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Characterization Theorems for the B-q-Binomial and the q-Poisson Distributions

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In this paper, the q-binomial and the q-hypergeometric distributions are redefined and re-introduced in the compact form. The redefined distributions are named B-q-binomial and B-q-hypergeometric. Furthermore, the generalization of the well-known Patil and Seshadri characterization is reported in the q-calculus. The characterizations of B-q-binomial and B-q-hypergeometric distributions are presented by using a conditional q-distribution. A necessary and sufficient condition identifying the q-Poisson distribution is outlined.

Key words: q-calculus, q-addition operator, characterization theorem

Mathematical Subject Classification 2010: 62E10, 62H05, 60D05

1. Introduction

A deep investigation in the mathematics history unveils that quantum calculus, often called q-calculus, was initially developed by the pioneer Euler in Arithmetic around 1748 and dealt with the enumeration problems.

Later on, Heine and Gauss studied the q-analogues of hypergeometric functions and demonstrated that these q-functions are solutions of linear q-differential equations of second order.

In the early twentieth century, Jackson [11] initiated a q-analysis by identifying the q-derivative, the q-integrals, also called Jackson integral, and systematically defined the analogues of some common functions such as the q-exponential, the q-Bessel functions, the q-gamma, and the q-beta functions, etc.

Recently, many researchers showed a keen interest in exploring q-calculus and highlighted its applications in interdisciplinary subjects such as various areas of pure and applied mathematics, computer sciences and theoretical physics [13,16]. The q-calculus is viewed as a bridge between Mathematics and Physics.

In mathematical physics and probability, the q-distribution was introduced by Díaz [6–8] in the continuous case and by Kupershmidt [12] in the discrete case. The hypothesis that the probability of success (or failure) at a trial varies geometrically, with rate (proportion) q, yields the introduction of discrete q-distributions. Exploring these distributions is substantially facilitated by the wealth existing with q-sequences and q-functions, in q-combinatorics, and the theory of q-hypergeometric series.

The characterization of the q-distributions remains under investigation due to its fundamental role in the linkage between different q-distributions. For that, a large number of researching papers have been published in the recent years [1–3, 10, 13]. To the best of our knowledge, Boutouria et al. [1, 10] were the first to study the characterization of the q-distribution in the continuous case. Otherwise, the investigation of the q-distribution in the discrete case is still a virgin researching area. In this paper, we report for the first time the generalization of the Patil and Seshardi [14] characterizations in the q-calculus theory. The paper is structured as follow: in Section 2, some preliminary concepts related to q-addition operator and certain essential results are presented to build up our work. In Section 3, based on the q-addition operator and the special q-functions, the q-Binomial and the q-hypergeometric distributions are redefined and re-introduced in the compact form and named as Bouzida q-binomial and Bouzida q-hypergeometric (B-q-binomial and B-q-hypergeometric). In section 4, the generalization of the well-known Patil and Seshardi characterization [14] is reported in the q-calculus. The characterizations of B-q-binomial and B-q-hypergeometric distributions are presented by using conditional q-distribution. A necessary and sufficient condition identifying the q-Poisson distribution is outlined. The last section wraps up the conclusion and provides new perspectives for future works.

2. Preliminaries

In this section, some useful basic definitions [11] are introduced. We shall start with the notion of $[n]_q$ given by

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1}.$$

We also denote, for all $n \in \mathbb{N}$,

$$[n]_q! = \begin{cases} 1 & \text{if } n = 0, \\ [n]_q[n-1]_q \ldots [1]_q & \text{otherwise.} \end{cases}$$

For $x \in \mathbb{R}$,

$$[x]_q = \frac{1 - q^x}{1 - q}.$$

If x goes to ∞, we obtain $[\infty]_q = \frac{1}{1-q}$ that is called a q-analogue of ∞.

Note that $[\infty]_q$ approaches 1 if q goes to 0 and goes to $+\infty$ if q approaches 1.

We recall some usual notations used in the q-theory.

