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A Weak Solution to the Complex Hessian Equation Associated to an m-Positive Closed Current

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The aim of this paper is to study the existence of a solution to the complex Hessian equation associated to an m-positive closed current T. We give a sufficient condition on T and the measure μ so that the equation $T \wedge \beta^{n-m} \wedge (dd^c.)^{m-p} = \mu$ has a solution on the set of m-subharmonic functions. For this we establish a connection between the convergence in $cap_{m,T}$ of a sequence of m-subharmonic functions and the weak convergence of the associated Hessian measure.

Key words: m-positive closed current, m-subharmonic function, Capacity, Hessian operator

Mathematical Subject Classification 2010: 32U40; 32U05; 32U20

1. Introduction

The Dirichlet problem for the complex Monge–Ampere operator was studied by Bedford and Taylor [1] who proved first that the operator $(dd^c.)^n$ is well defined on the set of locally bounded plurisubharmonic functions in a bounded domain Ω of $\mathbb{C}^n$ and then solved the Monge–Ampere equation $(dd^c.)^n = 0$. This problem achieved a considerable progress when several researchers studied the case of non-degenerated Monge–Ampere equation and the regularity of its solution. Recently Błocki [2] introduced the notion of m-subharmonic function denoted by $SH_m(\Omega)$ for $1 \leq m \leq n$ and developed the pluripotential theory for the complex Hessian operator. This allows [2, 8, 10, 13] to study the Dirichlet problem for the Hessian equation using pluripotential techniques adapted to the complex Hessian equation to settle the question of the existence of its weak solutions. In 2013, Dhouib and Elkhadhra [7] introduced analogous Cegrell classes for studying the complex Hessian operator with respect to an m-positive closed current T. The purpose of our paper is to study the existence of a solution to the Hessian equation with respect to T which is given as follows:

$$T \wedge \beta^{n-m} \wedge (dd^c.)^{m-p} = \mu, \tag{1.1}$$

where $\beta := dd^c|z|^2$.

Using the notion of m-capacity $\text{cap}_{m,T}$ introduced by [7] and under some conditions on the given current and measure μ, we prove the existence of a solution for equation (1.1). This result is given by the following main result.

Theorem 1.1. *Assume that all $\|T\wedge\beta^{n-m}\|-$negligible sets are negligible for the Lebesgue measure and that:*

1) *There exists $v\in SH_m(\Omega)\cap L^\infty(\Omega)$ such that $T\wedge\beta^{n-m}\wedge(dd^cv)^{m-p}\geq\mu$.*

2) *There exists a sequence of measures $\mu_j := T\wedge\beta^{n-m}\wedge(dd^cu_j)^{m-p}$ such that $\|\mu_j-\mu\|_\Omega\to 0$, where $u_j\in SH_m(\Omega)\cap\mathcal{C}(\overline{\Omega})$, $u_j=u_1$ on $\partial\Omega$ for all $j\in\mathbb{N}$.*

3) *For all $j\in\mathbb{N}$, one has $\text{cap}_{m,T}\left(\sup\{u_k \mid k\geq j\}<\sup^\star\{u_k\mid k\geq j\}\right)=0$.*

Then there exists $u\in SH_m(\Omega)\cap L^\infty(\Omega)$ such that $T\wedge\beta^{n-m}\wedge(dd^cu)^{m-p}=\mu$.

The main tool for proving the above theorem is to find a suitable condition on the convergence of a sequence u_j to u to ensure the weak convergence of the measures $T\wedge\beta^{n-m}\wedge(dd^cu_j)^{m-p}$ to $T\wedge\beta^{n-m}\wedge(dd^cu)^{m-p}$. In the case $T=1$ and $m=n$, Cegrell [3] and Lelong [12] observed that the $\mathbb{L}^1_{loc}$-convergence of u_j to u is not sufficient for obtaining the weak convergence of the measure $(dd^cu_j)^n$ to $(dd^cu)^n$. In 1996, Xing [14] gave a sharp sufficient condition to ensure the convergence of $(dd^cu_j)^n$ to $(dd^cu)^n$. Here we treat the problem in the case of m-positive currents and we prove that convergence with respect to capacity $\text{cap}_{m,T}$ is sufficient for obtaining the required convergence and also, under some conditions, we show that the converse is true. We study also the convergence of $dd^cu_k\wedge T_k\wedge\beta^{n-m}$ to $dd^c\wedge uT\wedge\beta^{n-m}$, where $(T_k)_k$ is a sequence of m-positive closed currents that converges to T. We prove, under a suitable condition on the growth of the mass of $T_k\wedge\beta^{n-m}$ with respect to $\text{cap}_{m,T}$, that such convergence holds.

2. Preliminaries

Let us recall first the notion of m-subharmonicity introduced by Błocki in [2].

Definition 2.1. A real form α of bidegree $(1,1)$ in a domain Ω of $\mathbb{C}^n$ is said to be m-positive if at every point of Ω one has

$$\alpha^j\wedge\beta^{n-j}\geq 0,\quad j=1,\dots,m.$$

The above definition coincides with the standard definition of positivity introduced by Lelong for the case $m=n$. To obtain a similar analogy, Dhouib and Elkhadhra [7] introduced the following definition of positivity for (p,p)-forms.

Definition 2.2. Let φ be a real (p,p)-form defined on an open subset Ω of $\mathbb{C}^n$ and let m be an integer such that $p\leq m\leq n$.

1. The form φ is said to be m-positive on Ω if at every point of Ω one has

$$\varphi\wedge\beta^{m-n}\wedge\alpha_1\wedge\cdots\wedge\alpha_{m-p}\geq 0$$

for every m-positive $(1,1)$-form $\alpha_1,\dots,\alpha_{m-p}$.

2. The form φ is said to be m-strongly positive on Ω if it can be written as follows:

$$\varphi = \sum_{k=1}^{N} \lambda_k \alpha_1^k \wedge \cdots \wedge \alpha_p^k,$$

where $\alpha_1^k, \ldots, \alpha_p^k$ are m-positive forms on Ω and $\lambda_k \geq 0$.

By duality, one can define the notion of m-positive currents as follows.

Definition 2.3. Let T be a current of bidimension $(n-p, n-p)$ on Ω and let m be an integer satisfying $p \leq m \leq n$.

1. The current T is called m-positive if $\langle T, \beta^{n-m} \wedge \varphi \rangle \geq 0$ for every m-strongly positive form φ of bidegree $(m-p, m-p)$.

2. The current T is called m-strongly positive if $\langle T, \beta^{n-m} \wedge \varphi \rangle \geq 0$ for every m-positive form φ of bidegree $(m-p, m-p)$.

Remark 2.4.

1. The above definitions generalize the standard definition of positivity for forms and currents, it suffices to take the case $m = n$ to recover them.

2. If T is an m-positive current, then the current $T \wedge \beta^{n-m}$ is positive.

3. There is no link between s-positive currents and $r-$positive currents for every $r \neq s$.

Definition 2.5. A function $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ is called m-subharmonic if it is subharmonic and

$$dd^c u \wedge \beta^{n-m} \wedge \alpha_1 \wedge \cdots \wedge \alpha_{m-1} \geq 0$$

for all m-positive forms $\alpha_1, \ldots, \alpha_{m-1}$. We denote by $SH_m(\Omega)$ the set of all m-subharmonic functions defined on Ω.

We cite below some well-known properties of m-subharmonicity. For more details, one can refer to [2, 7, 13].

Proposition 2.6.

