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Darboux Transformation for the Hirota Equation

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The Hirota equation is an integrable higher order nonlinear Schrödinger type equation which describes the propagation of ultrashort light pulses in optical fibers. We present a standard Darboux transformation for the Hirota equation and then construct its quasideterminant solutions. As examples, the multi-soliton, breather and rogue wave solutions of the Hirota equation are given explicitly.

Key words: Hirota equation, Darboux transformation, quasideterminant solutions, multisoliton solutions, breather solutions, rogue wave solutions

Mathematical Subject Classification 2010: 35C08, 35Q55, 37K10

1. Introduction

There exists a large class of nonlinear evolution equations which can be solved analytically. Such equations are called integrable. Integrable equations constitute an important part of the nonlinear wave theory. The simplest integrable equation which describes the dynamics of deep-water gravity waves is the nonlinear Schrödinger (NLS) equation

$$iq_t + q_{xx} + 2|q|^2 q = 0. \tag{1.1}$$

In 1967, it was first discussed in the general context of nonlinear dispersive waves by Benney and Newell [4]. In 1968, this equation was also derived by Zakharov in his study of modulational stability of deep water waves [38]. In 1972, Zakharov and Shabat found that the NLS equation had a Lax pair and could be solved by the inverse scattering transform (IST) method [40]. This equation plays an important role in different physical systems as wide as plasma physics [39], water waves [4, 5, 38], and nonlinear optics [14, 15]. One of the most interesting applications of the NLS equation is that it can be employed to model short soliton pulses in optical fibres [18]. However, as the pulses get shorter, various additional effects become important and the NLS model is no longer appropriate. In order to understand these additional effects, Kodama and Hasegawa [19, 20] suggested a higher-order NLS equation

$$iq_t + \alpha_1 q_{xx} + \alpha_2 |q|^2 q + i\beta \left[\gamma_1 q_{xxx} + \gamma_2 |q|^2 q_x + \gamma_3 q \left(|q|^2\right)_x\right] = 0, \tag{1.2}$$

where the α_i, γ_i are real constants, β is a real spectral parameter and q is a complex-valued function of x and t. By choosing $\beta = 0$ and $\alpha_2 = 2\alpha_1 = 2$ in this equation, we can easily see that the first three terms form the standard NLS equation (1.1). Generally, the Kodama–Hasegawa higher-order NLS equation (1.2) may not be completely integrable if some restrictions are not imposed on the real constants γ_i $(i = 1, 2, 3)$. Until now it is known that, besides the NLS equation (1.1) itself, there are four cases in which integrability can be proved via the IST. These are the Chen–Lee–Liu [6] derivative NLS equation ($\gamma_1 : \gamma_2 : \gamma_3 = 0 : 1 : 0$), the Kaup–Newell [17] derivative NLS equation ($\gamma_1 : \gamma_2 : \gamma_3 = 0 : 1 : 1$), the Hirota [16] NLS equation ($\gamma_1 : \gamma_2 : \gamma_3 = 1 : 6 : 0$), and the Sasa–Satsuma [33] NLS equation ($\gamma_1 : \gamma_2 : \gamma_3 = 1 : 6 : 3$).

In this paper, we consider the Hirota [16] NLS equation

$$iq_t + \alpha\left(q_{xx} + 2|q|^2 q\right) + i\beta\left(q_{xxx} + 6|q|^2 q_x\right) = 0, \quad \alpha, \beta \in \mathbb{R}, \tag{1.3}$$

in which $\alpha_2 = 2\alpha_1 = 2\alpha$. This equation is commonly known as the Hirota equation (HE), and we will denote it as such from now on. The HE (1.3) can be used to describe the wave propagation of ultrashort light pulses in optical fibers [1, 19, 20, 23, 26, 36]. It is very interesting to see that the Hirota equation (1.3) is the sum of the NLS (1.1) equation ($\alpha = 1, \beta = 0$) and the complex version of the modified Korteweg-de Vries (mKdV) equation ($\alpha = 0, \beta = 1$),

$$q_t + q_{xxx} + 6|q|^2 q_x = 0 \tag{1.4}$$

which is completely integrable [16, 34]. In the resent years, there has been some interest in solutions of the HE (1.3) obtained by *Darboux-type* transformations [3, 22, 32]. These solutions are often written in terms of determinants.

In 1882, the French mathematician Jean Gaston Darboux [7] introduced a method for solving the Sturm–Liouville equation called Darboux transformation (DT) afterwards. Almost a century later, in 1979, Matveev [24] realized that the method given by Darboux for the spectral problem of second order ordinary differential equations can be extended to some important soliton equations. Darboux transformations are one of important tools in studying integrable systems. They provide a universal algorithmic procedure to derive exact solutions of integrable systems.

In the present paper, for the first time we construct a standard Darboux transformation for the Hirota equation (1.3). We underline that the method we use here is based on Darboux's [7] and Matveev's original ideas [24, 25]. Therefore, our approach should be considered on its own merits. Furthermore, our solutions for the HE are written in terms of quasideterminants [8, 9] rather than determinants. It has been proved that quasideterminants are very useful for constructing exact solutions of integrable equations [10, 11, 21, 29, 31, 35, 41], enabling these solutions to be expressed in a simple and compact form. Finally, we prove that the DT we present here can be used to obtain various solutions of the HE (1.3) such as multi-solitons, breathers and *rogue waves*.

In recent years, rogue waves (RW) have been widely studied. It is not easy to give a full definition of the RW due to its complex phenomenon. In the ocean environment, the RW is defined as a surface wave that is abnormally large. The amplitude of this wave is two or three times higher than those of its surrounding waves. On mathematical aspect, rogue waves can be expressed as the rational function solutions of nonlinear evolution equations. A typical example of rogue wave is the Peregrine soliton [30]. In 1983, this solution was first presented by the British mathematician Howell Peregrine as the fundamental rational solution of the NLS equation (1.1) in the following form:

$$q = e^{2it}\left[1 - \frac{4(1+4it)}{1+4x^2+16t^2}\right]. \tag{1.5}$$

For the NLS equation, the second-order rogue wave solution was constructed by Akhmediev and his co-workers [2]. In general, the DT cannot be directly used to construct rational solutions for evolution equations. In [13], Gue *et al.* proposed a simple method (the generalized DT) for constructing higher-order rogue wave solutions of the NLS equation (1.1). In this paper, we construct the first-order RW solutions of the Hirota equation (1.3) from the Taylor expansion of the breather solutions as a particular example for non-zero seed solutions. The higher-order RW solutions of the Hirota equation are constructed basing on the gDT [3, 13].

This paper is structured as follows. In Section 1.1 below, we give a brief review of quasideterminants. In Section 2, we establish a 2×2 eigenfunction and a corresponding constant 2×2 square matrix for the eigenvalue problems of the Hirota equation (1.3) using two symmetries of the Lax pair of the HE. In Section 3, we state a standard Darboux theorem for the Hirota system. We review the reduced DTs for the HE, which can be considered as a dimensional reduction from $(2+1)$ to $(1+1)$ dimensions. In Section 4, we present the quasideterminant solutions for the HE constructed by the DT. In Section 5, the multi-soliton and breather solutions of the Hirota equation are given for both zero and non-zero seed solutions as particular solutions of the HE. Section 6 is devoted to the construction of the first-order rogue wave solutions of the HE (1.3). The conclusion is given in the final Section 7.

1.1. Quasideterminants. In this short section we will list some of the key elementary properties of quasideterminants used in the paper. The reader is referred to the original papers [8, 9] for more detailed and general treatments.

