the general solution of the equation $3x_{n+1} = x_n + 1$ over $\mathbb{Q}$ has the form $x_n = \frac{c}{3^n} + \frac{1}{2}$, where $c \in \mathbb{Q}$, $n = 0, 1, 2, \ldots$. It is obvious that for any value of a constant c we cannot obtain an integer solution (see Example 2.1 in [7]). But it is not difficult to verify that if a is not divisible by b, then (1.1) has at most one solution in integers [7]. Therefore the mapping, which assigns to any sequence of integers $\{x_n\}_{n=0}^{\infty}$ the sequence $\{f_n\}_{n=0}^{\infty}$, where $f_n = bx_{n+1} + ax_n$, is injective.

Hence, there exists an uncountable set of sequences of integers $\{f_n\}_{n=0}^{\infty}$ such that equation (1.1) has an integer solution. We show in Section 2.1 that this set of sequences, i.e., the solvability set for equation (1.1), is a dense meagre set in the natural topological sense (see Corollary 2.4).

Now, let $\{f_n\}_{n=0}^{\infty}$ be a sequence of independent identically distributed (i.i.d.) integer-valued random variables which are defined on a complete probability space $(\Omega, \mathcal{F}, P)$. For $\omega \in \Omega$, consider the equation

$$bx_{n+1} + ax_n = f_n(\omega), \quad n = 0, 1, 2, \dots \tag{1.2}$$

The main result of this paper shows that in this case equation (1.2) has no integer solutions almost surely (see Theorem 3.1, Remark 3.2 and Theorem 3.5). It means that under random choice of integers $f_0, f_1, f_2, \dots$ the implicit difference equation (1.1) has no integer solutions.

Note that in the classical (real or complex) situation difference equations with random parameters were considered in various works (see, for example, [1, 12]). Implicit linear difference equations over $\mathbb{Z}$ were studied in [2, 5, 7, 10]. Some analogues of results of these papers for implicit difference equations in Fréchet spaces were obtained in [6]. Integer solutions of nonlinear difference equations were considered in [4] in connection with the Laurent phenomenon.

2. Preliminary

2.1. Topological properties of the solvability set for equation (1.1)

Denote by $\mathbb{Z}^{\mathbb{N}_0}$ the countable degree of the space $\mathbb{Z}$. It is a compete metric space with the metric

$$d\big(\{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty}\big) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|},$$

and the convergence in this space coincides with the coordinate-wise stabilization [3, Section 4, §2].

The set $\mathbb{Z}^{\infty}$ of all finite sequences of integers is dense in the space $\mathbb{Z}^{\mathbb{N}_0}$. As shown in [7, Proof of Theorem 4.1], for $a = \pm 1$, equation (1.1) has a unique integer solution for all sequences $\{f_n\}_{n=0}^{\infty} \in \mathbb{Z}^{\infty}$. The following example shows that for $a \neq \pm 1$ this assertion can fail even for the co-prime integers a and b.

Example 2.1. Let $b \neq \pm 1$, a, b be co-prime integers, $f_n = 0$ for $n \geq 1$ and f_0 is not divisible by a. Then $bx_{n+1} + ax_n = 0, \quad n = 1, 2, 3, \dots$. Since a and b are co-prime, the uniqueness of an integer solution for equation (1.1) holds [7, Theorem 3.1]. Therefore, $x_n = 0, \quad n = 1, 2, \dots$. Then x_0 should be an integer solution of the linear equation $ax_0 = f_0$. However, this equation has no integer solutions because f_0 is not divisible by a.

The following lemma establishes the criterion of the existence and uniqueness of an integer solution of equation (1.1) for $\{f_n\}_{n=0}^{\infty} \in \mathbb{Z}^{\infty}$.

Lemma 2.2. *Let $b \neq \pm 1$ and a, b be co-prime integers. Assume that there exists $m \in \mathbb{N}$ such that $f_n = 0$ for $n > m$. Then equation (1.1) has an integer solution if and only if the number $\sum_{k=0}^{m} b^k(-a)^{m-k} f_k$ is divisible by a^{m+1}. Moreover, this integer solution is unique and has the form*

$$x_n = \begin{cases} \sum_{k=0}^{m-n} \frac{(-1)^k b^k}{a^{k+1}} f_{k+n}, & n = 0, 1, 2, \ldots, m \\ 0, & n = m+1, m+2, \ldots \end{cases}. \tag{2.1}$$