1. *If $u \in \mathcal{C}^2(\Omega)$, then $u \in SH_m(\Omega)$ if and only if the form $dd^c u$ is m-positive on Ω.*

2. *If $u \in SH_m(\Omega)$, then the current $dd^c u$ is m-positive.*

3. *If $u, v \in SH_m(\Omega)$, then $\lambda u + \mu v \in SH_m(\Omega), \forall \lambda, \mu > 0$.*

4. $PSH(\Omega) = SH_n(\Omega) \subsetneq \cdots \subsetneq SH_m(\Omega) \subsetneq \cdots \subsetneq SH_1(\Omega) = SH(\Omega)$.

5. *If u is m-subharmonic on Ω, then the standard regularization $u * \chi_\varepsilon$ is also m-subharmonic on $\Omega_\varepsilon := \{x \in \Omega \mid d(x, \partial\Omega) > \varepsilon\}$.*

6. *If $(u_i)_j$ is a decreasing sequence of m-subharmonic functions, then $u := \lim u_j$ is either m-subharmonic or identically equal to $-\infty$.*

The m-capacity of a subset E in Ω with respect to a given current T is defined as follows.

Definition 2.7. For every compact K of Ω, the m-capacity of K relatively to an m-positive current T denoted by $\mathrm{cap}_{m,T}(K)$ is defined by

$$\begin{aligned}\mathrm{cap}_{m,T}(K,\Omega) &= \mathrm{cap}_{m,T}(K)\\ &:= \sup\left\{\int_K (dd^c v)^{m-p}\wedge T\wedge \beta^{n-m} \mid v\in SH_m(\Omega), 0\le v\le 1\right\},\end{aligned}$$

and for every $E\subset\Omega$, $\mathrm{cap}_{m,T}(E,\Omega)=\sup\{\mathrm{cap}_{m,T}(K)\mid K \text{ is a compact of } \Omega\}$.

Basing on the definitions cited below, Dhouib and Elkhadhra [7] defined the Hessian operator with respect to a given m-positive closed current of bidegree (p,p) to generalize the well-known works of Bedford and Taylor [1], Błocki [2], Abdullaev and Sadullaev [13] and Lu [11]. They proved that the Hessian operator $(dd^c.)^p\wedge T\wedge\beta^{n-m}$ is well defined on the set of bounded m-subharmonic functions (eventually, also for m-subharmonic functions which are bounded near $\partial\Omega\cap \mathrm{Supp}\, T$) and studied its pluripotential properties. An essential tool used in their work is the convergence with respect to the capacity $\mathrm{cap}_{m,T}$ defined by using the complex Hessian measure associated to T. In the next section, we are intending to give a link between the weak convergence and the convergence with respect to $\mathrm{cap}_{m,T}$.

3. Weak convergence and convergence with respect to $\mathrm{cap}_{m,T}$

The notion of the convergence in capacity $\mathrm{cap}_{m,T}$ was introduced in [7] as follows.

Definition 3.1. Let Ω be an open subset of $\mathbb{C}^n$, $E\subset\Omega$ and let T be an m-positive closed current of bidimension $(n-p,n-p)$, $p\le m\le n$. A sequence of functions $(u_j)_j$ defined on Ω is said to be convergent with respect to $\mathrm{cap}_{m,T}$ to u on E if for all $t>0$ one has

$$\lim_{j\to+\infty}\mathrm{cap}_{m,T}(E\cap\{|u-u_j|>t\})=0.$$

We will prove first that every sequence of bounded m-subharmonic functions $(u_j)_j$ that decreases to a function u converges to u with respect to capacity $\mathrm{cap}_{m,T}$. This generalizes the result of Bedford and Taylor [1] for the limit case $m=n$ and $T=1$ and Lu [10] for the case $T=1$. To prove this, we will generalize first a result due to Dabbek and Elkhadra [6]. This result is given by the following lemma.

Lemma 3.2. *Let $u_1, u_2, v_1, v_2, w_1, \dots, w_{p-1} \in SH_m(\Omega) \cap L^\infty(\Omega)$ and let T be an m-positive closed current on Ω of bidimension $(n-p, n-p)$, $p \leq m \leq n$. Assume that $\{u_1 \neq u_2\} \Subset \Omega$ and let $0 \leq \psi \in \mathcal{D}(\Omega)$, $\psi = 1$ on $\{u_1 \neq u_2\}$. Then,*

$$\left(\int_\Omega d(u_1 - u_2) \wedge d^c(v_1 - v_2) \wedge \chi\right)^2 \leq \left(\int_\Omega d(u_1 - u_2) \wedge d^c(u_1 - u_2) \wedge \chi\right) \times \left(\int_\Omega \psi d(v_1 - v_2) \wedge d^c(v_1 - v_2) \wedge \chi\right),$$

where $\chi = T \wedge \beta^{n-m} \wedge dd^c w_1 \wedge dd^c w_2 \wedge \cdots \wedge dd^c w_{m-p-1}$.

Proof. Using Theorem 2 from [7], it suffices to prove the result for the case when $u_1 - u_2, v_1 - v_2$ are smooth. It is easy to check that $(u, v) \mapsto \int_\Omega \psi du \wedge d^c v \wedge \chi$ is a positive and symmetric bilinear form on $\mathcal{C}^\infty(\Omega) \times \mathcal{C}^\infty(\Omega)$. Using the Cauchy–Schwartz inequality on $(u_1 - u_2, v_1 - v_2)$, when $u_i, v_i \in \mathcal{S}H_m(\Omega) \cap \mathcal{C}^\infty(\Omega)$, we get the desired inequality. □

Theorem 3.3. *Let Ω be a bounded open subset of $\mathbb{C}^n$ and let T be an m-positive closed current on Ω of bidimension $(n-p, n-p)$, $p \leq m \leq n$. If $u_j, u \in SH_m(\Omega) \cap L^\infty_{loc}(\Omega)$ such that $u_j = u$ on a fixed neighborhood of $\partial\Omega$ and u_j decreases to u, then for all $\delta > 0$ one has*

$$\lim_{j \to +\infty} \mathrm{cap}_{m,T} \{z \in \Omega \mid u_j(z) > u(z) + \delta\} = 0.$$

Proof. Without loss of generality, one can assume that $\delta = 1$. We consider

$$\Omega_j = \{z \in \Omega \mid u_j(z) > u(z) + 1\}$$

and $\mathcal{U}$ such that $\{u_j \neq u\} \subset \mathcal{U} \Subset \Omega$. Let $v \in \mathcal{S}H_m(\Omega, [0, 1])$, using the Stokes formula, we obtain

$$\int_{\Omega_j} (dd^c v)^{m-p} \wedge T \wedge \beta^{n-m} \leq \int_{\mathcal{U}} (u_j - u)(dd^c v)^{m-p} \wedge T \wedge \beta^{n-m} = -\int_{\mathcal{U}} d(u_j - u) \wedge d^c v \wedge (dd^c v)^{m-p-1} \wedge T \wedge \beta^{n-m}.$$