Jackson in [11] proposed a q-analogue of the exponential function e^x indicated by

$$e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}.$$

The q-analogue of the identity $e^x e^{-x} = 1$ is $e_q^x E_q^{-x} = 1$, where the function E_q^x is defined by $e_{1/q}^x$, is given also by

$$e_{1/q}^x = E_q^x = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{x^n}{[n]_q!}.$$

The q-logarithm function $\log_q(x)$ is the inverse of the q-exponential function e_q^x, and the function $\mathrm{Log}_q(x)$ is the inverse function of E_q^x.

Chung *et al.* [5] proposed the q-addition operator and discussed its properties. The q-addition operator is defined by

$$\begin{cases} (a \oplus_q b)^n = \sum_{k=0}^{n} {}_qC_k^n a^k b^{n-k}, \quad n \in \mathbb{N},\ a \neq b, \\ (a \oplus_q a)^n = (a+a)^n = 2^n a^n, \end{cases}$$

where

$${}_qC_k^n = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Equivalently, $\oplus_q$ is defined as

$$\oplus_q : \mathbb{R}^2 \to \mathbb{R}, \quad (a,b) \mapsto a \oplus_q b$$

such that $a \oplus_q b$ is the unique real verifying $e_q^{a\oplus_q b} = e_q^a e_q^b$.

From the above definition, we get the property

$$k(a \oplus_q b) = ka \oplus_q kb, \quad k \in \mathbb{R}.$$

It is easy to see that this operator is commutative, i.e., $a \oplus_q b = b \oplus_q a$. In addition, if we take $b = a$, then we have $a \oplus_q a = a + a = 2a$. Finally, if we take $b = 0$, we obtain $a \oplus_q 0 = 0 \oplus_q a = a$.

The q-addition operator has the property x, y, $z \in \mathbb{C}$,

$$(x \oplus_q y) \oplus_q z = x \oplus_q (y \oplus_q z)].$$

This property (associativity) is proved as follows. We must prove that

$$[(x \oplus_q y) \oplus_q z]^n = [x \oplus_q (y \oplus_q z)]^n.$$

But this is equivalent to

$$\sum_{k=0}^{n} {}_qC_k^n \sum_{l=0}^{k} {}_qC_l^k x^l y^{k-l} z^{n-k} = \sum_{k'=0}^{n} {}_qC_{k'}^n \sum_{l'=0}^{n-k'} {}_qC_{l'}^{n-k'} y^{l'} z^{n-k'-l'}.$$

Now we put $l = k'$ and $l' = k - l$ to conclude the proof. The proof of the distributive law is obvious.

This operator permits to express the properties of the q-logarithm and q-exponential functions in more compact forms:

(i) $e_q^a e_q^b = e_q^{a \oplus_q b}$;

(ii) $e_q^{na} = (e_q^a)^n$, $n \in \mathbb{N}$;

(iii) $\log_q(ab) = \log_q(a) \oplus_q \log_q(b)$;

(iv) $\log_q(a^n) = n \log_q(a)$, $n \in \mathbb{N}$.

The q-subtraction is defined as $a \ominus_q b = a \oplus_q (-b)$, $a \neq b$, and if we take $b = a$, then we have $a \ominus_q a = a - a = 0$.

Chung *et al.* [5] defined the power function and used the q-operator in its properties.

A power function based on q-addition is defined by $a_q^x = E_q^{x \operatorname{Log}_q(a)}$ for all $a > 0$.

This function satisfies $a_q^x a_q^y = a_q^{x \oplus_q y}$, $\frac{a_q^x}{a_q^y} = a_q^{x \ominus_q y}$, $(a^x)_q^y = a_q^{xy}$ and $(ab)_q^x = a_q^x b_q^x$ for all $a, b > 0$.

3. The discrete q-distributions

In this section, some classical q-distributions and their properties are exhibited. We shall start by the the q-Bernoulli distribution.