Let $M = (m_{ij})$ be an $n\times n$ matrix with entries over a ring (noncommutative, in general) having n^2 quasideterminants written as $|M|_{ij}$ for $i, j = 1, \ldots, n$. They are defined recursively by

$$|M|_{ij} = m_{ij} - r_i^j \left(M^{ij}\right)^{-1} c_j^i, \tag{1.6}$$

where r_i^j represents the row vector obtained from the i^{th} row of M with the j^{th} element removed, c_j^i is the column vector obtained from the j^{th} column of M with the i^{th} element removed and M^{ij} is the $(n-1)\times(n-1)$ submatrix obtained by deleting the i^{th} row and the j^{th} column from M. Quasideterminants can be also denoted as shown below by boxing the entry about which the expansion is made,

$$|M|_{ij} = \begin{vmatrix} M^{ij} & c_j^i \\ r_i^j & \boxed{m_{ij}} \end{vmatrix}.$$

If the entries in M commute, then the quasideterminant $|M|_{ij}$ can be expressed as a ratio of determinants

$$|M|_{ij} = (-1)^{i+j} \frac{\det M}{\det M^{ij}}. \tag{1.7}$$

2. Hirota equation

Let us consider the coupled Hirota equations:

$$q_t - i\alpha\left(q_{xx} + 2q^2 r\right) + \beta\left(q_{xxx} + 6qrq_x\right) = 0, \tag{2.1}$$
$$r_t + i\alpha\left(r_{xx} + 2qr^2\right) + \beta\left(r_{xxx} + 6qrr_x\right) = 0, \tag{2.2}$$

where $q = q(x,t)$ and $r = r(x,t)$ are complex valued functions. Equations (2.1) and (2.2) reduce to the Hirota equation (1.3) when $r = q^*$. Here the asterisk superscript on q denotes the complex conjugate.

The Lax pair [32] for the coupled Hirota equations (2.1)–(2.2) is given by

$$L = \partial_x + J\lambda - R, \tag{2.3}$$
$$M = \partial_t + 4\beta J\lambda^3 + 2U\lambda^2 - 2V\lambda - W, \tag{2.4}$$

where J, R, U, V and W are 2×2 matrices

$$J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad R = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \quad U = \begin{pmatrix} i\alpha & -2\beta q \\ 2\beta r & -i\alpha \end{pmatrix}, \tag{2.5}$$
$$V = \begin{pmatrix} i\beta qr & \alpha q + i\beta q_x \\ -\alpha r + i\beta r_x & -i\beta qr \end{pmatrix},$$
$$W = \begin{pmatrix} i\alpha qr + \beta\left(qr_x - rq_x\right) & i\alpha q_x - \beta\left(q_{xx} + 2q^2 r\right) \\ i\alpha r_x + \beta\left(r_{xx} + 2qr^2\right) & -i\alpha qr - \beta\left(qr_x - rq_x\right) \end{pmatrix}. \tag{2.6}$$

Here λ is a spectral parameter. It can be seen that the potential matrix R in (2.5) has two symmetry properties. One of them is that it is skew-Hermitian: $R^\dagger = -R$. The other one is that $SRS^{-1} = R^*$, where

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let $\phi = (\varphi, \psi)^T$ be a vector eigenfunction for (2.3)–(2.4) for the eigenvalue λ so that $L_\lambda(\phi) = M_\lambda(\phi) = 0$. Using the second symmetry, it can be seen that $\tilde{\phi} = S\phi = (\psi^*, -\varphi^*)^T$ is another eigenfunction for eigenvalue λ^* such that $L_{\lambda^*}(\tilde{\phi}) =$

$M_{\lambda^*}(\tilde{\phi}) = 0$. Using these vector eigenfunctions we can define a square 2×2 matrix eigenfunction θ with 2×2 eigenvalue Λ

$$\theta = \begin{pmatrix} \varphi & \psi^* \\ \psi & -\varphi^* \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^* \end{pmatrix}, \tag{2.7}$$

satisfying

$$\theta_x + J\theta\Lambda - R\theta = 0, \tag{2.8}$$

$$\theta_t + 4\beta J\theta\Lambda^3 + 2U\theta\Lambda^2 - 2V\theta\Lambda - W\theta = 0. \tag{2.9}$$

3. Darboux transformations and dimensional reductions

3.1. Darboux transformation. Let us consider the linear operators

$$L = \partial_x + \sum_{i=0}^{n} u_i \partial_y^i, \quad M = \partial_t + \sum_{i=0}^{n} v_i \partial_y^i, \tag{3.1}$$

where u_i, v_i are $m \times m$ matrices. The standard approach to Darboux transformations [7, 24, 25] involves a gauge operator $G_\theta = \theta \partial_y \theta^{-1}$, where $\theta = \theta(x, y, t)$ is an invertible $m \times m$ matrix solution to a linear system

$$L(\phi) = M(\phi) = 0.$$

If ϕ is any eigenfunction of L and M, then $\tilde{\phi} = G_\theta(\phi)$ satisfies the transformed system

$$\tilde{L}(\tilde{\phi}) = \tilde{M}(\tilde{\phi}) = 0,$$

where the linear operators $\tilde{L} = G_\theta L G_\theta^{-1}$ and $\tilde{M} = G_\theta M G_\theta^{-1}$ have the same forms as L and M:

$$\tilde{L} = \partial_x + \sum_{i=0}^{n} \tilde{u}_i \partial_y^i, \quad \tilde{M} = \partial_t + \sum_{i=0}^{n} \tilde{v}_i \partial_y^i. \tag{3.2}$$

3.2. Dimensional reduction of Darboux transformation Here we describe a reduction of the Darboux transformation from $(2+1)$ to $(1+1)$ dimensions. We choose to eliminate the y-dependence by employing a 'separation of variables' technique. The reader is referred to the paper [27] for a more detailed treatment. We make the ansatz

$$\phi = \phi^r(x, t) e^{\lambda y}, \quad \theta = \theta^r(x, t) e^{\Lambda y},$$

where λ is a constant scalar and Λ is an $N \times N$ constant matrix and the superscript r denotes reduced functions, independent of y. Hence, in the dimensional reduction we obtain $\partial_y^i(\phi) = \lambda^i \phi$ and $\partial_y^i(\theta) = \theta \Lambda^i$, and thus the operator L and the Darboux transformation G become

$$L^r = \partial_x + \sum_{i=0}^{n} u_i \lambda^i, \quad G^r = \lambda - \theta^r \Lambda (\theta^r)^{-1}, \tag{3.3}$$

where θ^r is a matrix eigenfunction of L^r such that $L^r(\theta^r) = 0$, with λ replaced by the matrix Λ, that is,

$$\theta^r_x + \sum_{i=0}^{n} u_i \theta^r \Lambda^i = 0.$$

Below we omit the superscript r for ease of notation.

3.3. Iteration of reduced Darboux transformations. In this section, we consider the iteration of the Darboux transformations and find closed form expressions for them in terms of quasideterminants.

Let L be an operator, form invariant under the reduced Darboux transformation $G_\theta = \lambda - \theta\Lambda\theta^{-1}$ discussed above.