Proof. Let $f_n = 0$ for $n > m$. Substituting (2.1) into (1.1), we obtain that the sequence $\{x_n\}_{n=0}^{\infty}$ defined by (2.1) is a solution of equation (1.1) over the ring $\mathbb{Z}\left[\frac{1}{a}\right]$. Moreover, $x_0 \in \mathbb{Z}$ if and only if $\sum_{k=0}^{m} \frac{(-1)^k b^k}{a^{k+1}} f_k \in \mathbb{Z}$ or, equivalently, $\sum_{k=0}^{m} b^k(-a)^{m-k} f_k$ is divisible by a^{m+1}. We show that if $x_0 \in \mathbb{Z}$, then $x_1 \in \mathbb{Z}$. Since $x_1 \in \mathbb{Z}\left[\frac{1}{a}\right]$, we have that $x_1 = \frac{u}{a^j}$, where $u \in \mathbb{Z}$, $j \in \mathbb{N}_0$. It follows from equation (1.1) that $bx_1 = f_0 - ax_0 \in \mathbb{Z}$, i.e., $\frac{bu}{a^j} \in \mathbb{Z}$. Since a and b are co-prime, we obtain that u is divisible by a^j, i.e., $x_1 \in \mathbb{Z}$. Repeating the above argument, we find that x_n defined by (2.1) is integer, where $n = 2, 3, \ldots$. The uniqueness of the integer solution of equation (1.1) follows from [7, Theorem 3.1].

Now we prove the converse assertion of Lemma 2.2. Let $\{x_n\}_{n=0}^{\infty}$ be an integer solution of equation (1.1). Since $f_n = 0$ for $n > m$, we have

$$bx_{n+1} + ax_n = 0, \quad n = m+1, m+2, \ldots. \tag{2.2}$$

By the uniqueness of the integer solution of equation (2.2) [7, Theorem 3.1], we obtain $x_n = 0$ for $n = m+1, m+2, \ldots$. Now equation (1.1) implies

$$\begin{aligned} x_n &= \frac{f_n}{a} - \frac{bx_{n+1}}{a} = \frac{f_n}{a} - \frac{bf_{n+1}}{a^2} + \frac{b^2 x_{n+2}}{a^2} = \ldots \\ &= \sum_{k=0}^{m-n} \frac{(-1)^k b^k}{a^{k+1}} f_{k+n}, \quad n = 0, 1, 2, \ldots, m. \end{aligned}$$

The lemma is proved. □

Denote by $H \subset \mathbb{Z}^{\mathbb{N}_0}$ the solvability set for equation (1.1), i.e., the set of sequences $\{f_n\}_{n=0}^{\infty}$ such that there exists an integer solution of equation (1.1):

$$H = \left\{ \{f_n\}_{n=0}^{\infty} \in \mathbb{Z}^{\mathbb{N}_0} : \exists \{x_n\}_{n=0}^{\infty} \in \mathbb{Z}^{\mathbb{N}_0} \ \forall n = 0, 1, 2, \ldots \quad bx_{n+1} + ax_n = f_n \right\}.$$

The following theorem establishes sufficient conditions of the density of H and its complement $G = \mathbb{Z}^{\mathbb{N}_0} \setminus H$ in the space $\mathbb{Z}^{\mathbb{N}_0}$.

Theorem 2.3. *Let $b \neq \pm 1$ and let a, b be co-prime integers. Then the sets H and G are dense in the space $\mathbb{Z}^{\mathbb{N}_0}$.*

Proof. Let $\{g_n\}_{n=0}^{\infty} \in \mathbb{Z}^{\mathbb{N}_0}$ and $m \in \mathbb{N}$. We set $f_n = 0$ for $n > m$ and $f_n = g_n$ if $n < m$. Let us show that we can define the value f_m such that $\{f_n\}_{n=0}^{\infty} \in H$. By Lemma 2.2, the sequence $\{f_n\}_{n=0}^{\infty}$ has to satisfy the following property: $\sum_{k=0}^{m} b^k(-a)^{m-k} f_k$ is divisible by a^{m+1}. Therefore, it is necessary to define an element $f_m \in \mathbb{Z}$ such that this condition holds. Consider the following Diophantine equation with respect to unknown integers f_m, z_m:

$$b^m f_m + a^{m+1} z_m = -\sum_{k=0}^{m-1} b^k(-a)^{m-k} g_k. \tag{2.3}$$

By Theorem 1 [11, Chapter 5], the linear Diophantine equation

$$b^m f_m + a^{m+1} z_m = c$$

has a solution for any integer c, because a^{m+1} and b^m are co-prime. Then, by (2.3), we obtain the required sequence $\{f_n\}_{n=0}^{\infty} \in H \cap \mathbb{Z}^{\infty}$. Hence the set $H \cap \mathbb{Z}^{\infty}$ is dense in $\mathbb{Z}^{\mathbb{N}_0}$.

Now we prove that G is dense in $\mathbb{Z}^{\mathbb{N}_0}$. Let $a \neq \pm 1$. Note that for every $m \in \mathbb{N}$ and every sequence $\{f_n\}_{n=0}^{\infty} \in H \cap \mathbb{Z}^{\infty}$ we can find the sequence $\{g_n\}_{n=0}^{\infty} \in \mathbb{Z}^{\infty} \cap G$ such that $g_n = 0$ for $n > m$ and $g_n = f_n$ if $n < m$. For this purpose, it is sufficient to choose g_m such that g_m is not divisible by a (see Lemma 2.2). It implies that for $a \neq \pm 1$ the set $\mathbb{Z}^{\infty} \cap G$ is dense in $\mathbb{Z}^{\infty}$, and therefore in the space $\mathbb{Z}^{\mathbb{N}_0}$.