3.1. The q-Bernoulli distribution. The Bernoulli distribution of a random variable takes the value 1 with probability p and 0 value with probability $1-p$, that is, the probability distribution of any single experiment giving success or failure. The problem here is that if there is an error in this experiment, then the probability seems to be inaccurate. The solution of this problem can be derived from the theory of q-calculus. For this reason, we change the parameter p by $[p]_q$ with $[p]_q = \frac{1-q^p}{1-q}$. From this perspective, the q-Bernoulli distribution is expressed by

$$P_q(X = 1) = [p]_q \quad \text{and} \quad P_q(X = 0) = 1 - [p]_q.$$

The q-Bernoulli distribution is more general than the classical Bernoulli distribution. In fact, if q goes to 1, then $[p]_q$ approaches p.

We can imagine that the coin is not really well balanced, and we can translate the default by "q" with $0 < q < 1$. Basing on this definition, we can propose the B-q-Binomial distribution.

3.2. The B-q-binomial distribution. Kupershmidt [12] proposed a q-analogue of basic discrete probability distributions as the q-Binomial distribution (K-q-binomial) using the following expression:

$$(a \dotplus b)^n = \prod_{i=0}^{n-1} (a + q^i b), \quad n \in \mathbb{N}.$$

Bouzida *et al.* [9] defined the B-q-binomial distribution with parameters n and $[p]_q$ as a discrete probability distribution of the number of successes in a sequence of n

independent experiments. A random variable contains a single bit of information: success, with probability $[p]_q$, or failure, with probability $1-[p]_q$. By referring to the definition of the q-addition operator and the formulae of the q-binomial polynomial, we define the B-q-binomial distribution by

$$P_q(X=k) =_q C_n^k [p]^k (1 \ominus_q [p]_q)^{n-k}, \quad \text{with } k \in \{0, \ldots n\}.$$

In fact,

$$\sum_{k=0}^{n} P_q(X=k) = \sum_{k=0}^{n} {}_qC_n^k [p]_q^k (1 \ominus_q [p]_q)^{n-k} = ([p]_q \oplus_q 1 \ominus_q [p]_q)^n = 1.$$

It is noteworthy to highlight that the B-q-binomial distribution introduced has a compact form comparing to the K-q-binomial distribution.

3.3. The q-hypergeometric distribution. The q-analogue of the hypergeometric distribution presented by Kupershmidt [12] is defined according to the q-Vandermonde formula

$$\sum_{r=0}^{n} q_q^{(n-r)(M-r)} C_r^M {}_qC_{n-1}^{N-M} = \sum_{r=0}^{n} q_q^{r(r+N-M-n)} C_r^M {}_qC_{n-1}^{N-M} =_q C_n^N.$$

In this section, we introduce the B-q-hypergeometric distribution concerning the probability of r successes and random draws. In this respect, the drawn object has a specific feature in n draws without replacement. Within this framework, we may identify an error of minimization noted q, with $0<q<1$. Within this framework, we can identify an error of minimization noted by q, with $0<q<1$. In contrast, the B-q-binomial distribution portrays the probability of r successes in n draws with replacement.

The B-q-hypergeometric distribution is based on the the formula of the q-binomial theorem and the properties of the q-addition operator. It is given by

$$P(X=r) = \frac{{}_qC_r^M {}_qC_{n\ominus_q r}^{N\ominus_q M}}{{}_qC_n^N}.$$

Our goal is to prove that

$$\sum_{r=0}^{M} {}_qC_r^M {}_qC_{n\ominus_q r}^{N\ominus_q M} =_q C_n^N.$$

The proof is based on the Cauchy product and the q-binomial theorem. Let $M = n \oplus_q m$. In fact,

$$\begin{aligned}\sum_{r=0}^{M} {}_qC_r^M x^r &= (x \oplus_q 1)^M = (x \oplus_q 1)^{n \oplus_q m} = (x \oplus_q 1)^n (x \oplus_q 1)^m \\ &= \sum_{j=0}^{n} {}_qC_j^n x^j \sum_{s=0}^{m} {}_qC_s^m x^s = \sum_r \left[\sum_k {}_qC_k^n {}_qC_{r\ominus_q k}^m \right] x^r.\end{aligned}$$

Thus the proof is complete.