Now, by Lemma 3.2, the right-hand side is dominated by

$$C \left(\int_{\mathcal{U}} d(u_j - u) \wedge d^c(u_j - u) \wedge (dd^c v)^{m-p-1} \wedge T \wedge \beta^{n-m}\right)^{\frac{1}{2}},$$

where

$$C = \left(\int_{\mathcal{U}} dv \wedge d^c v \wedge (dd^c v)^{m-p-1} \wedge T \wedge \beta^{n-m}\right)^{\frac{1}{2}} \leq M < +\infty$$

and M is a constant independent on v using the Chern–Levine–Nirenberg inequality [7]. Again, by the Stokes formula, we get

$$\int_{\mathcal{U}} d(u_j - u) \wedge d^c(u_j - u) \wedge (dd^c v)^{m-p-1} \wedge T \wedge \beta^{n-m}$$

$$= -\int_{\mathcal{U}} (u_j - u) dd^c(u_j - u) \wedge (dd^c v)^{m-p-1} \wedge T \wedge \beta^{n-m}$$
$$= \int_{\mathcal{U}} (u - u_j)(dd^c u_j - dd^c u) \wedge (dd^c v)^{m-p-1} \wedge T \wedge \beta^{n-m}$$
$$\leq \int_{\mathcal{U}} (u_j - u) dd^c u \wedge (dd^c v)^{m-p-1} \wedge T \wedge \beta^{n-m}.$$

It follows that

$$\int_{\Omega_j} (dd^c v)^{m-p} \wedge T \wedge \beta^{n-m} \leq C \left(\int_{\mathcal{U}} (u_j - u) dd^c u \wedge (dd^c v)^{m-p-1} \wedge T \wedge \beta^{n-m} \right)^{\frac{1}{2}}.$$

By repeating the process $(m-p-1)$-times, we get the following estimate:

$$\int_{\Omega_j} (dd^c v)^{m-p} \wedge T \wedge \beta^{n-m} \leq C_1 \left(\int_{\Omega} (u_j - u)(dd^c u)^{m-p} \wedge T \wedge \beta^{n-m} \right)^{\frac{1}{2^p}},$$

where C_1 is a constant which does not depend on j and v. As v is arbitrarily chosen, we deduce that $\lim_{j\to+\infty} \mathrm{cap}_{m,T}(\Omega_j) = 0$. □

The following theorem was proved in [7] and it will be useful later on.

Theorem 3.4. *Let Ω be a bounded subset of $\mathbb{C}^n$, $u \in \mathcal{SH}_m(\Omega) \cap L^\infty_{loc}(\Omega)$ and let T be an m-positive closed current on Ω of bidimension $(n-p, n-p)$, $p \leq m \leq n$. Then, for all $\varepsilon > 0$, there exists an open set $\mathcal{O}_\varepsilon$ of Ω such that $\mathrm{cap}_{m,T}(\mathcal{O}_\varepsilon, \Omega) < \varepsilon$ and u is continuous $\Omega \setminus \mathcal{O}_\varepsilon$.*

Now we will establish the connection between the convergence in capacity of a sequence of m-subharmonic functions u_j and the weak convergence of the associated Hessian measure. A similar version of the first assertion in the theorem below was proved in [7] for m-subharmonic functions that are bounded only near the boundary of Ω, but with an additional sufficient condition (each of the Hessian measure of u_j is absolutely continuous with respect to $\mathrm{cap}_{m,T}$). Here we give a different proof for the case of locally bounded m-subharmonic functions and without any assumption on the Hessian measure of u_j. We will also prove the converse.

Theorem 3.5. *Let Ω be an open subset of $\mathbb{C}^n$ and let T be an m-positive closed current on Ω of bidimension $(n-p, n-p)$ and $(u_j)_j$ be a sequence of locally uniformly bounded m-subharmonic functions and $u \in \mathcal{SH}_m(\Omega) \cap L^\infty_{loc}(\Omega)$. Then*

1. *If u_j converges to u in capacity $\mathrm{cap}_{m,T}$ on every $E \Subset \Omega$, then the sequence of currents $(dd^c u_j)^{m-p} \wedge T \wedge \beta^{n-m}$ converges weakly to $(dd^c u)^{m-p} \wedge T \wedge \beta^{n-m}$.*
2. *Assume that there exist $E \Subset \Omega$ such that for all j, $u_j = u$ on $\Omega \setminus E$ and that the sequences $u(dd^c u_j)^{m-p} \wedge T \wedge \beta^{n-m}$, $u_j(dd^c u)^{m-p} \wedge T \wedge \beta^{n-m}$ and $u_j(dd^c u_j)^{m-p} \wedge T \wedge \beta^{n-m}$ converge weakly to $u(dd^c u)^{m-p} \wedge T \wedge \beta^{n-m}$. Then u_j converges to u with respect to $\mathrm{cap}_{m,T}$ on E.*

Proof. 1. We proceed by induction on $m-p$. The case $m-p=1$ will be proved if we show that $u_jT\wedge\beta^{n-m}$ converges to $uT\wedge\beta^{n-m}$. Let φ be a smooth form with compact support in Ω ($\varphi\in\mathcal{D}_{m-p,m-p}(\Omega)$) such that $supp\varphi\subset\Omega_1\Subset\Omega$. Then,

$$\begin{aligned}\left|\int_\Omega (u_jT-uT)\wedge\beta^{n-m}\wedge\varphi\right| &\le C\int_{\Omega_1}|u_j-u|T\wedge\beta^{n-p}\\ &= C\int_{\{|u_j-u|\le\delta\}\cap\Omega_1}|u_j-u|T\wedge\beta^{n-p}+C\int_{\{|u_j-u|>\delta\}\cap\Omega_1}|u_j-u|T\wedge\beta^{n-p}\\ &\le C\delta||T\wedge\beta^{n-m}||_{\Omega_1}+C||u_j-u||_{L^\infty(\Omega_1)}\int_{\{|u_j-u|\ge\delta\}\cap\Omega_1}T\wedge\beta^{n-p}\\ &\le C\delta||T\wedge\beta^{n-m}||_{\Omega_1}+M\,\mathrm{cap}_{m,T}(\{z\in\Omega_1;|u_j(z)-u(z)|>\delta\}).\end{aligned}$$

This proves the case $m-p=1$ since δ is arbitrary, u_j converges to u in capacity $\mathrm{cap}_{m,T}$ and M is independent on j.

Assume by induction that the sequence $(dd^cu_j)^s\wedge T\wedge\beta^{n-m}$ converges weakly to $(dd^cu)^s\wedge T\wedge\beta^{n-m}$ for $s<m-p$. It suffices to prove that $u_j(dd^cu_j)^s\wedge T\wedge\beta^{n-m}$ converges weakly to $u(dd^cu)^s\wedge T\wedge\beta^{n-m}$. By Theorem 3.4, for all $\varepsilon>0$, there exists an open subset O_ε such that $\mathrm{cap}_{m,T}(O_\varepsilon)<\varepsilon$ and $u=\varphi+\psi$, where φ is continuous on Ω and $\psi=0$ on $\Omega\setminus O_\varepsilon$. Note that

$$\begin{aligned}&u_j(dd^cu_j)^s\wedge T\wedge\beta^{n-m}-u(dd^cu)^s\wedge T\wedge\beta^{n-m}\\ &\qquad=(u_j-u)(dd^cu_j)^s\wedge T\wedge\beta^{n-m}\\ &\qquad\quad+\psi((dd^cu_j)^s\wedge T\wedge\beta^{n-m}-(dd^cu)^s\wedge T\wedge\beta^{n-m})\\ &\qquad\quad+\varphi((dd^cu_j)^s\wedge T\wedge\beta^{n-m}-(dd^cu)^s\wedge T\wedge\beta^{n-m}).\end{aligned}$$

Denote the first, second, and third summands in the right-hand side of this equality by (1), (2), and (3), respectively. Since φ is continuous on Ω and using induction's hypothesis, we get that (3) tends weakly to 0 when $j\to\infty$.