Now consider the case $a = \pm 1$. Let $f = \{f_n\}_{n=0}^{\infty}$ be an arbitrary sequence from $\mathbb{Z}^{\infty}$. Then, in every neighbourhood of the element f, there exists a sequence $g = \{g_n\}_{n=0}^{\infty} \in \mathbb{Z}^{\mathbb{N}_0}$ of 0 and 1 for sufficiently large numbers n such that 0 and 1 occur in this sequence infinitely many times. Therefore $g \in G$ (see [7, Example 4.1]). Hence, for $a = \pm 1$, the set G is dense in the set $\mathbb{Z}^{\infty}$ and in the space $\mathbb{Z}^{\mathbb{N}_0}$. The proof is complete. □

Corollary 2.4. *Under the conditions of Theorem 2.3, the set H is an uncountable dense meagre F_σ-subset in the complete metric space $\mathbb{Z}^{\mathbb{N}_0}$.*

Proof. Since a is not divisible by b, equation (1.1) can have at most one integer solution [7, Theorem 3.1]. Therefore, the mapping, which assigns to any sequence of integers $\{x_n\}_{n=0}^{\infty}$ the sequence $\{f_n\}_{n=0}^{\infty}$, where $f_n = bx_{n+1} + ax_n$, is injective. Consequently, the set H is uncountable.

For $x_0 \in \mathbb{Z}$ and $m \in \mathbb{N}_0$, we define the sets

$$H_{x_0} = \big\{\{f_n\}_{n=0}^{\infty} \in \mathbb{Z}^{\mathbb{N}_0} : \exists \{x_n\}_{n=1}^{\infty} \in \mathbb{Z}^{\mathbb{N}}\ \forall n = 0,1,2,\dots \quad bx_{n+1} + ax_n = f_n\big\},$$
$$H_{x_0}^m = \big\{\{f_n\}_{n=0}^{\infty} \in \mathbb{Z}^{\mathbb{N}_0} : \exists \{x_n\}_{n=1}^{m+1} \in \mathbb{Z}^{m+1}\ \forall n = 0,1,2,\dots,m \quad bx_{n+1} + ax_n = f_n\big\}.$$

We have

$$H = \bigcup_{x_0=-\infty}^{\infty} H_{x_0}, \quad H_{x_0} = \bigcap_{m=1}^{\infty} H_{x_0}^m. \tag{2.4}$$

Since the sets $H_{x_0}^m$ are closed in the space $\mathbb{Z}^{\mathbb{N}_0}$, by the second relation in (2.4), the sets H_{x_0} are also closed. Therefore, we obtain from the first relation in (2.4) that H is an F_σ-set.

To end the proof, it is sufficient to show that for any $x_0 \in \mathbb{Z}$ the set H_{x_0} is nowhere dense in $\mathbb{Z}^{\mathbb{N}_0}$. Since the complement of H_{x_0} is dense (see Theorem 2.3), the set H_{x_0} has no interior points. Therefore H_{x_0} is nowhere dense. The corollary is proved. □

Remark 2.5. Under the conditions of Theorem 2.3, the set $H \cap \mathbb{Z}^\infty$ is dense in the space $\mathbb{Z}^{\mathbb{N}_0}$, and for $a \neq \pm 1$ the set $G \cap \mathbb{Z}^\infty$ is also dense.

Remark 2.6. If the coefficients a and b of the implicit equation (1.1) are not co-prime, then the set H cannot be dense in $\mathbb{Z}^{\mathbb{N}_0}$. Let $d \neq \pm 1$ be the greatest common divisor of a, b and $f = \{f_n\}_{n=0}^\infty \in H$. Then the numbers f_n $(n = 0, 1, 2, \ldots)$ have to be divisible by d. However, the set of all sequences of integers dividing by $d \neq \pm 1$ is not dense in $\mathbb{Z}^{\mathbb{N}_0}$.

2.2. Implicit stochastic first-order difference equation Consider the following stochastic analogue of equation (1.1). Let $\{f_n\}_{n=0}^\infty$ be a given sequence of integer-valued random variables defined on a complete probability space $(\Omega, \mathcal{F}, P)$. The sequence of integer-valued random variables $\{\xi_n\}_{n=0}^\infty$ defined on this space is said to be a solution of the implicit stochastic difference equation

$$b\xi_{n+1} + a\xi_n = f_n, \quad n = 0, 1, 2, \ldots \tag{2.5}$$

if

$$P\left(\omega \in \Omega : b\xi_{n+1}(\omega) + a\xi_n(\omega) = f_n(\omega),\ n = 0, 1, 2, \ldots\right) = 1.$$

The following lemma establishes the connection between the solutions to the stochastic difference equation (2.5) and the solutions for the family of equations of the form (1.2).