3.4. The q-Poisson distribution. The q-Poisson distribution was introduced by Kupershmidt [12] and it was defined as a q-analogue of the Poisson distribution. Two major definitions of the q-Poisson distribution will be discussed in this section.

Definition 3.1. The distribution of a random variable X is called Euler or q-Poisson distribution with parameters λ and q if its probability function is given by

$$P_q(X = x) = E_q(-\lambda)\frac{\lambda^x}{[x]_q!}, \quad x = 0, 1, \dots,$$

where $\lambda > 0$.

Definition 3.2. Let X be a discrete random variable with probability function

$$P(X = x) = e_q(-\lambda)\frac{q^{\binom{x}{2}}\lambda^x}{[x]_q!}, \quad x = 0, 1, \dots, \tag{3.1}$$

where $\lambda > 0$ and $0 < q < 1$.

The distribution of the random variable X is called the Heine distribution with parameters λ and q.

The Heine distribution is a q-Poisson distribution since the probability function (3.1), for $q \to 1$, converges to the probability function of the Poisson distribution. And using the expansion of the q-exponential function $E_q(t)$ into a power series, we have

$$E_q(t) = \sum_{x=0}^{+\infty} q^{\binom{x}{2}} \frac{t^x}{[x]_q!}.$$

Together with the relation

$$e_q(-t)E_q(t) = 1,$$

it yelds that it sums to unity as required by the definition of a probability function.

4. Characterization Theorems

In 1964, Patil and Seshadri characterized the Poisson distribution which, for independent variables X and Y, holds non-negative integral values and states that if the conditional distribution of X given the total $X + Y$ is a binomial distribution with a common parameter p for all given values of $X + Y$, and if there exists at least one integer i so that $P(X = i)$ and $P(Y = i)$ are both positive, then X and Y are individually distributed in Poisson distributions. Within the same framework, we explore results of this nature to characterize the B-q-binomial and the q-Poisson distributions among the discrete distributions.

Suppose that the following assumptions are carried out about the random variables X and Y, whose probability q-distributions are denoted by $f(x)$ and $g(y)$:

1. X and Y are independent;

2. X and Y are both discrete;

3. the conditional q-distribution of X given $(X \oplus_q Y)$, denoted by

$$c(x, x \oplus_q y),$$

is such that

$$\frac{c(x \oplus_q y, x \oplus_q y)c(0, y)}{c(x, x \oplus_q y)c(y, y)}$$

is of the form $h(x \oplus_q y)/h(x)h(y)$, where h is an arbitrary non-negative function.

Theorem 4.1. *Under the assumptions 1, 2, and 3,*

$$f(x) = f(0)h(x)E_q^{ax},$$

where a is an arbitrary constant and $f(0)$ is a suitable normalizer which makes $f(x)$ a probability function

$$g(y) = g(0)k(y)E_q^{ay},$$

where

$$k(y) = \frac{h(y)c(0, y)}{c(y, y)}$$

and $g(0)$ is the corresponding normalizer for $g(y)$.

Proof. One has

$$c(x, x \oplus_q y) = \frac{f(x)g(y)}{r(x \oplus_q y)},$$

where $r(s) = \sum f(x)g(s \ominus_q x)$. Hence,

$$f(x)g(y) = r(x \oplus_q y)c(x, x \oplus_q y). \tag{4.1}$$

Put $y = 0$ in (4.1) and change x to $x \oplus_q y$ to obtain

$$f(x \oplus_q y)g(0) = r(x \oplus_q y)c(x \oplus_q y, x \oplus_q y). \tag{4.2}$$

In this step, dividing by $f(x)g(y)$ and using (4.1), we obtain

$$\frac{f(x \oplus_q y)g(0)}{f(x)g(y)} = \frac{c(x \oplus_q y, x \oplus_q y)}{c(x, x \oplus_q y)}. \tag{4.3}$$