For (1), let $\varphi\in\mathcal{D}_{m-p-s,m-p-s}(\Omega)$ such that $supp\,\varphi\subset\Omega_1\Subset\Omega_2\Subset\Omega$. Then,

$$\begin{aligned}&\left|\int_\Omega (u_j-u)T\wedge\beta^{n-m}\wedge(dd^cu_j)^s\wedge\varphi\right|\\ &\quad\le C\int_{\Omega_1}|u_j-u|T\wedge\beta^{n-m}\wedge(dd^cu_j)^s\wedge(dd^c|z|^2)^{m-p-s}\\ &\quad\le C\int_{\Omega_1}|u_j-u|T\wedge\beta^{n-m}\wedge(dd^c(u_j+|z|^2))^{m-p}\\ &\quad\le C\int_{\{|u_j-u|>\delta\}\cap\Omega_1}|u_j-u|T\wedge\beta^{n-m}\wedge dd^c(u_j+|z|^2)^{m-p}\\ &\quad+C\int_{\{|u_j-u|\le\delta\}\cap\Omega_1}|u_j-u|T\wedge\beta^{n-m}\wedge dd^c(u_j+|z|^2)^{m-p}\\ &\quad\le C_1\,\mathrm{cap}_{m,T}(z\in\Omega_1;|u_j(z)-u(z)|>\delta)+\delta M||T\wedge\beta^{n-m}||_{\Omega_2}.\end{aligned}$$

Since the sequence $(u_j)_j$ is uniformly bounded, M and C_1 do not depend on j and $u_j\to u$ in capacity $\mathrm{cap}_{m,T}$, we get that (1) tends to 0.

The same reason for (2) gives

$$\left|\int_{\Omega_1\cap O_\varepsilon}\psi T\wedge\beta^{n-m}\wedge(dd^c u_j)^s\wedge\varphi\right|\leq A\int_{\Omega_1\cap O_\varepsilon}T\wedge\beta^{n-m}\wedge(dd^c(u_j+|z^2|))^{m-p}$$
$$\leq B_1\,\mathrm{cap}_{m,T}(O_\varepsilon)\leq\varepsilon B_1.$$

Using the same reasoning as above, one can obtain that

$$\left|\int_{\Omega_1\cap O_\varepsilon}\psi T\wedge\beta^{n-m}\wedge(dd^c u)^s\wedge\varphi\right|\leq\varepsilon B_2.$$

2. Let Ω' be an open subset such that $E\Subset\Omega'\Subset\Omega$, $\varphi\in SH_m(\Omega,[0,1])$ and $\delta>0$. By the Stokes formula and Lemma 3.2, we obtain

$$\begin{aligned}
&\int_{\{|u_j-u|>\delta\}}T\wedge\beta^{n-m}\wedge(dd^c\varphi)^{m-p}\\
&\quad\leq\frac{1}{\delta^2}\int_{\Omega'}(u_j-u)^2T\wedge\beta^{n-m}\wedge(dd^c\varphi)^{m-p}\\
&\quad=\frac{-1}{\delta^2}\int_{\Omega'}T\wedge\beta^{n-m}\wedge d(u_j-u)^2\wedge d^c\varphi\wedge(dd^c\varphi)^{m-p-1}\\
&\quad\leq C_1\left(\int_{\Omega'}T\wedge\beta^{n-m}\wedge d(u_j-u)^2\wedge d^c(u_j-u)^2\wedge(dd^c\varphi)^{m-p-1}\right)^{\frac{1}{2}}\\
&\quad\leq 2C_1C_2\left(\int_{\Omega'}T\wedge\beta^{n-m}\wedge d(u_j-u)\wedge d^c(u_j-u)\wedge(dd^c\varphi)^{m-p-1}\right)^{\frac{1}{2}},
\end{aligned}$$

where

$$C_1:=\frac{1}{\delta^2}\int_{\Omega'}T\wedge\beta^{n-m}\wedge d\varphi\wedge d^c\varphi\wedge(dd^c\varphi)^{m-p-1}<+\infty$$

and $C_2:=||u_j-u||_\infty<\infty$. As

$$dd^c(u_j-u)\wedge T\wedge\beta^{n-m}\leq dd^c(u_j+u)\wedge T\wedge\beta^{n-m},$$

then, by repeating the same operation $(m-p-1)-$times, we get

$$\begin{aligned}
&\int_{\Omega'}T\wedge\beta^{n-m}\wedge d(u_j-u)\wedge d^c(u_j-u)\wedge(dd^c\varphi)^{m-p-1}\\
&\quad=\int_{\Omega'}T\wedge\beta^{n-m}\wedge d(u_j-u)\wedge d^c\varphi\wedge dd^c(u_j-u)\wedge(dd^c\varphi)^{m-p-2}\\
&\quad\leq B\left(\int_{\Omega'}T\wedge\beta^{n-m}\wedge d(u_j-u)\wedge d^c(u_j-u)\wedge dd^c(u_j+u)\wedge(dd^c\varphi)^{m-p-2}\right)^{\frac{1}{2}}\\
&\quad\leq\cdots\\
&\quad\leq B_1\left(\int_{\Omega'}T\wedge\beta^{n-m}\wedge d(u_j-u)\wedge d^c(u_j-u)\wedge(dd^c(u_j+u))^{m-p-1}\right)^{\frac{1}{2(m-p)}}\\
&\quad\leq B_2\Bigg(\int_{\Omega'}T\wedge\beta^{n-m}\wedge d(u_j-u)\wedge d^c(u_j-u)
\end{aligned}$$

$$\wedge \sum_{k=0}^{m-p-1} (dd^c u_j)^{m-p-k-1} \wedge (dd^c u)^k \Bigg)^{\frac{1}{2(m-p)}}$$

$$= B_2 \left(\int_{\Omega'} (u_j - u) \Bigg[T \wedge \beta^{n-m} \wedge (dd^c u_j - dd^c u) \right.$$

$$\left. \wedge \sum_{k=0}^{m-p-1} (dd^c u_j)^{m-p-k-1} \wedge (dd^c u)^k \Bigg] \right)^{\frac{1}{2(m-p)}}$$

$$= B_2 \left(\int_{\Omega'} (u_j - u) \big[T \wedge \beta^{n-m} \wedge (dd^c u_j)^{m-p} \right.$$

$$\left. - T \wedge \beta^{n-m} \wedge (dd^c u)^{m-p} \big] \right)^{\frac{1}{2(m-p)}},$$

where B_2 does not depends on j and φ. As $u_j = u$ on $\Omega' \backslash E$ and the sequences $uT \wedge \beta^{n-m} \wedge (dd^c u_j)^{m-p}$, $u_j T \wedge \beta^{n-m} \wedge (dd^c u)^{m-p}$, $u_j T \wedge \beta^{n-m} \wedge (dd^c u_j)^{m-p}$ converge to $uT \wedge \beta^{n-m} \wedge (dd^c u)^{m-p}$, then we get

$$\lim_{j \to +\infty} \int_{\Omega'} (u_j - u) T \wedge \beta^{n-m} \wedge \left[(dd^c u_j)^{m-p} - (dd^c u)^{m-p} \right] = 0.$$

It follows that

$$\operatorname{cap}_{m,T}(|u_j - u| > \delta, \Omega) = 0. \qquad \square$$

Proposition 3.6. *Let T be an m-positive closed current on Ω of bidimension $(n-p, n-p)$, $v_1, \ldots, v_{m-p} \in SH_m(\Omega) \bigcap L^\infty(\Omega)$; $v_1^j, \ldots, v_{m-p}^j \in SH_m(\Omega)$. Assume that the sequence $(v_k^j)_j$ is locally uniformly bounded and increases almost everywhere to v_k with respect to $\operatorname{cap}_{m,T}$. Then*

$$T \wedge \beta^{n-m} \wedge dd^c v_1^j \wedge \cdots \wedge dd^c v_{m-p}^j \to T \wedge \beta^{n-m} \wedge dd^c v_1 \wedge \cdots \wedge dd^c v_{m-p}$$

weakly in Ω.