Let $\phi = \phi(x,t)$ be a general eigenfunction of L such that $L(\phi) = 0$. Then

$$\tilde{\phi} = G_\theta(\phi) = \lambda\phi - \theta\Lambda\theta^{-1}\phi = \begin{vmatrix} \theta & \phi \\ \theta\Lambda & \boxed{\lambda\phi} \end{vmatrix}$$

is an eigenfunction of $\tilde{L} = G_\theta L G_\theta^{-1}$ so that $\tilde{L}(\tilde{\phi}) = \lambda\tilde{\phi}$. Let θ_i for $i = 1, \ldots, n$, be a particular set of invertible eigenfunctions of L so that $L(\theta_i) = 0$ for $\lambda = \Lambda_i$, and introduce the notation $\Theta = (\theta_1, \ldots, \theta_n)$. To apply the Darboux transformation the second time, let $\theta_{[1]} = \theta_1$ and $\phi_{[1]} = \phi$ be a general eigenfunction of $L_{[1]} = L$. Then $\phi_{[2]} = G_{\theta_{[1]}}\left(\phi_{[1]}\right)$ and $\theta_{[2]} = \phi_{[2]}|_{\phi\to\theta_2}$ are eigenfunctions for $L_{[2]} = G_{\theta_{[1]}} L_{[1]} G^{-1}_{\theta_{[1]}}$.

In general, for $n \geq 1$, we define the nth Darboux transform of ϕ by

$$\phi_{[n+1]} = \lambda\phi_{[n]} - \theta_{[n]}\Lambda_n\theta^{-1}_{[n]}\phi_{[n]}, \tag{3.4}$$

in which $\theta_{[k]} = \phi_{[k]}|_{\phi\to\theta_k}$. For example,

$$\phi_{[2]} = \lambda\phi - \theta_1\Lambda_1\theta_1^{-1}\phi = \begin{vmatrix} \theta_1 & \phi \\ \theta_1\Lambda_1 & \boxed{\lambda\phi} \end{vmatrix},$$

$$\phi_{[3]} = \lambda\phi_{[2]} - \theta_{[2]}\Lambda_2\theta^{-1}_{[2]}\phi_{[2]} = \begin{vmatrix} \theta_1 & \theta_2 & \phi \\ \theta_1\Lambda_1 & \theta_2\Lambda_2 & \lambda\phi \\ \theta_1\Lambda_1^2 & \theta_2\Lambda_2^2 & \boxed{\lambda^2\phi} \end{vmatrix}.$$

After n iterations, we get

$$\phi_{[n+1]} = \begin{vmatrix} \theta_1 & \theta_2 \ \ldots \ \theta_n & \phi \\ \theta_1\Lambda_1 & \theta_2\Lambda_2 \ldots \theta_n\Lambda_n & \lambda\phi \\ \theta_1\Lambda_1^2 & \theta_2\Lambda_2^2 \ldots \theta_n\Lambda_n^2 & \lambda^2\phi \\ \vdots & \vdots \ \ldots \ \vdots & \vdots \\ \theta_1\Lambda_1^n & \theta_2\Lambda_2^n \ldots \theta_n\Lambda_n^n & \boxed{\lambda^n\phi} \end{vmatrix}.$$

4. Constructing solutions for Hirota equation

In this section, we determine the specific effect of the Darboux transformation $G_\theta = \lambda - \theta\Lambda\theta^{-1}$ on the operator L given by (2.3). Corresponding results hold for the operator M given by (2.4). Here the eigenfunction θ, the solution of the linear system (2.8)–(2.9), is given explicitly with the eigenvalue Λ in (2.7). From $\tilde{L}G_\theta = G_\theta L$, the operator $L = \partial_x + J\lambda - R$ is transformed to a new operator $\tilde{L}$ in which J is unchanged and

$$\tilde{R} = R - \left[J, \theta\Lambda\theta^{-1}\right]. \tag{4.1}$$

For notational convenience, we introduce a 2×2 matrix Q such that $R = [J, Q]$, and hence

$$Q = \frac{1}{2i}\begin{pmatrix} & q \\ r & \end{pmatrix},$$

where the entries left blank are arbitrary and do not contribute to R. From (4.1), it follows that

$$\tilde{Q} = Q - \theta\Lambda\theta^{-1} \tag{4.2}$$

which can be written in a quasideterminant structure as

$$\tilde{Q} = Q + \begin{vmatrix} \theta & I_2 \\ \theta\Lambda & \boxed{0_2} \end{vmatrix}.$$

We rewrite (4.2) as

$$Q_{[2]} = Q_{[1]} - \theta_{[1]}\Lambda_1\theta_{[1]}^{-1},$$

where $Q_{[1]} = Q$, $Q_{[2]} = \tilde{Q}$, $\theta_{[1]} = \theta_1 = \theta$ and $\Lambda_1 = \Lambda$. Then, after repeating Darboux transformations n times, we have

$$Q_{[n+1]} = Q_{[n]} - \theta_{[n]}\Lambda_n\theta_{[n]}^{-1}, \tag{4.3}$$

in which $\theta_{[k]} = \phi_{[k]}\,|_{\phi\to\theta_k}$. We express $P_{[n+1]}$ in the quasideterminant form as

$$Q_{[n+1]} = Q + \begin{vmatrix} \theta_1 & \theta_2 \;\dots\; \theta_n & 0_2 \\ \theta_1\Lambda_1 & \theta_2\Lambda_2 \dots \theta_n\Lambda_n & 0_2 \\ \vdots & \vdots \;\dots\; \vdots & \vdots \\ \theta_1\Lambda_1^{n-2} & \theta_2\Lambda_2^{n-2} \dots \theta_n\Lambda_n^{n-2} & 0_2 \\ \theta_1\Lambda_1^{n-1} & \theta_2\Lambda_2^{n-1} \dots \theta_n\Lambda_n^{n-1} & I_2 \\ \theta_1\Lambda_1^{n} & \theta_2\Lambda_2^{n} \dots \theta_n\Lambda_n^{n} & \boxed{0_2} \end{vmatrix}, \tag{4.4}$$

where each θ_i, Λ_i as a 2×2 matrix

$$\theta_i = \begin{pmatrix} \varphi_i & \psi_i^* \\ \psi_i & -\varphi_i^* \end{pmatrix}, \quad \Lambda_i = \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i^* \end{pmatrix} \tag{4.5}$$

in which $i = 1, ..., n$. Now let $\Theta^{(n)}$ be a $2 \times 2n$ matrix such that

$$\Theta^{(n)} = (\theta_1 \Lambda_1^n, \ldots, \theta_n \Lambda_n^n) = \begin{pmatrix} \varphi^{(n)} \\ \psi^{(n)} \end{pmatrix},$$

where

$$\varphi^{(n)} = (\lambda_1^n \varphi_1, \lambda_1^{*n} \psi_1^*, \ldots, \lambda_n^n \varphi_n, \lambda_n^{*n} \psi_n^*),$$
$$\psi^{(n)} = (\lambda_1^n \psi_1, -\lambda_1^{*n} \varphi_1^*, \ldots, \lambda_n^n \psi_n, -\lambda_n^{*n} \varphi_n^*)$$

denote $1 \times 2n$ row vectors. Thus, (4.4) can be written as

$$Q_{[n+1]} = Q + \begin{vmatrix} \widehat{\Theta} & E \\ \Theta^{(n)} & \boxed{0_2} \end{vmatrix},$$

where $\widehat{\Theta} = \left(\theta_i \Lambda_i^{j-1}\right)_{i,j=1,\ldots,n}$ and $E = (e_{2n-1}, e_{2n})$ denote the $2n \times 2n$ and the $2n \times 2$ matrices respectively, in which e_i represents a column vector with 1 in the i^{th} row and zeros elsewhere. Hence, we obtain

$$Q_{[n+1]} = Q + \begin{pmatrix} \begin{vmatrix} \widehat{\Theta} & e_{2n-1} \\ \varphi^{(n)} & \boxed{0} \end{vmatrix} & \begin{vmatrix} \widehat{\Theta} & e_{2n} \\ \varphi^{(n)} & \boxed{0} \end{vmatrix} \\ \begin{vmatrix} \widehat{\Theta} & e_{2n-1} \\ \psi^{(n)} & \boxed{0} \end{vmatrix} & \begin{vmatrix} \widehat{\Theta} & e_{2n} \\ \psi^{(n)} & \boxed{0} \end{vmatrix} \end{pmatrix}.$$

Here we immediately see that a quasideterminant solution $q_{[n+1]}$ of the Hirota equation (1.3) along with its complex conjugate $r_{[n+1]}$ can be expressed as

$$q_{[n+1]} = q + 2i \begin{vmatrix} \widehat{\Theta} & e_{2n} \\ \varphi^{(n)} & \boxed{0} \end{vmatrix}, \quad r_{[n+1]} = r + 2i \begin{vmatrix} \widehat{\Theta} & e_{2n-1} \\ \psi^{(n)} & \boxed{0} \end{vmatrix}, \tag{4.6}$$

where it can be easily shown that the reduction $r_{[n+1]} = q^*_{[n+1]}$ holds.