Setting $x = 0$ in (4.3) yields

$$\frac{f(y)g(0)}{f(0)g(y)} = \frac{c(y, y)}{c(0, y)}. \tag{4.4}$$

Departing from (4.3) and (4.4) makes it clear that

$$\frac{f(x \oplus_q y)f(0)}{f(x)f(y)} = \frac{c(x \oplus_q y, x \oplus_q y)c(0, y)}{c(x, x \oplus_q y)c(y, y)}. \tag{4.5}$$

Then, using the assumption 3, we obtain

$$\frac{f(x \oplus_q y)f(0)}{f(x)f(y)} = \frac{h(x \oplus_q y)}{h(x)h(y)}. \tag{4.6}$$

We suppose that $\Phi(x) = \frac{f(x)}{f(0)h(x)}$. Then (4.6) becomes $\Phi(x \oplus_q y) = \Phi(x)\Phi(y)$. The solution of this equation is $\Phi(x) = E_q^{ax}$, with a being an arbitrary constant. Therefore, $f(x) = f(0)h(x)E_q^{ay}$. Using (4.4)

$$g(y) = \frac{g(0)g(y)c(0,y)}{f(0)c(y,y)}.$$

Hence,

$$g(y)g(0)k(y)E_q^{ay},$$

where

$$k(y) = \frac{h(y)c(0,y)}{c(y,y)}.$$

If there are two functions h_1 and h_2 satisfying 4.3, it is easy to infer that $h_2(x) = h_1(x) = E_q^{bx}$, with b being an arbitrary constant. It means that f and g remain unchanged. ☐

Now, we will characterize the B-q-binomial and the q-Poisson distributions. These characterizations are based on Theorem 4.1 and the following two technical lemmas.

Lemma 4.2. *Let X and Y be two independent random variables according to B-q-binomial distributions with respective parameters m and n. Then $X \oplus_q Y$ is B-q-binomial distributed with parameter $m \oplus_q n$.*

Proof. Let G_X be the generating function of X. Then

$$G_X = \sum_{k=0}^{n} P(X=k)t^k = \sum_{k=0}^{n} {}_qC_n^k([p]_q t)^k(1 \ominus_q [p]_q)^{\ominus_q k} = ([p]_q t \oplus_q 1 \ominus_q [p]_q)^n.$$

Therefore, the generating function of $X \oplus_q Y$ is given by

$$G_{X\oplus_q Y}(t) = ([p]_q t \oplus_q 1 \ominus_q [p]_q)^n([p]_q t \oplus_q 1 \ominus_q [p]_q)^m = ([p]_q t \oplus_q 1 \ominus_q [p]_q)^{n\oplus_q m}.$$

The lemma is proved. ☐

Lemma 4.3. *Let X and Y be two independent random variables according to the q-Poisson distribution with respective parameters λ and μ. Then $X \oplus_q Y$ is q-Poisson distributed with parameter $\lambda \oplus_q \mu$.*

Proof. Let $Z = X \oplus_q Y$ and let $\Sigma_{n\geq 0} a_n t^n$, $\Sigma_{n\geq 0} b_n t^n$ and $\Sigma_{n\geq 0} c_n t^n$ be the generating functions of X, Y and Z respectively. Then we obtain

$$a_n = \frac{E_q^{-\lambda}\lambda^n}{[n]_q!} \quad \text{and} \quad b_n = \frac{E_q^{-\mu}\mu^n}{[n]_q!},$$

c_n is obtained by performing the Cauchy product of these two series. Therefore, we have

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = \frac{E_q^{-(\lambda \oplus_q \mu)}}{[n]_q!} \sum_{k=0}^{n} [n]! \frac{\lambda^k \mu^{n-k}}{[k]_q! [n-k]_q!}$$
$$= \frac{E_q^{-(\lambda \oplus_q \mu)}}{[n]_q!} \sum_{k=0}^{n} {}_qC_n^k \lambda^k \mu^{n-k} = \frac{E_q^{-(\lambda \oplus_q \mu)} (\lambda \oplus_q \mu)^n}{[n]_q!}.$$

Then Z is q-Poisson distributed with parameter $\lambda \oplus_q \mu$. □

Theorem 4.4. *Let X and Y be two positive independent real random variables.*

Then X and Y have the B-q-binomial distribution with the same parameters p and n if and only if the conditional q-distribution of X given the total $X \oplus_q Y$ is the B-q-hypergeometric distribution with parameters n and $2n$.