Proof. We proceed as in [4]. Let φ and $\eta \in \mathcal{D}(\Omega)$ be such that $\eta \geq 0$ and $\eta \equiv 1$ in a neighborhood of supp φ. Let $\varphi_1, \varphi_2 \in SH_m(\Omega) \cap \mathcal{C}^\infty(\Omega)$ be such that $\varphi = \varphi_1 - \varphi_2$. We have

$$\int_\Omega \varphi T \wedge \beta^{n-m} \wedge dd^c v_1^j \wedge \cdots \wedge dd^c v_{m-p}^j$$

$$= \int_\Omega v_1^j T \wedge \beta^{n-m} \wedge dd^c v_2^j \wedge \cdots \wedge dd^c v_{m-p}^j \wedge dd^c \varphi$$

$$= \int_\Omega v_1^j T \wedge \beta^{n-m} \wedge dd^c v_2^j \wedge \cdots \wedge dd^c v_{m-p}^j \wedge (dd^c \varphi_1 - dd^c \varphi_2).$$

By induction, we assume that we have the following weak convergence:

$$\lim_{j \to +\infty} T \wedge \beta^{n-m} \wedge dd^c v_2^j \wedge \cdots \wedge dd^c v_{m-p}^j \wedge dd^c \varphi_1$$

$$= T \wedge \beta^{n-m} \wedge dd^c v_2 \wedge \ldots \wedge dd^c v_{m-p} \wedge dd^c \varphi_1. \tag{3.1}$$

As $(v_1^j) \uparrow v_1$, for all $k \leq j$, one has

$$\int_\Omega \eta v_1^k T \wedge \beta^{n-m} \wedge dd^c v_2^j \wedge \cdots \wedge dd^c v_{m-p}^j \wedge dd^c \varphi_1$$
$$\leq \int_\Omega \eta v_1^j T \wedge \beta^{n-m} \wedge dd^c v_2^j \wedge \cdots \wedge dd^c v_{m-p}^j \wedge dd^c \varphi_1$$
$$\leq \int_\Omega \eta v_1 T \wedge \beta^{n-m} \wedge dd^c v_2^j \wedge \cdots \wedge dd^c v_{m-p}^j \wedge dd^c \varphi_1.$$

Using (3.1) and Theorem 3.4, one can prove that

$$\lim_{j\to+\infty} \int_\Omega \eta v_1^k T \wedge \beta^{n-m} \wedge dd^c v_2^j \wedge \cdots \wedge dd^c v_{m-p}^j \wedge dd^c \varphi_1$$
$$= \int_\Omega \eta v_1^k T \wedge \beta^{n-m} \wedge dd^c v_2 \wedge \cdots . \wedge dd^c v_{m-p} \wedge dd^c \varphi_1,$$
$$\lim_{j\to+\infty} \int_\Omega \eta v_1 T \wedge \beta^{n-m} \wedge dd^c v_2^j \wedge \cdots \wedge dd^c v_{m-p}^j \wedge dd^c \varphi_1$$
$$= \int_\Omega \eta v_1 T \wedge \beta^{n-m} \wedge dd^c v_2 \wedge \cdots \wedge dd^c v_{m-p} \wedge dd^c \varphi_1.$$

It follows that

$$\int_\Omega \eta v_1^k T \wedge \beta^{n-m} \wedge dd^c v_2 \wedge \cdots \wedge dd^c v_{m-p} \wedge dd^c \varphi_1$$
$$\leq \liminf_{j\to+\infty} \int_\Omega \eta v_1^j T \wedge \beta^{n-m} \wedge dd^c v_2^j \wedge \cdots \wedge dd^c v_{m-p}^j \wedge dd^c \varphi_1$$
$$\leq \limsup_{j\to+\infty} \int_\Omega \eta v_1^j T \wedge \beta^{n-m} \wedge dd^c v_2^j \wedge \cdots \wedge dd^c v_{m-p}^j \wedge dd^c \varphi_1$$
$$\leq \int_\Omega \eta v_1 T \wedge \beta^{n-m} \wedge dd^c v_2 \wedge \cdots \wedge dd^c v_{m-p} \wedge dd^c \varphi_1. \tag{3.2}$$

To finish the proof, it suffices to prove that

$$\lim_{k\to+\infty} \int_\Omega \eta v_1^k T \wedge \beta^{n-m} \wedge dd^c v_2 \wedge \cdots \wedge dd^c v_{m-p} \wedge dd^c \varphi_1$$
$$= \int_\Omega \eta v_1 T \wedge \beta^{n-m} \wedge dd^c v_2 \wedge \cdots \wedge dd^c v_{m-p} \wedge dd^c \varphi_1.$$

Let $v_2^\varepsilon = v_2 * \chi_\varepsilon$, where χ_ε is a regularizing kernel. We can assume that $v_2^\varepsilon = v_2$ on $\Omega \setminus \operatorname{supp} \eta$ and that $\eta = 1$ in a neighborhood of $\{v_2^\varepsilon \neq v_2\}$. It is easy to check that

$$\int_\Omega \eta v_1^k dd^c(v_2 - v_2^\varepsilon) \wedge dd^c v_3 \wedge \cdots \wedge dd^c v_{m-p} \wedge T \wedge \beta^{n-m} \wedge dd^c \varphi_1$$
$$= \int_\Omega v_1^k dd^c(v_2 - v_2^\varepsilon) \wedge dd^c v_3 \wedge \cdots \wedge dd^c v_{m-p} \wedge T \wedge \beta^{n-m} \wedge dd^c \varphi_1$$

$$= \int_\Omega (v_2 - v_2^\varepsilon) dd^c v_1^k \wedge dd^c v_3 \wedge \cdots \wedge dd^c v_{m-p} \wedge T \wedge \beta^{n-m} \wedge dd^c \varphi_1.$$

For all k, v_1^k is locally uniformly bounded. Using Theorem 3.4 and the fact that $v_2^\varepsilon \downarrow v_2$, the last integral tends to 0 uniformly in k. It follows that

$$\begin{aligned}
&\int_\Omega \eta v_1^k dd^c v_2 \wedge \cdots \wedge dd^c v_{m-p} \wedge T \wedge \beta^{n-m} \wedge dd^c \varphi_1 \\
&\qquad \geq -\varepsilon + \int_\Omega \eta v_1^k dd^c v_2^\varepsilon \wedge \cdots \wedge dd^c v_{m-p} \wedge T \wedge \beta^{n-m} \wedge dd^c \varphi_1 \geq \cdots \\
&\qquad \geq -(p-1)\varepsilon \\
&\qquad\quad + \int_\Omega \eta v_1^k dd^c v_2^\varepsilon \wedge dd^c v_3^{\varepsilon_1} \wedge \cdots \wedge dd^c v_{m-p}^{\varepsilon_{m-p-2}} \wedge T \wedge \beta^{n-m} \wedge dd^c \varphi_1,
\end{aligned}$$

where $0 < \varepsilon_{m-p-2} < \cdots < \varepsilon_1 < \varepsilon$. Since the sequence (v_1^k) is increasing almost everywhere to v_1 with respect to $\mathrm{cap}_{m,T}$, we get

$$\begin{aligned}
&\liminf_{k \to +\infty} \int_\Omega \eta v_1^k dd^c v_2 \wedge \cdots \wedge dd^c v_{m-p} \wedge T \wedge \beta^{n-m} \wedge dd^c \varphi_1 \\
&\qquad \geq -(p-1)\varepsilon + \int_\Omega \eta v_1 dd^c v_2^\varepsilon \wedge \cdots \wedge dd^c v_{m-p}^{\varepsilon_{m-p-2}} \wedge T \wedge \beta^{n-m} \wedge dd^c \varphi_1.
\end{aligned}$$