Lemma 2.7. *Let $b \neq \pm 1$. Assume that a and b are co-prime integers. Equation (2.5) has a solution if and only if $P(A) = 1$, where*

$$A = \left\{\omega \in \Omega : \exists \{x_n\}_{n=0}^\infty \in \mathbb{Z}^{\mathbb{N}_0}\ \forall n = 0, 1, 2, \ldots \quad bx_{n+1} + ax_n = f_n(\omega)\right\}. \tag{2.6}$$

Proof. The necessity of $P(A) = 1$ is obvious. We prove the sufficiency. Since f_n is measurable, it is not difficult to show that $A \in \mathcal{F}$. Let $\omega \in A$. Since a and b are co-prime, it follows from [7, Corollary 2.1] that there exists a unique sequence $\{x_n\}_{n=0}^\infty \in \mathbb{Z}^{\mathbb{N}_0}$ such that

$$bx_{n+1} + ax_n = f_n(\omega), \quad n = 0, 1, 2, 3, \ldots.$$

Put $\xi_n(\omega) = x_n$, $n = 0, 1, 2, \ldots$. Then $\xi_n : \Omega \to \mathbb{Z}$ and the sequence $\{\xi_n\}_{n=0}^\infty$ satisfies equation (2.5) on the set A. We prove that ξ_n $(n = 0, 1, 2, \ldots)$ is a measurable mapping. By [7, Theorem 3.1],

$$\xi_0(\omega) = \sum_{k=0}^\infty \frac{(-b)^k}{a^{k+1}} f_k(\omega),$$

where the series converges in rings $\mathbb{Z}_p$ of p-adic integers [9, Part 1, Chapter 3, §5] for any prime divisor p of b. Then ξ_0 is a measurable mapping with values in $\mathbb{Z}_p$ [8, Lemma 1.10]. It follows from $\xi_0(\Omega) \subset \mathbb{Z}$ that ξ_0 is a measurable mapping of Ω into $\mathbb{Z}$. Therefore equation (2.5) implies the measurability of ξ_n for all n. Hence the sequence of random variables $\{\xi_n\}_{n=0}^{\infty}$ is a solution of equation (2.5). □

3. Main results

At first, we suppose that integer-valued random variables f_n, $n = 0, 1, 2, \ldots$, take every integer value with a positive probability. The following theorem shows that in this case the probability that there exists an integer solution of the implicit difference equation (1.2) is equal to zero.

Theorem 3.1. *Let $a, b \in \mathbb{Z}$, $b \neq 0, \pm 1$ and let $\{f_n\}_{n=0}^{\infty}$ be a sequence of i.i.d. integer-valued random variables such that $P(\omega : f_n(\omega) = z) > 0$ for any $z \in \mathbb{Z}$, $n = 0, 1, 2 \ldots$. Then $P(A) = 0$, where the event A is defined by (2.6).*

Proof. We define the probability measure μ on the set of integers $\mathbb{Z}$ as follows: $\mu(\{z\}) = P(\omega \in \Omega : f_n(\omega) = z)$, $z \in \mathbb{Z}$. Moreover,

$$\mu(\{z\}) > 0, \quad z \in \mathbb{Z}. \tag{3.1}$$

Define the events

$$A_{x_0} = \left\{\omega \in \Omega : \exists \{x_n\}_{n=1}^{\infty} \in \mathbb{Z}^{\mathbb{N}}\ \forall n = 0, 1, 2, \ldots \quad f_n(\omega) = b x_{n+1} + a x_n\right\}$$

and

$$A_{x_0}^m = \left\{\omega \in \Omega : \exists \{x_n\}_{n=1}^{m+1} \in \mathbb{Z}^{m+1}\ \forall n = 0, 1, 2, \ldots, m \quad f_n(\omega) = b x_{n+1} + a x_n\right\}$$

for numbers $x_0 \in \mathbb{Z}$ and $m \in \mathbb{N}_0$. We have

$$A = \bigcup_{x_0=-\infty}^{\infty} A_{x_0}, \quad A_{x_0} = \bigcap_{m=0}^{\infty} A_{x_0}^m, \quad A_{x_0}^m \supset A_{x_0}^{m+1}. \tag{3.2}$$

We show that for any $x_0 \in \mathbb{Z}$ the relation

$$P(A_{x_0}) = 0 \tag{3.3}$$

is fulfilled.

By the countable additivity property of the measure P, for any $m \in \mathbb{N}$, we obtain

$$P(A_{x_0}^m) = \sum_{x_1 \in \mathbb{Z}} \sum_{x_2 \in \mathbb{Z}} \cdots \sum_{x_{m+1} \in \mathbb{Z}} P\left(\omega \in \Omega : f_n(\omega) = b x_{n+1} + a x_n,\ n = 0, \ldots, m\right).$$

It follows from this equality and the independence of random variables $f_0, \ldots, f_m$ that

$$P(A_{x_0}^m) = \sum_{x_1 \in \mathbb{Z}} \sum_{x_2 \in \mathbb{Z}} \cdots \sum_{x_{m+1} \in \mathbb{Z}} \prod_{n=0}^{m} P\left(\omega \in \Omega : f_n(\omega) = b x_{n+1} + a x_n\right)$$

$$= \sum_{x_1 \in \mathbb{Z}} \mu(\{bx_1 + ax_0\}) \cdots \sum_{x_m \in \mathbb{Z}} \mu(\{bx_m + ax_{m-1}\}) \sum_{x_{m+1} \in \mathbb{Z}} \mu(\{bx_{m+1} + ax_m\}). \tag{3.4}$$