4.1. Explicit solutions. In order to construct explicit solutions for the Hirota equation (1.3), we consider the quasideterminant solution given by (4.6) in which we obtain

$$q_{[n+1]} = q + 2i \begin{vmatrix} \varphi_1 & \psi_1^* & \cdots & \varphi_n & \psi_n^* & 0 \\ \psi_1 & -\varphi_1^* & \cdots & \psi_n & -\varphi_n^* & 0 \\ \varphi_1\lambda_1 & \psi_1^*\lambda_1^* & \cdots & \varphi_n\lambda_n & \psi_n^*\lambda_n^* & 0 \\ \psi_1\lambda_1 & -\varphi_1^*\lambda_1^* & \cdots & \psi_n\lambda_n & -\varphi_n^*\lambda_n^* & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ \varphi_1\lambda_1^{n-1} & \psi_1^*\lambda_1^{*n-1} & \cdots & \varphi_n\lambda_n^{n-1} & \psi_n^*\lambda_n^{*n-1} & 0 \\ \psi_1\lambda_1^{n-1} & -\varphi_1^*\lambda_1^{*n-1} & \cdots & \psi_n\lambda_n^{n-1} & -\varphi_n^*\lambda_n^{*n-1} & 1 \\ \varphi_1\lambda_1^n & \psi_1^*\lambda_1^{*n} & \cdots & \varphi_n\lambda_n^n & \psi_n^*\lambda_n^{*n} & \boxed{0} \end{vmatrix}. \tag{4.7}$$

Here φ_j and ψ_j are scalar functions such that the eigenfunction $\phi_j = (\varphi_j, \psi_j)^T$ denotes n distinct solutions of the spectral problem $L(\phi_j) = M(\phi_j) = 0$ with the associated eigenvalue λ_j, where the operators L, M are given by (2.3)–(2.4) so that

$$\begin{aligned} \phi_{j,x} + J\phi_j\lambda_j - R\phi_j = 0, \\ \phi_{j,t} + 4\beta J\phi_j\lambda_j^3 + 2U\phi_j\lambda_j^2 - 2V\phi_j\lambda_j - W\phi_j = 0, \end{aligned} \tag{4.8}$$

in which $j = 1, \ldots, n$ and J, R, U, V, W are the 2×2 matrices given by (2.5)–(2.6). In the next section, we will present some explicit solutions of the equation (1.3) for the cases $n = 1, \ldots, 3$. For the one-fold ($n = 1$), two-fold ($n = 2$) and three-fold ($n = 3$) Darboux transformations, the solution (4.7) yields

$$q_{[2]} = q + 2i \begin{vmatrix} \varphi_1 & \psi_1^* & 0 \\ \psi_1 & -\varphi_1^* & 1 \\ \varphi_1\lambda_1 & \psi_1^*\lambda_1^* & \boxed{0} \end{vmatrix}, \tag{4.9}$$

$$q_{[3]} = q + 2i \begin{vmatrix} \varphi_1 & \psi_1^* & \varphi_2 & \psi_2^* & 0 \\ \psi_1 & -\varphi_1^* & \psi_2 & -\varphi_2^* & 0 \\ \varphi_1\lambda_1 & \psi_1^*\lambda_1^* & \varphi_2\lambda_2 & \psi_2^*\lambda_2^* & 0 \\ \psi_1\lambda_1 & -\varphi_1^*\lambda_1^* & \psi_2\lambda_2 & -\varphi_2^*\lambda_2^* & 1 \\ \varphi_1\lambda_1^2 & \psi_1^*\lambda_1^{*2} & \varphi_2\lambda_2^2 & \psi_2^*\lambda_2^{*2} & \boxed{0} \end{vmatrix} \tag{4.10}$$

and

$$q_{[4]} = q + 2i \begin{vmatrix} \varphi_1 & \psi_1^* & \varphi_2 & \psi_2^* & \varphi_3 & \psi_3^* & 0 \\ \psi_1 & -\varphi_1^* & \psi_2 & -\varphi_2^* & \psi_3 & -\varphi_3^* & 0 \\ \varphi_1\lambda_1 & \psi_1^*\lambda_1^* & \varphi_2\lambda_2 & \psi_2^*\lambda_2^* & \varphi_3\lambda_3 & \psi_3^*\lambda_3^* & 0 \\ \psi_1\lambda_1 & -\varphi_1^*\lambda_1^* & \psi_2\lambda_2 & -\varphi_2^*\lambda_2^* & \psi_3\lambda_3 & -\varphi_3^*\lambda_3^* & 0 \\ \varphi_1\lambda_1^2 & \psi_1^*\lambda_1^{*2} & \varphi_2\lambda_2^2 & \psi_2^*\lambda_2^{*2} & \varphi_3\lambda_3^2 & \psi_3^*\lambda_3^{*2} & 0 \\ \psi_1\lambda_1^2 & -\varphi_1^*\lambda_1^{*2} & \psi_2\lambda_2^2 & -\varphi_2^*\lambda_2^{*2} & \psi_3\lambda_3^2 & -\varphi_3^*\lambda_3^{*2} & 1 \\ \varphi_1\lambda_1^3 & \psi_1^*\lambda_1^{*3} & \varphi_2\lambda_2^3 & \psi_2^*\lambda_2^{*3} & \varphi_3\lambda_3^3 & \psi_3^*\lambda_3^{*3} & \boxed{0} \end{vmatrix} \tag{4.11}$$

respectively. The quasideterminant solutions (4.9)–(4.10) can be expanded as

$$q_{[2]} = q - 2i\,(\lambda_1 - \lambda_1^*)\,\frac{\varphi_1\psi_1^*}{|\varphi_1|^2 + |\psi_1|^2} \tag{4.12}$$

and

$$q_{[3]} = q - 2i\frac{\Lambda_{11}\left(\Pi_{12}\,|\varphi_2|^2 + \Pi_{12}^*\,|\psi_2|^2\right)\varphi_1\psi_1^* + \Lambda_{22}\left(\Lambda_{12}\,|\varphi_1|^2 + \Lambda_{12}^*\,|\psi_1|^2\right)\varphi_2\psi_2^*}{|\lambda_1 - \lambda_2|^2\,|\varphi_1\varphi_2^* + \psi_1\psi_2^*|^2 + |\lambda_1 - \lambda_2^*|^2\,|\varphi_1\psi_2 - \varphi_2\psi_1|^2}, \tag{4.13}$$