By taking the limit when $\varepsilon \downarrow 0$, we obtain

$$\begin{aligned}
&\liminf_{k \to +\infty} \int_\Omega \eta v_1^k dd^c v_2 \wedge \cdots \wedge dd^c v_{m-p} \wedge T \wedge \beta^{n-m} \wedge dd^c \varphi_1 \\
&\qquad \geq \int_\Omega \eta v_1 dd^c v_2 \wedge \cdots \wedge dd^c v_{m-p} \wedge T \wedge \beta^{n-m} \wedge dd^c \varphi_1.
\end{aligned}$$

Using (3.2), we obtain the following weak convergence:

$$\begin{aligned}
&\lim_{j \to +\infty} v_1^j T \wedge \beta^{n-m} \wedge dd^c v_2^j \wedge \cdots \wedge dd^c v_{m-p}^j \wedge dd^c \varphi_1 \\
&\qquad = v_1 T \wedge \beta^{n-m} \wedge dd^c v_2 \wedge \cdots \wedge dd^c v_{m-p} \wedge dd^c \varphi_1.
\end{aligned}$$

The same reason applied to φ_2 gives the desired result. □

In the following theorem we will prove the convergence of the sequence $(dd^c u_k \wedge T_k \wedge \beta^{n-m})_k$ (here the current T is no longer fixed and is replaced by a sequence of currents that converges towards it). This result generalizes Elkhadhra's Theorem [9] proved for the limit case $m = n$.

Theorem 3.7. *Let T and T_k be closed m-positive currents of bidimension (p,p) in Ω such that T_k converges weakly to T in Ω. Let u and u_k be locally uniformly bounded $m-sh$ functions in Ω such that $u_k \to u$ in $\mathrm{cap}_{m,T}$ on each $E \Subset \Omega$. Assume that*

$$\|T_k \wedge \beta^{n-m}\| \ll \mathrm{cap}_{m,T}$$

on each $E \Subset \Omega$ uniformly as $k \to \infty$. Then

$$dd^c u_k \wedge T_k \wedge \beta^{n-m} \to dd^c u \wedge T \wedge \beta^{n-m} \quad \textit{weakly in } \Omega.$$

Proof. It suffices to prove that $u_kT_k \wedge \beta^{n-m} \to uT \wedge \beta^{n-m}$ weakly in Ω. For this, let φ be a test form on Ω, $E = supp(\varphi)$ and let K be a compact subset of Ω such that $E \subset K^\circ$. Since $\|T_k \wedge \beta^{n-m}\| \ll \mathrm{cap}_{m,T}$ on K uniformly for all k, we get that for every $\varepsilon > 0$ there exists $\delta > 0$ and $k_0 \in \mathbb{N}$ such that for any Borel subset $K_1 \subset K^\circ$ with $\mathrm{cap}_{m,T}(K_1) < \delta$, we have $||T_k \wedge \beta^{n-m}||(K_1) < \varepsilon$ uniformly for $k \geq k_0$. Now, by Theorem 3.4, there exists an open set $\mathcal{O} \subset \Omega$ with $\mathrm{cap}_{m,T}(\mathcal{O}) < \delta$ such that u is continuous on $\Omega \setminus \mathcal{O}$. Thus, we can write $u_k = v_k + w_k$, $u = v + w$, where v is a continuous function in Ω, $w = w_k = 0$ on $\Omega \setminus \mathcal{O}$, for each k and all v_k, w_k, v, w are locally uniformly bounded on Ω by a constant independent of δ. It is easy to check that

$$\left|\int_\Omega (u_kT_k - uT) \wedge \beta^{n-m} \wedge \varphi\right| \leq \left|\int_\Omega (v_k - v)T_k \wedge \beta^{n-m} \wedge \varphi\right| + \left|\int_\Omega v(T_k - T) \wedge \beta^{n-m} \wedge \varphi\right| + \left|\int_\Omega (w_kT_k - wT) \wedge \beta^{n-m} \wedge \varphi\right|. \quad (3.3)$$

As $v = u$, $v_k = u_k$ on $\Omega \setminus \mathcal{O}$, the first term in the right-hand side of inequality is bounded by

$$\left|\int_{E\setminus\mathcal{O}} (u_k - u)T_k \wedge \beta^{n-m} \wedge \varphi\right| + \left|\int_{\mathcal{O}\cap E} (v_k - v)T_k \wedge \beta^{n-m} \wedge \varphi\right|.$$

As the functions u_k, u are locally uniformly bounded, there exist A, B independent of k and ε such that

$$\begin{aligned}\left|\int_{E\setminus\mathcal{O}} (u_k - u)T_k \wedge \beta^{n-m} \wedge \varphi\right| &\leq A_1 \int_{E\setminus\mathcal{O}} \mid u_k - u \mid T_k \wedge \beta^p \\ &= A_1\left(\int_{(E\setminus\mathcal{O})\cap\{|u_k-u|<\varepsilon\}} |u_k - u|T_k \wedge \beta^p \right.\\ &\quad \left. + \int_{(E\setminus\mathcal{O})\cap\{|u_k-u|\geq\varepsilon\}} \mid u_k - u \mid T_k \wedge \beta^p\right) \\ &\leq A_1\varepsilon||T_k \wedge \beta^{n-m}||(E) + A_2||T_k \wedge \beta^{n-m}||(E \cap \{|u_k - u| \geq \varepsilon\}).\end{aligned}$$

Now, using the fact that $T_k \wedge \beta^{n-m} \to T \wedge \beta^{n-m}$ weakly in Ω, we get that $\|T_k \wedge \beta^{n-m}\|(E)$ is uniformly bounded.

On the other hand, since $\mathrm{cap}_{m,T}(E \cap \{|u_k - u| \geq \varepsilon\}) \to 0$ as $k \to \infty$, then for $k \geq k_1$ large enough we deduce that $\mathrm{cap}_{m,T}(E \cap \{|u_k - u| \geq \varepsilon\}) < \delta$. It follows that $\|T_k \wedge \beta^{n-m}\|(E \cap \{|u_k - u| \geq \varepsilon\}) < \varepsilon$ for all $k \geq \max(k_0, k_1)$. Hence we get that

$$\lim_{k\to+\infty} \left|\int_{E\setminus\mathcal{O}} (u_k - u)T_k \wedge \beta^{n-m} \wedge \varphi\right| \leq A_3\varepsilon.$$

Since v_k, v are locally uniformly bounded, there exists a constant A_4 which does not depend on ε and k such that

$$\left|\int_{\mathcal{O}\cap E} (v_k - v)T_k \wedge \beta^{n-m} \wedge \varphi\right| \leq A_4\|T_k \wedge \beta^{n-m}\|(\mathcal{O} \cap E).$$