Since $b \neq 0, \pm 1$, for any $x_m \in \mathbb{Z}$, there are integers $k = k(x_m) \in \mathbb{Z}$ and $r = r(x_m) \in \mathbb{N}_0$ such that $0 \leq l \leq |b| - 1$ and

$$ax_m = bk + r. \tag{3.5}$$

Then, for any $x_m \in \mathbb{Z}$, we have

$$\begin{aligned} \sum_{x_{m+1} \in \mathbb{Z}} \mu(\{bx_{m+1} + ax_m\}) &= \sum_{x_{m+1} \in \mathbb{Z}} \mu(\{bx_{m+1} + bk + r\}) \\ &= \sum_{j \in \mathbb{Z}} \mu(\{bj + r\}) \leq q, \end{aligned} \tag{3.6}$$

where

$$q = \max_{r=0,\ldots,|b|-1} \sum_{j \in \mathbb{Z}} \mu(\{bj + r\}) \in (0, 1)$$

by the property (3.1) on the measure μ and by the condition $b \neq \pm 1$. Now we estimate the right-hand part of equality (3.4) by means of (3.6). We obtain

$$\begin{aligned} P(A_{x_0}^m) \leq q \sum_{x_1 \in \mathbb{Z}} \mu(\{bx_1 + ax_0\}) \sum_{x_2 \in \mathbb{Z}} \mu(\{bx_2 + ax_1\}) \cdots \sum_{x_m \in \mathbb{Z}} \mu(\{bx_m + ax_{m-1}\}) \\ = qP(A_{x_0}^{m-1}), \quad m = 2, 3, \ldots. \end{aligned}$$

Therefore $P(A_{x_0}^m) \leq q^{m-1} P(A_{x_0}^1)$ and, by the continuity from above of the measure P (see the second relation in (3.2)),

$$P(A_{x_0}) = \lim_{m \to \infty} P(A_{x_0}^m) = 0.$$

Equality (3.3) is proved. Taking into account the countable additivity property of the measure P and the first equality in (3.2), we obtain

$$P(A) = \sum_{x_0 = -\infty}^{\infty} P(A_{x_0}) = 0.$$

The theorem is proved. □

Remark 3.2. We assumed in Theorem 3.1 that

$$\mu(\{z\}) = P(\omega \in \Omega : f_n(\omega) = z) > 0, \ z \in \mathbb{Z}, \ n = 0, 1, 2, \ldots.$$

For proving this theorem, the following property of the measure μ was used:

$$\sum_{j \in \mathbb{Z}} \mu(\{bj + r\}) < 1, \quad r = 0, \ldots, |b| - 1.$$

This property can be rewritten in the form

$$\sum_{j\in\mathbb{Z}} P(\omega\in\Omega : f_n(\omega)=bj+r)<1, \quad r=0,\dots,|b|-1,\ n=0,1,2,\dots. \tag{3.7}$$

Note that Theorem 3.1 and its proof are valid if the probability distribution of random variables f_n, $n=0,1,2,\dots$, satisfies the restriction (3.7). For example, this condition is fulfilled if

$$P(\omega\in\Omega : f_n(\omega)=z)=\begin{cases} 0, & z<0,\\ p_z, & z\geq 0\end{cases},$$

where $p_z\in(0,1)$. In particular, it is true when the random variables f_n are Poisson distributed:

$$p_z=P(\omega\in\Omega : f_n(\omega)=z)=\frac{\lambda^z e^{-\lambda}}{z!}, \quad z=0,1,2,\dots,$$

where $\lambda>0$.

We consider another example which satisfies the condition (3.7). Let $\alpha,\beta\in\mathbb{Z}$, $\alpha<\beta$ and the random variables f_n take each value from $[\alpha,\beta]\cap\mathbb{Z}$ with the same probability, i.e.,

$$P(\omega\in\Omega : f_n(\omega)=z)=\frac{1}{\beta-\alpha+1}, \quad z\in[\alpha,\beta]\cap\mathbb{Z}, \quad n=0,1,2,\dots.$$

Then the random variables f_n satisfy the condition (3.7).

Theorem 3.1 and Remark 3.2 imply the following theorem.