where

$$\Lambda_{11} = \lambda_1 - \lambda_1^*, \quad \Lambda_{22} = \lambda_2 - \lambda_2^*, \quad \Lambda_{12} = (\lambda_1 - \lambda_2)\,(\lambda_1 - \lambda_2^*)\,,$$

$$\Pi_{12} = (\lambda_1 - \lambda_2)(\lambda_1^* - \lambda_2).$$

Moreover, the solution (4.11) can be expressed in terms of determinants such that

$$q_{[4]} = q - 2i\frac{D}{\Delta}, \tag{4.14}$$

in which

$$D = \begin{vmatrix} \varphi_1 & \psi_1^* & \varphi_2 & \psi_2^* & \varphi_3 & \psi_3^* \\ \psi_1 & -\varphi_1^* & \psi_2 & -\varphi_2^* & \psi_3 & -\varphi_3^* \\ \varphi_1\lambda_1 & \psi_1^*\lambda_1^* & \varphi_2\lambda_2 & \psi_2^*\lambda_2^* & \varphi_3\lambda_3 & \psi_3^*\lambda_3^* \\ \psi_1\lambda_1 & -\varphi_1^*\lambda_1^* & \psi_2\lambda_2 & -\varphi_2^*\lambda_2^* & \psi_3\lambda_3 & -\varphi_3^*\lambda_3^* \\ \varphi_1\lambda_1^2 & \psi_1^*\lambda_1^{*2} & \varphi_2\lambda_2^2 & \psi_2^*\lambda_2^{*2} & \varphi_3\lambda_3^2 & \psi_3^*\lambda_3^{*2} \\ \varphi_1\lambda_1^3 & \psi_1^*\lambda_1^{*3} & \varphi_2\lambda_2^3 & \psi_2^*\lambda_2^{*3} & \varphi_3\lambda_3^3 & \psi_3^*\lambda_3^{*3} \end{vmatrix},$$

$$\Delta = \begin{vmatrix} \varphi_1 & \psi_1^* & \varphi_2 & \psi_2^* & \varphi_3 & \psi_3^* \\ \psi_1 & -\varphi_1^* & \psi_2 & -\varphi_2^* & \psi_3 & -\varphi_3^* \\ \varphi_1\lambda_1 & \psi_1^*\lambda_1^* & \varphi_2\lambda_2 & \psi_2^*\lambda_2^* & \varphi_3\lambda_3 & \psi_3^*\lambda_3^* \\ \psi_1\lambda_1 & -\varphi_1^*\lambda_1^* & \psi_2\lambda_2 & -\varphi_2^*\lambda_2^* & \psi_3\lambda_3 & -\varphi_3^*\lambda_3^* \\ \varphi_1\lambda_1^2 & \psi_1^*\lambda_1^{*2} & \varphi_2\lambda_2^2 & \psi_2^*\lambda_2^{*2} & \varphi_3\lambda_3^2 & \psi_3^*\lambda_3^{*2} \\ \psi_1\lambda_1^2 & -\varphi_1^*\lambda_1^{*2} & \psi_2\lambda_2^2 & -\varphi_2^*\lambda_2^{*2} & \psi_3\lambda_3^2 & -\varphi_3^*\lambda_3^{*2} \end{vmatrix}.$$

5. Particular solutions

5.1. Solutions for zero seed. For $q = r = 0$, the spectral problem (4.8) becomes

$$\phi_{j,x} + J\phi_j\lambda_j = 0,$$
$$\phi_{j,t} + \left(4\beta\lambda_j^3 + 2\alpha\lambda_j^2\right)J\phi_j = 0,$$

which has a solution $\phi_j = (\varphi_j, \psi_j)^T$ such that

$$\varphi_j(x,t,\lambda_j) = e^{-i\left[\lambda_j x + \left(2\alpha\lambda_j^2 + 4\beta\lambda_j^3\right)t\right]}, \quad \psi_j(x,t,\lambda_j) = e^{i\left[\lambda_j x + \left(2\alpha\lambda_j^2 + 4\beta\lambda_j^3\right)t\right]}, \tag{5.1}$$

where $j = 1, \ldots, n$.

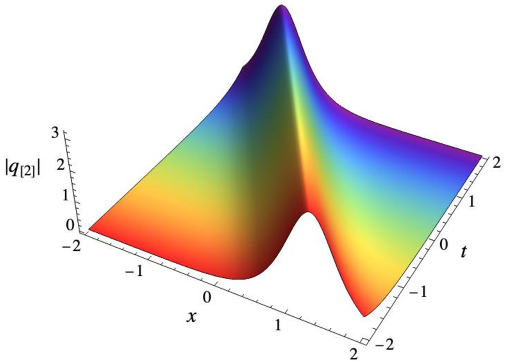
Case I ($n = 1$). By letting $\lambda_1 = \xi + i\eta$ and substituting the functions φ_1 and ψ_1 given by (5.1) into (4.12), we obtain the one-soliton solution of the Hirota equation (1.3) as

$$q_{[2]} = 2\eta e^{-2i\left[\xi x + 2\left(\alpha\left[\xi^2-\eta^2\right] + 2\beta\left[\xi^3 - 3\xi\eta^2\right]\right)t\right]}\,\mathrm{sech}\left(2\eta x + 8\left[\alpha\xi\eta + \beta\left(3\xi^2\eta - \eta^3\right)t\right]\right)$$

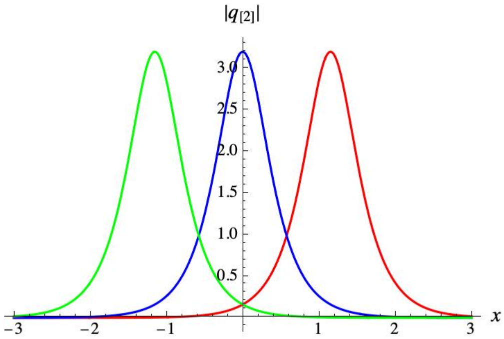
which yields

$$\left|q_{[2]}\right|^2 = 4\eta^2\,\mathrm{sech}^2\left(2\eta x + 8\left[\alpha\xi\eta + \beta\left(3\xi^2\eta - \eta^3\right)\right]t\right).$$

This solution is plotted in Fig. 5.1.



(a)



(b)

Fig. 5.1: One-soliton solution $|q_{[2]}|$ of the HE (1.3) when $\alpha = \beta = 1$, $\xi = 0.8, \eta = 1.6$. Figure (a) describes its surface and (b) gives its profiles at different times $t = -1.8$ (red), $t = 0$ (blue), $t = 1.8$ (green).