Since $\mathcal{O} \cap E \subset K$ and $\text{cap}_{m,T}(\mathcal{O} \cap E) < \delta$, we have

$$\left| \int_{\mathcal{O}\cap E} (v_k - v) T_k \wedge \beta^{n-m} \wedge \varphi \right| \leq A_4 \|T_k \wedge \beta^{n-m}\|(\mathcal{O} \cap E) < A_4 \varepsilon \textit{ for all } k \geq k_0.$$

It follows that if $k \to +\infty$, then the first term in the right-hand side of inequality (3.3) is less than $(A_3 + A_4)\varepsilon$, while the second one converges to zero because of the continuity of v and the fact that $T_k \wedge \beta^{n-m} \to T \wedge \beta^{n-m}$ weakly in Ω. For the third term, since w_k, w are locally uniformly bounded on Ω and are vanishing on $\Omega \smallsetminus \mathcal{O}$, there exists a constant $A_5 > 0$ such that

$$\left| \int_{\Omega} (w_k T_k - wT) \wedge \beta^{n-m} \wedge \varphi \right| \leq A_5 (\|T_k \wedge \beta^{n-m}\|(\mathcal{O} \cap E) + \|T \wedge \beta^{n-m}\|(\mathcal{O} \cap E)).$$

As explained above, we have $\|T_k \wedge \beta^{n-m}\|(\mathcal{O} \cap E) < \varepsilon$ for all $k \geq k_0$. On the other hand, since $\mathcal{O} \cap K^\circ$ is open and $T_k \wedge \beta^{n-m} \to T \wedge \beta^{n-m}$ as currents in Ω, we can easily prove that

$$\|T \wedge \beta^{n-m}\|(\mathcal{O} \cap E) \leq \|T \wedge \beta^{n-m}\|(\mathcal{O} \cap K^\circ) \leq \varliminf_{k \to +\infty} \|T_k \wedge \beta^{n-m}\|(\mathcal{O} \cap K^\circ) \leq \varepsilon.$$

The last inequality follows from the fact that $\text{cap}_{m,T}(\mathcal{O} \cap K^\circ) < \delta$. Finally, by summing up the three terms in the right-hand side of inequality (3.3), we obtain the estimate

$$\lim_{k \to +\infty} \left| \int_{\Omega} (u_k T_k - uT) \wedge \beta^{n-m} \wedge \varphi \right| \leq A_6 \varepsilon,$$

where A_6 is a constant not depending on ε. Since ε is arbitrary, the result follows. □

4. Range of the operator $T \wedge \beta^{n-m} \wedge (dd^c.)^{m-p}$

Proposition 4.1. *Let Ω be a bounded open subset of $\mathbb{C}^n$ and let T be an m-positive closed current of bidimension $(n-p, n-p)$, $p \leq m \leq n$, defined on Ω. Let also u, $v \in SH_m(\Omega) \cap L^\infty(\Omega)$ such that*

$$\limsup_{\substack{\xi \to \partial\Omega \\ \xi \in \operatorname{Supp} T}} |u(\xi) - v(\xi)| = 0.$$

Then, for all $\delta > 0$ and $0 < k < 1$, one has

$$\text{cap}_{m,T}(\{|u - v| \geq \delta\}) \leq \frac{[(m-p)!]^2}{(1-k)^{m-p} \delta^{m-p}} \times \left\| T \wedge \beta^{n-m} \wedge (dd^c u)^{m-p} - T \wedge \beta^{n-m} \wedge (dd^c v)^{m-p} \right\|_{\{|u-v| > k\delta\}}.$$

In particular, if $T \wedge \beta^{n-m} \wedge (dd^c u)^{m-p} = T \wedge \beta^{n-m} \wedge (dd^c v)^{m-p}$, then $u = v$ almost everywhere with respect to $\text{cap}_{m,T}$.

Proof. Let $w \in SH_m(\Omega, [0,1])$, $\delta > 0$ and $k \in]0,1[$. Using Lemma 3 in [7] and the fact that
$$\{| u - v |\geq \delta\} \subset \{|u - v + \delta k| \geq (1-k)\delta\},$$
we can obtain

$$\begin{aligned}
&\int_{\{|u-v|\geq\delta\}} T\wedge\beta^{n-m}\wedge(dd^c w)^{m-p}\\
&\quad\leq \frac{1}{(1-k)^{m-p}\delta^{m-p}}\int_{\{u+\delta\leq v\}}(v-u-k\delta)^{m-p}T\wedge\beta^{n-m}\wedge(dd^c w)^{m-p}\\
&\qquad+\frac{1}{(1-k)^{m-p}\delta^{m-p}}\int_{\{v+\delta\leq u\}}(u-v-k\delta)^{m-p}T\wedge\beta^{n-m}\wedge(dd^c w)^{m-p}\\
&\quad\leq \frac{1}{(1-k)^{m-p}\delta^{m-p}}\int_{\{u+k\delta< v\}}(v-u-k\delta)^{m-p}T\wedge\beta^{n-m}\wedge(dd^c w)^{m-p}\\
&\qquad+\frac{1}{(1-k)^{m-p}\delta^{m-p}}\int_{\{v+k\delta< u\}}(u-v-k\delta)^{m-p}T\wedge\beta^{n-m}\wedge(dd^c w)^{m-p}\\
&\quad\leq \frac{[(m-p)!]^2}{(1-k)^{m-p}\delta^{m-p}}\\
&\qquad\times\int_{\{|u-v|>k\delta\}}(1-w)(\chi_{\{u+k\delta<v\}}-\chi_{\{v+k\delta<u\}})T\wedge\beta^{n-m}\wedge(dd^c u)^{m-p}\\
&\qquad-\frac{[(m-p)!]^2}{(1-k)^{m-p}\delta^{m-p}}\\
&\qquad\times\int_{\{|u-v|>k\delta\}}(1-w)(\chi_{\{u+k\delta<v\}}-\chi_{\{v+k\delta<u\}})T\wedge\beta^{n-m}\wedge(dd^c v)^{m-p}\\
&\quad\leq \frac{[(m-p)!]^2}{(1-k)^{m-p}\delta^{m-p}}\\
&\qquad\times\left\|T\wedge\beta^{n-m}\wedge(dd^c u)^{m-p}-T\wedge\beta^{n-m}\wedge(dd^c v)^{m-p}\right\|_{\{|u-v|>k\delta\}}.
\end{aligned}$$

The result follows. □

Corollary 4.2. *Let Ω be a bounded open subset of $\mathbb{C}^n$ and let T be an m-positive closed current of bidimension $(n-p, n-p)$ $(p \leq m \leq n)$ defined on Ω and u, $u_j \in SH_m(\Omega)\cap L^\infty(\Omega)$. Assume that:*

i) $\limsup_{\substack{\xi\to\partial\Omega\\ \xi\in\operatorname{Supp}T}} |u_j(\xi) - u(\xi)| = 0$ *uniformly on* j.

ii) *For all $E \Subset \Omega$, one has*
$$\left\|T\wedge\beta^{n-m}\wedge(dd^c u_j)^{m-p} - T\wedge\beta^{n-m}\wedge(dd^c u)^{m-p}\right\|_E \to 0.$$

Then u_j converges to u with respect to capacity $\operatorname{cap}_{m,T}$ on Ω.