Theorem 3.3. *Let $a,b\in\mathbb{Z}$, $b\neq 0,\pm1$ and let $\{f_n\}_{n=0}^\infty$ be a sequence of i.i.d. integer-valued random variables such that $P(\omega : f_n(\omega)=z)=\frac{1}{\beta-\alpha+1}$ for all $z\in[\alpha,\beta]\cap\mathbb{Z}$, $n=0,1,2\dots.$ Then the probability that there exists an integer solution of the difference equation (1.2) is equal to zero, i.e., $P(A)=0$.*

Remark 3.4. Consider the case $\alpha=\beta$ in Theorem 3.3. Then random variables f_n, $n=0,1,2,\dots$, have a degenerate distribution, i.e., f_n take a fixed integer with probability 1. In this case, $P(A)$ can be equal to 1. For example, let $b=2$, $a=-1$ and $f_n(\omega)=1$ for all $n\in\mathbb{N}_0$ and $\omega\in\Omega$. Equation (1.2) has the solution $x_n=1$ $(n=0,1,2,\dots.)$. Hence $P(A)=1$. Therefore, in Theorems 3.1,3.3 the restrictions on the distribution law of random variables f_n are essential.

The following main theorem shows then Theorem 3.1 can be extended to the general case when random variables f_n, $n=0,1,2,\dots$, have an arbitrary non-degenerate distribution.

Theorem 3.5. *Let $a,b\in\mathbb{Z}$, $b\neq 0,\pm1$. Assume that a is not divisible by b and $\{f_n\}_{n=0}^\infty$ is a sequence of i.i.d. integer-valued random variables which have a non-degenerate distribution. Then the probability that there exists an integer solution of the difference equation (1.2) is equal to zero, i.e.,*

$$P\Big(\omega\in\Omega : \exists\{x_n\}_{n=0}^\infty\in\mathbb{Z}^{\mathbb{N}_0} : bx_{n+1}+ax_n=f_n(\omega),\ n=0,1,2,\dots\Big)=0.$$

The proof of this theorem is based on the following lemma.

Lemma 3.6. *Let $b \in \mathbb{N}$, $b \neq 1$ and $M \subset \mathbb{Z}$ be a set containing at least two elements. Assume that all elements of M belong to the same residue class modulo b. Then there exist $k \in \mathbb{N}$ and $r_0 \in \mathbb{Z}$ such that b^k divide all numbers $l - r_0$ ($l \in M$) and at least two numbers $\frac{c-r_0}{b^k}$, $\frac{d-r_0}{b^k}$ ($c, d \in M$) belong to different residue classes modulo b.*

Proof. Divide every number $l \in M$ by b with a remainder. By the conditions of the lemma, all numbers l have the same remainder r_1 modulo b. Therefore, there exist numbers $q_1(l) \in \mathbb{Z}$ such that

$$l = q_1(l)b + r_1, \quad l \in M.$$

If the numbers $q_1(l)$ belong to different residue classes modulo b, then put $k = 1$, $r_0 = r_1$ and the proof is complete. If the numbers $q_1(l)$ belong to the same residue class modulo b, then divide these numbers by b with a remainder. Denote by r_2 a common remainder from the division of numbers $q_1(l)$ ($l \in M$) by b. Therefore, there exist numbers $q_2(l) \in \mathbb{Z}$ such that $q_1(l) = q_2(l)b + r_2$, $\quad l \in M$ and

$$l = q_2(l)b^2 + br_2 + r_1, \quad l \in M.$$

If the numbers $q_2(l)$ belong to different residue classes modulo b, then put $k = 2$, $r_0 = br_2 + r_1$ and the proof of the lemma is complete. If the numbers $q_2(l)$ belong to the same residue class modulo b, then we continue to divide these numbers by b with a remainder etc. By the kth step, we obtain the decomposition

$$l = b^k q_k(l) + \sum_{j=1}^{k} b^{j-1} r_j, \quad l \in M, \tag{3.8}$$

where the remainders $r_1, \ldots, r_k \in \{0, \ldots, b-1\}$ do not depend on l. Denote by $\mathbb{Z}_b$ the ring of b-adic integers [9, Part 1, Chapter 3, §5]. Passing to the limit for $k \to \infty$ in (3.8) in the topology of the space $\mathbb{Z}_b$, we obtain the same canonical decomposition [9, Part 1, Chapter 3, §4] for all numbers $l \in M$ into the series which converges in the topology of this space:

$$l = \sum_{j=1}^{\infty} b^{j-1} r_j, \quad l \in M,$$

But it contradicts the uniqueness of this decomposition [9, Part 1, Chapter 3, §4]. The lemma is proved. □

Proof of Theorem 3.5. Without loss of generality, we assume that $b \in \mathbb{N}$, i.e., $b \geq 2$. By Remark 3.2, it is sufficient to prove Theorem 3.5 for the case when the condition (3.7) is not fulfilled. It means that for some $r = 0, \ldots, b-1$ the distribution law of random values f_n satisfies the condition

$$\sum_{j \in \mathbb{Z}} P(\omega \in \Omega : f_n(\omega) = bj + r) = 1, \quad n = 0, 1, 2, \ldots. \tag{3.9}$$