Case II ($n = 2$). Let $\lambda_1 = \xi + \eta_1$ and $\lambda_2 = \xi + \eta_2$ such that $\eta_1\eta_2 \neq 0$. By substituting the corresponding eigenfunctions φ_1, ψ_1 and φ_2, ψ_2, given by (5.1), into (4.13), we obtain the two-soliton solution of the Hirota equation (1.3) as

$$q_{[3]} = 4\left(\eta_1^2 - \eta_2^2\right) \frac{\eta_1 e^{-ig_1}\cosh f_2 - \eta_2 e^{-ig_2}\cosh f_1}{\left(\eta_1 - \eta_2\right)^2 \cosh F_1 + \left(\eta_1 + \eta_2\right)^2 \cosh F_2 - 4\eta_1\eta_2 \cos F_3}$$

which yields

$$\left|q_{[3]}\right|^2 = 16\left(\eta_1^2 - \eta_2^2\right)^2 \frac{\eta_2^2\cosh^2 f_1 + \eta_1^2\cosh^2 f_2 - 2\eta_1\eta_2\cosh f_1 \cosh f_2 \cos F_3}{\left[\left(\eta_1 - \eta_2\right)^2 \cosh F_1 + \left(\eta_1 + \eta_2\right)^2 \cosh F_2 - 4\eta_1\eta_2 \cos F_3\right]^2},$$

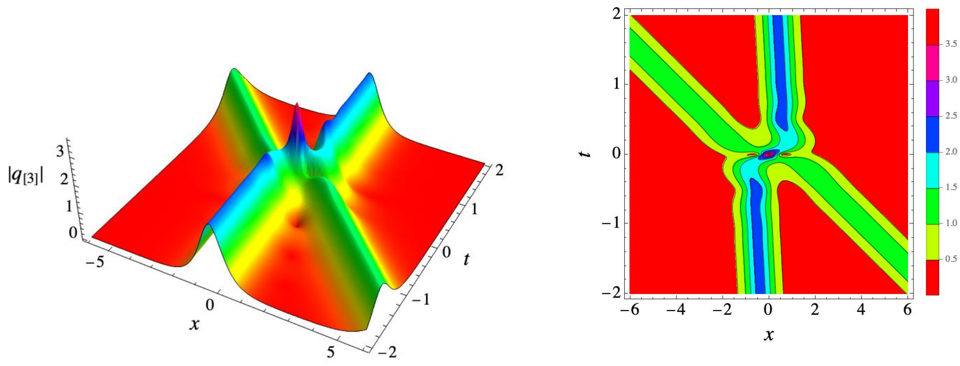
where

$$\begin{aligned}
f_1 &= 2\eta_1\left[x + 4\left(\alpha\xi + \beta\left[3\xi^2 - \eta_1^2\right]\right)t\right],\\
f_2 &= 2\eta_2\left[x + 4\left(\alpha\xi + \beta\left[3\xi^2 - \eta_2^2\right]\right)t\right],\\
g_1 &= 2\xi x + 4\left[\alpha\left(\xi^2 - \eta_1^2\right) + 2\beta\xi\left(\xi^2 - 3\eta_1^2\right)\right]t,\\
g_2 &= 2\xi x + 4\left[\alpha\left(\xi^2 - \eta_2^2\right) + 2\beta\xi\left(\xi^2 - 3\eta_2^2\right)\right]t
\end{aligned}$$

and $F_1 = f_1 + f_2$, $F_2 = f_1 - f_2$, $F_3 = g_1 - g_2$ such that

$$\begin{aligned}
F_1 &= 2\left(\eta_1 + \eta_2\right)\left[x + 4\left(\alpha\xi + \beta\left[3\xi^2 + \eta_1\eta_2 - \eta_1^2 - \eta_2^2\right]\right)t\right],\\
F_2 &= 2\left(\eta_1 - \eta_2\right)\left[x + 4\left(\alpha\xi + \beta\left[3\xi^2 - \eta_1\eta_2 - \eta_1^2 - \eta_2^2\right]\right)t\right],\\
F_3 &= 4\left(\eta_2^2 - \eta_1^2\right)\left[\alpha + 6\beta\xi\right]t.
\end{aligned}$$

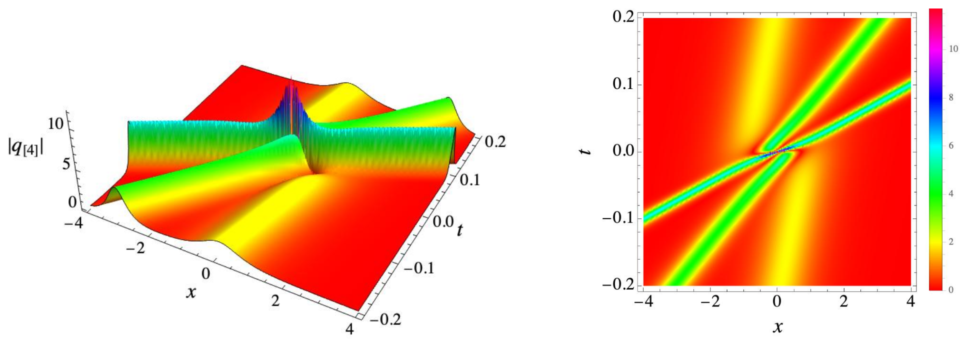
By choosing appropriate parameters, the two-soliton solution of the Hirota equation (1.3) is plotted in Fig. 5.2.



(a) (b)

Fig. 5.2: Two-soliton solution $|q_{[3]}|$ of the HE (1.3) when $\alpha = \beta = 1$, $\xi = 0.5$, $\eta_1 = 0.7$ and $\eta_2 = 1.1$. (a) Surface diagram. (b) Contour diagram.

Case III ($n = 3$). In this case, we have three eigenvalues λ_1, λ_2 and λ_3. Let us choose $\lambda_1 = i$, $\lambda_2 = 2i$ and $\lambda_3 = 3i$. By substituting the corresponding eigenfunctions $(\varphi_1, \psi_1)^T$, $(\varphi_2, \psi_2)^T$ and $(\varphi_3, \psi_3)^T$, given by (5.1), into (4.14), we obtain the three-soliton solution of the Hirota equation (1.3). By choosing appropriate parameters, this solution is plotted in Fig. 5.3.



(a) (b)

Fig. 5.3: Three-soliton solution $|q_{[4]}|$ of the HE (1.3) when $\lambda_1 = i$, $\lambda_2 = 2i$ and $\lambda_3 = 3i$. (a) Surface diagram. (b) Density diagram.

5.2. Solutions for non-zero seed. In this subsection, for $q, r \neq 0$ and $r = q^*$, we take $q = ce^{i\mu}$ as a plane wave solution of the Hirota equation (1.3), where $\mu = ax + bt$ in which $a, b, c \in \mathbb{R}$ under the condition $b = \alpha\left(2c^2 - a^2\right) + \beta\left(a^3 - 6ac^2\right)$. We use this as a seed solution. Substituting $q = ce^{i\mu}$ into the linear system (4.8) and then solving for the eigenfunction $\phi_j = (\varphi_j, \psi_j)^T$, we obtain

$$\varphi_j\left(x, t, \lambda_j\right) = e^{\frac{1}{2}i\mu}\left(c_j e^{\frac{1}{2}i\gamma_j} + e_j e^{-\frac{1}{2}i\gamma_j}\right),$$

$$\psi_j\left(x,t,\lambda_j\right)=e^{-\frac{1}{2}i\mu}\left(\widetilde{c_j}e^{\frac{1}{2}i\gamma_j}+\widetilde{e_j}e^{-\frac{1}{2}i\gamma_j}\right), \tag{5.2}$$

where

$$\gamma_j=s_j\left(x+k_jt\right),\quad \widetilde{c_j}=i\frac{c_j}{2c}\left(a+2\lambda_j+s_j\right),\quad \widetilde{e_j}=i\frac{e_j}{2c}\left(a+2\lambda_j-s_j\right)$$

in which $s_j=\sqrt{\left(a+2\lambda_j\right)^2+4c^2}$, $k_j=\alpha\left(2\lambda_j-a\right)+\beta\left(a^2-2a\lambda_j+4\lambda_j^2-2c^2\right)$ and c_j, e_j are arbitrary constants such that $j=1,\ldots,n$.