Throughout this section we denote by μ a positive measure on a bounded open set Ω, by λ, the Lebesgue measure and by T, an m-positive closed current of bidimension $(n-p, n-p)$ $(p \leq m \leq n)$. We will solve the following Hessian equation on the set of m-subharmonic functions

$$T \wedge \beta^{n-m} \wedge (dd^c.)^{m-p} = \mu.$$

Proof of Theorem 1.1. Let $A > 0$ such that for all $z \in \overline{\Omega}$, one has $A > |z|$. Take $c > 0$ such that $c \geq |v(z)| + |u_1(w)| + 1$ for all $z \in \Omega$ and $w \in \partial\Omega$. Using Lemma 3 from [7] and the hypothesis 1), we get

$$\begin{aligned}
&\int_{\{u_j<v-c\}} \left(1 - \frac{|z|^2}{A^2}\right) T \wedge \beta^{n-m} \wedge (dd^c u_j)^{m-p} \\
&\qquad \geq \int_{\{u_j<v-c\}} \left(1 - \frac{|z|^2}{A^2}\right) T \wedge \beta^{n-m} \wedge (dd^c v)^{m-p} \\
&\qquad\quad + \frac{1}{[(m-p)!]^2 A^{2(m-p)}} \int_{\{u_j<v-c\}} (v-c-u_j)^{m-p} T \wedge \beta^{n-p} \\
&\qquad \geq \int_{\{u_j<v-c\}} \left(1 - \frac{|z|^2}{A^2}\right) d\mu \\
&\qquad\quad + \frac{1}{([(m-p)!]^2 A^{2(m-p)}} \int_{\{u_j<v-c\}} (v-c-u_j)^{m-p} T \wedge \beta^{n-p}.
\end{aligned}$$

As $\|\mu_j - \mu\|_\Omega \to 0$, then

$$\begin{aligned}
0 &\geq \liminf_{j\to+\infty} \int_{\{u_j<v-c\}} (v-c-u_j)^{m-p} T \wedge \beta^{n-p} \\
&\geq \int_\Omega \liminf_{j\to+\infty} \left(\chi_{\{u_j<v-c\}} (v-c-u_j)^{m-p}\right) T \wedge \beta^{n-p} \\
&\geq \int_\Omega \chi_{\{\limsup_{j\to+\infty} u_j<v-c\}} \left(\liminf_{j\to+\infty} |v-c-u_j|\right)^{m-p} T \wedge \beta^{n-p} \\
&\geq \int_\Omega \chi_{\{\limsup_{j\to+\infty} u_j<v-c\}} \left(v-c-\limsup_{j\to+\infty} u_j\right)^{m-p} T \wedge \beta^{n-p}.
\end{aligned}$$

It follows that $\limsup_{j\to+\infty} u_j \geq v-c$ for $\|T \wedge \beta^{n-m}\|$-almost everywhere. Thus,

$$\limsup_{j\to+\infty} u_j \neq -\infty.$$

If we take

$$A := \bigcup_j \left(\sup\{u_j, u_{j+1}, \ldots\} < \sup^\star\{u_j, u_{j+1}, ...\}\right),$$

after using [2], we can see that there exists $g \in SH_m(\Omega)$ such that

$$\sup\{u_j, u_{j+1}, \ldots\} = \sup^\star\{u_j, u_{j+1}, \ldots\} \downarrow \limsup_{j\to+\infty} u_j = g \quad \text{on } \Omega \backslash A.$$

As $\mathrm{cap}_{m,T}(A) = 0$ and the $\|T \wedge \beta^{n-m}\|-$negligible set are λ-negilgeable, we get that $g \geq v-c$ almost everywhere. It follows that g is bounded on Ω. Using

Theorem 3.5, it suffices to prove that u_j converges to g with respect to capacity $\mathrm{cap}_{m,T}$. Letting $E \subset\subset \Omega$ and $\delta > 0$, one has

$$\mathrm{cap}_{m,T}\left(E \cap \{|g - u_j| \geq \delta\}\right) \geq \mathrm{cap}_{m,T}\left(E \cap \left\{|g - \sup\{u_j, u_{j+1}, \ldots\}| \geq \frac{\delta}{2}\right\}\right) + \mathrm{cap}_{m,T}\left(\left\{|\sup\{u_j, u_{j+1}, \ldots\} - u_j| \geq \frac{\delta}{2}\right\}\right).$$

By applying Theorem 3.4 and Dini's theorem on g, it is easy to check that the sequence $\sup\{u_j, u_{j+1}, ...\} \downarrow g$ uniformly on E outside a set of small capacity $\mathrm{cap}_{m,T}$. It follows that

$$\mathrm{cap}_{m,T}\left(E \cap \left\{|g - \sup\{u_j, u_{j+1}, \ldots\}| \geq \frac{\delta}{2}\right\}\right)$$

tends to 0 when j goes to $+\infty$.

Now, let us prove that

$$B := \left\{|\sup\{u_j, u_{j+1}, \ldots\} - u_j| \geq \frac{\delta}{2}\right\} \subset \bigcup_{l=0}^{+\infty}\left\{|u_{j+l+1} - u_{j+l}| \geq \frac{\delta}{2^{l+j+2}}\right\}.$$

We can assume that $[(m-p)!]^2\|\mu_j - \mu\| \leq \frac{1}{2^{(m-p+1)j}}$. So, by Proposition 4.1, one has for all $\delta > 0$,

$$\begin{aligned}\mathrm{cap}_{m,T}\{|u_{j+1} - u_j| > \delta\} &\leq \frac{(m-p)!^2}{\delta^{m-p}}\|\mu_{j+1} - \mu_j\| \\ &\leq \frac{(m-p)!^2}{\delta^{m-p}}(\|\mu_{j+1} - \mu\|_\Omega + \|\mu - \mu_j\|_\Omega) \leq \frac{2}{\delta^{m-p}2^{(m-p+1)j}}\end{aligned}$$

and we deduce that

$$\begin{aligned}&\mathrm{cap}_{m,T}\left(\left\{|\sup\{u_j, u_{j+1}, \ldots\} - u_j| \geq \frac{\delta}{2}\right\}\right) \\ &\qquad \leq \sum_{l=0}^{+\infty}\mathrm{cap}_{m,T}\{|u_{j+l+1} - u_{j+l}| \geq \frac{\delta}{2^{l+j+2}}\} \leq \frac{4^{m-p}}{\delta^{m-p}2^j}.\end{aligned}$$

Hence the sequence u_j converges to g with respect to capacity $\mathrm{cap}_{m,T}$ and, by Theorem 3.5, we get that the sequence $(dd^c u_j)^{m-p} \wedge T \wedge \beta^{n-m}$ converges weakly to $(dd^c g)^{m-p} \wedge T \wedge \beta^{n-m}$. □

Remark 4.3. Without the first hypothesis, we cannot have the existence of the solution to the equation $T \wedge \beta^{n-m} \wedge (dd^c.)^{m-p} = \mu$ even in the trivial case $T = 1$ and $m = n$. In fact, using [5], there exists $f \in L^1(\Omega)$ such that the equation $(dd^c u)^n = f d\lambda$ has no solution in $PSH(\Omega) \cap L^\infty(\Omega)$.

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Слабкий розв'язок комплексного рівняння гессіана, пов'язаного з m-позитивним замкнутим потоком

Jawhar Hbil and Mohamed Zaway

Метою даної статті є вивчення існування розв'язку комплексного рівняння гессіана, пов'язаного з m-позитивним замкнутим потоком T. Даємо достатню умову на T і міру μ, так що рівняння $T \wedge \beta^{n-m} \wedge (dd^c.)^{m-p} = \mu$ має розв'язок на множині m-субгармонічних функцій. Для цього встановлюємо зв'язок між збіжністю відносно $cap_{m,T}$ послідовності m-субгармонічних функцій та слабкою збіжністю асоційованої гессіанової міри.

Ключові слова: m-позитивний замкнутий потік, m-субгармонічна функція, ємність, оператор гессіана