Consider the set $M \subset \mathbb{Z}$ of all values of the random variable f_n. Since f_n have a non-degenerate distribution, this set contains at least two elements. By the assumption (3.9), every element of M has the same remainder r when divided by b. Therefore all elements of M belong to the same residue class modulo b. By Lemma 3.6, there are numbers $k \in \mathbb{N}$ and $r_0 \in \mathbb{Z}$ such that b^k divide all numbers $j - r_0$ $(j \in M)$ and at least two numbers $\frac{c-r_0}{b^k}$, $\frac{d-r_0}{b^k}$ $(c, d \in M)$ belong to different residue classes modulo b. It means that the random variables $g_n = \frac{f_n - r_0}{b^k}$, $n = 0, 1, 2, \ldots$, take at least two different values l_1, l_2, which belong to different residue classes modulo b. Moreover, the random variables g_n, $n = 0, 1, 2, \ldots$, are i.i.d. and have a non-degenerate distribution. Hence,

$$f_n(\omega) = b^k g_n(\omega) + r_0, \quad \omega \in \Omega,\ n = 0, 1, 2, \ldots, \tag{3.10}$$

$$P\big(\omega \in \Omega : g_n(\omega) = l_j\big) \in (0, 1), \quad j = 1, 2,\ n = 0, 1, 2, \ldots. \tag{3.11}$$

Substituting the expression (3.10) for f_n into equation (1.2), we obtain the difference equation

$$bx_{n+1} + ax_n = b^k g_n(\omega) + r_0, \quad n = 0, 1, 2, \ldots \tag{3.12}$$

It implies that $r_0 - ax_n$ is divisible by b, and therefore there exists a sequence $\{y_n\}_{n=0}^{\infty} \in \mathbb{Z}^{\mathbb{N}_0}$ such that $r_0 - ax_n = by_n$. Then, substituting $x_n = \frac{r_0 - by_n}{a}$ into equation (3.12), we obtain that this sequence $\{y_n\}_{n=0}^{\infty}$ is an integer solution for the difference equation

$$by_{n+1} + ay_n = -ab^{k-1} g_n(\omega) + r_0, \quad n = 0, 1, 2, \ldots$$

Therefore,

$$\begin{aligned} A &= \big\{\omega \in \Omega : \exists \{x_n\}_{n=0}^{\infty} \in \mathbb{Z}^{\mathbb{N}_0}\ \forall n = 0, 1, 2, \ldots \quad bx_{n+1} + ax_n = f_n(\omega)\big\} \\ &\subset \big\{\omega \in \Omega : \exists \{y_n\}_{n=0}^{\infty} \in \mathbb{Z}^{\mathbb{N}_0}\ \forall n = 0, 1, 2, \ldots \\ &\qquad\qquad\qquad by_{n+1} + ay_n = -ab^{k-1} g_n(\omega) + r_0\big\} \\ &\subset \big\{\omega \in \Omega : \exists \{y_n\}_{n=0}^{\infty} \in \mathbb{Z}^{\mathbb{N}_0}\ \forall n = 0, 1, 2, \ldots \\ &\qquad\qquad\qquad by_{n+1} + ay_n = (-a)^2 b^{k-2} g_n(\omega) + r_0\big\} \subset \ldots \\ &\subset \big\{\omega \in \Omega : \exists \{y_n\}_{n=0}^{\infty} \in \mathbb{Z}^{\mathbb{N}_0}\ \forall n = 0, 1, 2, \ldots \quad by_{n+1} + ay_n = (-a)^k g_n(\omega) + r_0\big\}, \end{aligned}$$

i.e., $A \subset B$, where

$$B = \big\{\omega \in \Omega : \exists \{y_n\}_{n=0}^{\infty} \in \mathbb{Z}^{\mathbb{N}_0}\ \forall n = 0, 1, 2, \ldots \quad by_{n+1} + ay_n = (-a)^k g_n(\omega) + r_0\big\}.$$

At first, let a and b be co-prime integers. Then the numbers $(-a)^k$ and b are co-prime integers too. We show that the numbers $l_1(-a)^k$ and $l_2(-a)^k$ belong to different residue classes modulo b. Assuming the contrary, we obtain that $(l_1 - l_2)(-a)^k$ is divisible by b. Since the numbers b and $(-a)^k$ are co-prime integers, the number $(l_1 - l_2)$ is also divisible by b. It contradicts the assumption that the numbers l_1 and l_2 belong to different residue classes modulo b. Hence the sequence of random variables $h_n = (-a)^k g_n$ $(n = 0, 1, 2, \dots)$ satisfies the restriction (3.7). By Remark 3.2, $P(B) = 0$, therefore $P(A) = 0$.

Now, let a and b not be co-prime and let d be the greatest common divisor of a, b. We assume that $d > 1$. By the conditions of the theorem, $d \neq b$. Then $A \subset C$, where $C = \{\omega \in \Omega : d | f_n(\omega),\ n = 0, 1, 2, \dots\}$. We note that either all numbers $bj + r$ $(j \in \mathbb{Z})$ are divisible by d or none of these numbers are divisible by d. It follows by the conditions (3.9) that either $P(C) = 0$ or $P(C) = 1$ respectively. In the first case, we have $P(A) = 0$ and in the second case, equation (1.2) can be reduced by division by d to the equivalent equation

$$b_1 x_{n+1} + a_1 x_n = h_n(\omega), \quad n = 0, 1, 2, \dots, \tag{3.13}$$

where the coefficients $a_1 = \frac{a}{d}$ and $b_1 = \frac{b}{d}$ are co-prime, and $h_n = \frac{f_n}{d}$, $n = 0, 1, 2, \dots$, is a sequence of i.i.d. integer-valued random variables. The assertion of Theorem 3.5 has already been established, moreover

$$A = \big\{\omega \in \Omega : \exists \{x_n\}_{n=0}^{\infty} \in \mathbb{Z}^{\mathbb{N}_0}\ \forall n = 0, 1, 2, \dots \quad b_1 x_{n+1} + a_1 x_n = h_n(\omega)\big\}.$$