Case IV ($n=1$). Let the eigenvalue $\lambda_1=\xi+i\eta$. For simplicity, choose $a=-2\xi$ and $e_1=\delta c_1=\delta c$ in which $\delta^2=1$. Substituting the seed solution $q=ce^{i\mu}$ and the functions φ_1, ψ_1, given by (5.2), into (4.12), we obtain the following breather solution:

$$q_{[2]}=ce^{i\mu}\left(1-2\eta\frac{\eta\cosh(\Omega t)-i\omega\sinh(\Omega t)+\delta\eta\cos[2\omega(x+\Gamma t)]+\delta\omega\sin[2\omega(x+\Gamma t)]}{c^2\cosh(\Omega t)+\delta\eta^2\cos[2\omega(x+\Gamma t)]+\delta\eta\omega\sin[2\omega(x+\Gamma t)]}\right), \tag{5.3}$$

where

$$\begin{aligned}
\mu&=-2\xi x+\left[2\alpha\left(c^2-2\xi^2\right)+4\beta\left(3c^2\xi-2\xi^3\right)\right]t,\\
\omega&=\sqrt{c^2-\eta^2},\\
\Omega&=4\eta\omega(\alpha+6\beta\xi),\\
\Gamma&=4\alpha\xi+2\beta\left(6\xi^2-2\eta^2-c^2\right).
\end{aligned}$$

Thus, we have $\left|q_{[2]}\right|^2=c^2\frac{F^2+G^2}{H^2}$, where

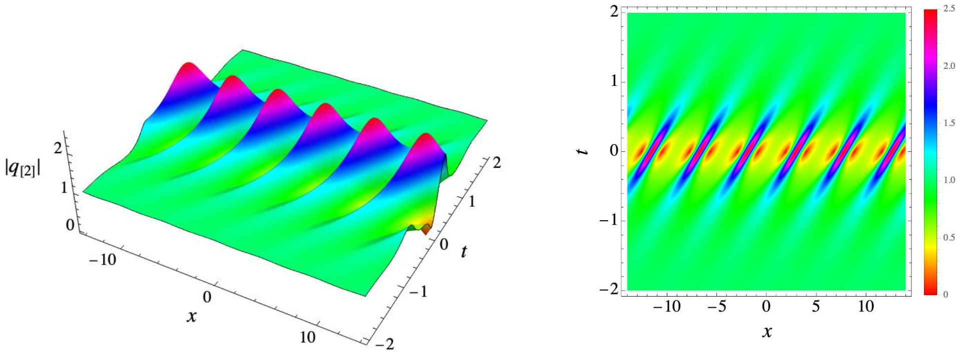
$$\begin{aligned}
F&=\left(2\eta^2-c^2\right)\cosh(\Omega t)+\delta\eta^2\cos[2w(x+\Gamma t)]+\delta\eta w\sin[2w(x+\Gamma t)],\\
G&=2\eta w\sinh(\Omega t)\\
H&=c^2\cosh(\Omega t)+\delta\eta^2\cos[2w(x+\Gamma t)]+\delta\eta w\sin[2w(x+\Gamma t)].
\end{aligned}$$

Fig. 5.4 shows the dynamical evolution of the breather solution of the Hirota equation (1.3).

6. Rogue wave solutions

In this section, we consider the breather solution (5.3) of the HE (1.3) with $\delta=-1$. Here we use the Taylor expansion approach [2, 32] in order to obtain the rogue wave solutions of the HE from the breather solution (5.3). The Taylor expansion of the breather solution (5.3) with the limit $c\to\eta$ gives us the first-order rogue wave solution of the Hirota equation:

$$q_{[2]}=\eta e^{i\mu}\left[-1+2\frac{1+4i\eta^2\left(\alpha+6\beta\xi\right)t}{1+8\eta^4\left(\alpha+6\beta\xi\right)^2t^2+2\eta^2\left(x+\Gamma t\right)^2-2\eta\left(x+\Gamma t\right)}\right], \tag{6.1}$$



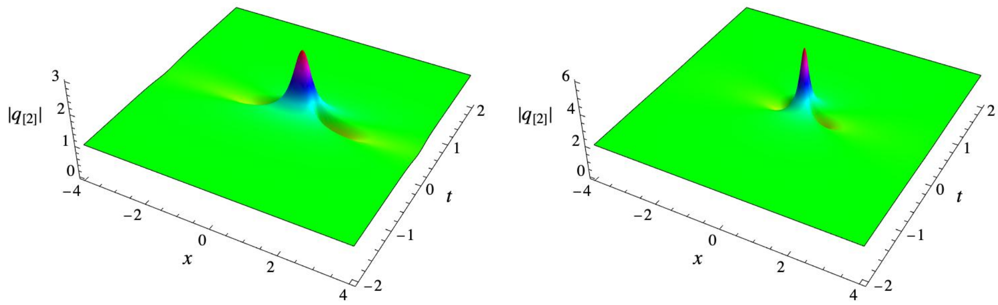
(a) (b)

Fig. 5.4: Breather solution $|q_{[2]}|$ of the HE (1.3) when $c = \delta = 1$, $\alpha = \beta = 1$, $\xi = 0.04$ and $\eta = 0.76$. (a) Surface diagram. (b) Density diagram.

where

$$\mu = -2\xi x + bt, \quad \Gamma = 4\alpha\xi + 6\beta\left(2\xi^2 - \eta^2\right).$$

in which $b = 2\alpha\left(\eta^2 - 2\xi^2\right) + 4\beta\xi\left(3\eta^2 - 2\xi^2\right)$. This is the simplest RW solution of the HE (1.3). This rogue wave solution of the Hirota equation is shown in Fig. 6.1(a). Furthermore, by choosing $\alpha = 1, \beta = 0$, the rogue wave solution (6.1)



(a) (b)

Fig. 6.1: The first-order rogue wave solutions of (a) the HE (1.3) with parameters $\alpha = \beta = 1$, $\xi = 0.5$ and $\eta = 1$. (b) the NLS (1.1) equation when $\xi = 0$ and $\eta = 2$.

reduces to the first-order RW solution of the NLS equation (1.1) :

$$q_{[2]} = \eta e^{i\mu}\left[-1 + 2\frac{1 + 4i\eta^2 t}{1 + 8\eta^4 t^2 + 2\eta^2\left(x + 4\xi t\right)^2 - 2\eta\left(x + 4\xi t\right)}\right]. \qquad (6.2)$$

This is the fundamental Peregrine-like rogue wave solution which is shown in Fig. 6.1(b). Finally, we should point out that the higher-order RW solutions of the HE (1.3) can be constructed by different methods, for example, by using gDT [3,13].

7. Conclusion

In conclusion, we have studied a standard Darboux transformation to construct quasideterminant solutions for the Hirota equation (1.3). These quasideterminants are expressed in terms of solutions of the linear partial differential equations given by (4.8). It should be highlighted that these quasideterminant solutions arise naturally from the Darboux transformation we present here. Furthermore, the multi-soliton, breather and rogue wave solutions for zero and nonzero seeds are given as particular examples for the HE. The examples of these particular solutions are plotted in the figures 5.1–6.1 with the chosen parameters. Finally, we point out that the method we present in this paper allows us to construct exact solutions for other integrable nonlinear evolution equations such as [12, 28, 37].

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References

[1] G.P. Agrawal, *Nonlinear Fiber Optics,* Academic Press, 2007.

[2] N. Akhmediev, A. Ankiewicz, and M. Taki, *Waves that appear from nowhere and disappear without a trace*, Phys. Lett. A **373** (2009), 675–678.

[3] A. Ankiewicz, J.M. Soto-Crespo, and N. Akhmediev, *Rogue waves and rational solutions of the Hirota equation*, Phys. Rev. E **81** (2010), 046602.

[4] D.J. Benney and A.C. Newell, *The propagation of nonlinear wave envelopes*, J. Math. Phys. **46** (1967), 133–139.