Therefore $P(A) = 0$, and the proof is complete. $\square$

Theorem 3.5 and Lemma 2.7 imply the following corollary.

Corollary 3.7. *Let the conditions of Theorem 3.5 be valid and, in addition, let a and b be co-prime integers. Then the stochastic difference equation (2.5) has no solutions.*

Remark 3.8. We show that the assumption $b \not| \, a$ of Theorem 3.5 is essential. Let a be divisible by b and let

$$\sum_{j \in \mathbb{Z}} P(\omega \in \Omega : f_n(\omega) = bj) = 1, \quad n = 0, 1, 2, \dots.$$

Then $P(A) = 1$, where the event A is defined by (2.6). Moreover, the stochastic difference equation (2.5) has a solution $\{\xi_n\}_{n=0}^{\infty}$, where $\xi_0 : \Omega \to \mathbb{Z}$ is an arbitrary random variable and other random variables $\xi_n : \Omega \to \mathbb{Z}$ $(n = 1, 2, \dots)$ can be defined by the following recurrence relation: $\xi_{n+1} = \frac{1}{b}(f_n - a\xi_n)$, $n = 0, 1, 2, \dots$.

Now, let μ be a probability measure on $\mathbb{Z}$ such that $\mu(\{m\}) \neq 1$ for all $m \in \mathbb{Z}$. Define the measure $\mathbf{P}$ on $\mathbb{Z}^{\mathbb{N}_0}$ as the countable product of measures μ.

Corollary 3.9. *Let $a, b \in \mathbb{Z}$, $b \neq 0, \pm 1$ and let a be not divisible by b. Then*

$$\mathbf{P}\left(\{f_n\}_{n=0}^{\infty} \in \mathbb{Z}^{\mathbb{N}_0} : \exists \{x_n\}_{n=1}^{\infty} \in \mathbb{Z}^{\mathbb{N}_0}\ \forall n = 0, 1, 2, \dots \quad b x_{n+1} + a x_n = f_n\right) = 0. \tag{3.14}$$

Proof. We consider $\mathbb{Z}^{\mathbb{N}_0}$ with the measure $\mathbf{P}$ as a probability space. For $\omega = \{f_k\}_{k=0}^{\infty} \in \mathbb{Z}^{\mathbb{N}_0}$, we set $g_n(\omega) = f_n$, $n = 0, 1, 2, \dots$. Then $\{g_n\}_{n=0}^{\infty}$ is a sequence of i.i.d. integer-valued random variables which have a non-degenerate distribution. It follows from Theorem 3.5 that equality (3.14) is true. The assertion of Corollary 3.9 is proved. $\square$

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S.L. Gefter,
V. N. Karazin Kharkiv National University, 4 Svobody Sq., Kharkiv, 61022, Ukraine,
E-mail: gefter@karazin.ua

A.L. Piven',
V. N. Karazin Kharkiv National University, 4 Svobody Sq., Kharkiv, 61022, Ukraine,
E-mail: aleksei.piven@karazin.ua

Неявне лінійне неоднорідне різницеве рівняння над $\mathbb{Z}$ з випадковою правою частиною

S.L. Gefter and A.L. Piven'

Нехай $\{f_n\}_{n=0}^{\infty}$ — послідовність незалежних цілозначних однаково розподілених випадкових величин, що визначені на ймовірнісному просторі $(\Omega, \mathcal{F}, P)$. Припускається, що ці величини мають невироджений розподіл. Нехай a та b — цілі числа, $b \neq 0, \pm 1$ та a не ділиться на b. Для кожного $\omega \in \Omega$ розглядається наступне неявне лінійне неоднорідне різницеве рівняння першого порядку: $bx_{n+1} + ax_n = f_n(\omega)$, $n = 0, 1, 2, \ldots$. Доведено, що ймовірність існування розв'язку в цілих числах цього неявного різницевого рівняння дорівнює нулю. Отже, при випадковому виборі цілих чисел $f_0, f_1, f_2, \ldots$ неявне різницеве рівняння $bx_{n+1} + ax_n = f_n$, $n = 0, 1, 2, \ldots$, не має розв'язків в цілих числах. Також доведено, що якщо a та b — взаємно прості числа, то множина розв'язності цього рівняння є незліченою щільною множиною першої категорії у просторі всіх послідовностей цілих чисел.

Ключові слова: різницеве рівняння, незалежні випадкові величини, множина розв'язності