[5] D.J. Benney and G.J. Roskes, *Wave instabilities*, Stud. Appl. Math. **48** (1969), 377–385.

[6] H.H. Chen, Y.C. Lee, and C.S. Liu, *Integrability of nonlinear Hamiltonian systems by inverse scattering method*, Physica Scripta **20** (1979), 490–492.

[7] G. Darboux, *Sur une proposition relative aux équations linéaires* C. R. Acad. Sci. **94** (1882), 1456–1459.

[8] I. Gelfand and V. Retakh, *Determinants of the matrices over noncomutative rings*, Funct. Anal. App. **25** (1991), 91–102.

[9] I. Gelfand, S. Gelfand, V. Retakh, and R.L. Wilson, *Quasideterminants*, Adv. Math. **193** (2005), 56–141.

[10] C.R. Gilson and J.J.C. Nimmo, *On a direct approach to quasideterminant solutions of a noncommutative KP equation*, J. Phys. A: Math. Theor. **40** (2007), 3839–3850.

[11] C.R. Gilson, M. Hamanaka, and S.C. Huang and J. J. C. Nimmo, *Soliton Solutions of Noncommutative Anti-Self-Dual Yang–Mills Equations*, J. Phys. A: Math. Theor. **53** (2020) 404002.

[12] C.R. Gilson and S.R. Macfarlane, *Dromion solutions of noncommutative Davey–Stewartson equations*, J. Phys. A: Math. Theor. **42** (2009), 235232.

[13] B.L. Guo, L.M. Ling, and Q.P. Liu, *Nonlinear Schrödinger Equation: Generalized Darboux Transformation and Rogue Wave Solutions*, Phys. Rev. E **85** (2012), 026607.

[14] A. Hasegawa and F. Tappert, *Transmission of stationary nonlinear optical pulses in dispersive dielectric fibres I. Anomalous dispersion*, Appl. Phys. Lett. **23** (1973), 142 .

[15] A. Hasegawa and F. Tappert, *Transmission of stationary nonlinear optical pulses in dispersive dielectric fibres II. Normal dispersion*, Appl. Phys. Lett. **23** (1973), 171.

[16] R. Hirota, *Exact envelope-soliton solutions of a nonlinear wave equation*, J. Math. Phys. **14** (1973), 805–809.

[17] D.J. Kaup and A.C. Newell, *An exact solution for a derivative nonlinear Schrödinger equation*, J. Math. Phys. **19** (1978), 798–801.

[18] Y. Kivshar and G. Agrawal, *Optical Solitons: From Fibers to Photonic Crystals*, Academic Press, 2003.

[19] Y. Kodama, *Optical solitons in a monomode fiber*, J. Stat. Phys. **39** (1985), 597–614.

[20] Y. Kodama and A. Hasegawa, *Nonlinear pulse propagation in a monomode dielectric guide*, IEEE J. Quantum Electron. **23** (1987), 510–524.

[21] C.X. Li and J.J.C. Nimmo, *Darboux transformations for a twisted derivation and quasideterminant solutions to the super KdV equation*, Proc. R. Soc. A **466** (2010), 2471–2493.

[22] L. Li, Z. Wu, L. Wang, and J. He, *Higher-order rogue waves for the Hirota equation*, Ann. Phys. **334** (2013), 198-211.

[23] B.A. Malomed and D. Mihalache, *Nonlinear waves in optical and matter-wavemedia: A topical survey of recent theoretical and experimental results*, Rom. J. Phys. **64** (2019), 106.

[24] V.B. Matveev, *Darboux transformation and explicit solutions of the Kadomtcev–Petviaschvily equation, depending on functional parameters*, Lett. Math. Phys. **3** (1979), 213–216.

[25] V.B. Matveev and M.A. Salle, *Darboux Transformations and Solitons.* Springer Series in Nonlinear Dynamics, Springer-Verlag, Berlin, 1991.

[26] D. Mihalache, N. Truta, and L.-C. Crasovan, *Painlevé analysis and bright solitary waves of the higher-order nonlinear Schrödinger equation containing third-order dispersion and self-steepening term*, Phys. Rev. E **56** (1997), 1064–1070.

[27] J.J.C. Nimmo, C.R. Gilson, and Y. Ohta, *Applications of Darboux transformations to the self-dual Yang–Mills equations*, Theor. Math. Phys. **122** (2000), 239–246.

[28] J.J.C. Nimmo and H. Yilmaz, *On Darboux Transformations for the derivative nonlinear Schrödinger equation*, J. Nonlinear Math. Phys. **21** (2014), 278–293.

[29] J.J.C. Nimmo and H. Yilmaz, *Binary Darboux transformation for the Sasa–Satsuma equation*, J. Phys. A: Math. Theor. **48** (2015), 425202.

[30] D.H. Peregrine, *Water waves, nonlinear Schrödinger equations and their solutions*, J. Aust. Math. Soc. Ser. B **25** (1983), 16–43.

[31] H.W.A. Riaz, *Noncommutative coupled complex modified Korteweg–de Vries equation: Darboux and binary Darboux transformations*, Mod. Phys. Lett. A **34** (2019), 1950054.

[32] Y. Tao and J. He, *Multisolitons, breathers and rogue waves for the Hirota equation generated by the Darboux transformation*, Phys. Rev. E **85** (2012), 026601.

[33] N. Sasa and J. Satsuma, *New type of soliton solutions for a higher-order nonlinear Schrödinger equation*, J. Phys. Soc. Japan **60** (1991), 409–417.

[34] M. Wadati, *The exact solution of the modified Korteweg–de Vries equation*, J. Phys. Soc. Japan **32** (1972), 1681.

[35] H. Wu, J. Liu, and C. Li, *Quasideterminant solutions of the extended noncommutative Kadomtsev–Petviashvili hierarchy*, Theor. Math. Phys. **192** (2017), 982–999.

[36] Z. Yan and C. Dai, *Optical rogue waves in the generalized inhomogeneous higher-order nonlinear Schrödinger equation with modulating coefficients*, J. Opt. **15** (2013), 064012.

[37] H. Yilmaz, *Exact solutions of the Gerdjikov–Ivanov equation using Darboux transformations*, J. Nonlinear Math. Phys. **22** (2015), 32–46.

[38] V.E. Zakharov, *Stability of periodic waves of finite amplitude on the surface of a deep fluid*, J. Appl. Mech. Tech. Phys. **9** (1968), 190–194.

[39] V.E. Zakharov, *Collapse of Langmuir waves*, Sov. Phys. JETP **35** (1972), 908–914.

[40] V.E. Zakharov and A.B. Shabat, *Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media*, Sov. Phys. JETP **34** (1972), 62–69.

[41] H.Q. Zhang, Y. Wang, and W.X. Ma, *Binary Darboux transformation for the coupled Sasa–Satsuma equations*, Chaos **27** (2017), 073102.

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Перетворення Дарбу для рівняння Хіроти

Halis Yilmaz

Рівняння Хіроти є інтегровним нелінійним рівнянням вищого порядку типу Шредінгера, яке описує поширення ультракоротких світлових імпульсів в оптичних волокнах. Ми представляємо стандартне перетворення Дарбу для рівняння Хіроти, а потім будуємо його квазідетермінантні розв'язки. В якості прикладів наведено мультисолітонні і бризерні розв'язки, а також розв'язки у вигляді поодиноких хвиль для рівняння Хіроти в явному вигляді.

Ключові слова: рівняння Хіроти, перетворення Дарбу, квазідетермінантні розв'язки, мультисолітонні розв'язки, бризерні розв'язки, розв'язки у вигляді поодиноких хвиль