Proof. Necessity. Let X and Y have the q-binomial distribution. Then, using Lemma 4.2, $X \oplus_q Y$ has the q-binomial distribution with parameter $2n$. Therefore, we have

$$P(X = k / X \oplus_q Y = n) = \frac{P(X = k) P(y = n \ominus_q k)}{P(X \oplus_q Y = n)}$$
$$= \frac{{}_qC_n^k [p]_q^k (1 \ominus_q [p]_q)^{n \ominus_q k} {}_qC_n^{n \ominus_q k} [p]_q^{n-k} (1 \ominus_q [p]_q)^k}{{}_qC_{2n}^n [p]_q^n (1 \ominus_q [p]_q)^n}$$
$$= \frac{{}_qC_n^k {}_qC_n^{n \ominus_q k} (1 \ominus_q [p]_q)^{n \ominus_q k} (1 \ominus_q [p]_q)^k}{{}_qC_{2n}^n (1 \ominus_q [p]_q)^n}$$
$$= \frac{{}_qC_n^k {}_qC_n^{n \ominus_q k}}{{}_qC_{2n}^n}.$$

Sufficiency. In Theorem 4.1, we take

$$c(x, x \oplus_q y) = \frac{{}_qC_x^n {}_qC_{z \ominus_q x}^n}{{}_qC_z^{2n}}.$$

Then,

$$h(x) = {}_qC_x^n, \quad k(y) = {}_qC_y^n, \quad f(x) = f(0) {}_qC_x^n E_q^{ax}, \quad g(y) = g(0) {}_qC_y^n E_q^{ay}.$$

Let $E_q^a = [\alpha]_q$. It is clear that

$$f(0) = (1 \oplus_q [\alpha])^{-n} \quad \text{and} \quad g(0) = (1 \oplus_q [\alpha]_q)^{-n}.$$

Therefore,

$$f(x) = {}_qC_x^n \theta^x (1 \ominus_q [\theta]_q)^{n \ominus_q x} \quad \text{and} \quad g(y) = {}_qC_y^n \theta^y (1 \ominus_q [\theta]_q)^{n \ominus_q y},$$

where

$$[\theta]_q = \frac{[\alpha]_q}{1 \oplus_q [\alpha]_q}.$$

From the last equality, we get

$$[\alpha]_q = \frac{[\theta]_q}{1 \ominus_q [\theta]_q},$$

which completes the proof. □

The characterization of the q-Poisson distribution is discussed in the following theorem.

Theorem 4.5. *Let X and Y be two positive independent real random variables. Then X and Y have q-Poisson distributions with parameter λ if and only if the conditional q-distribution of X given the total $X \oplus_q Y$ is a B-q-binomial distribution with a common parameter p.*

Proof. Necessity. Let X and Y have the q-Poisson distribution. By using Lemma 4.3, we get that $X \oplus_q Y$ has the q-Poisson distribution with parameter 2λ. Thus we obtain

$$\begin{aligned} P(X = k/X \oplus_q Y = n) &= \frac{P(X = k)P(y = n \ominus_q k)}{P(X \oplus_q Y = n)} \\ &= \frac{\frac{E_q^{-\lambda}\lambda^k}{[k]_q!}\frac{E_q^{-\lambda}\lambda^{n-k}}{[n-k]_q!}}{\frac{E_q^{-2\lambda}(2\lambda)^n}{[n]_q!}} = \frac{[n]_q!}{[k]_q![n-k]_q!2^n}. \end{aligned}$$

Hence, the conditional distribution of X given the total $X \oplus_q Y$ is B-q-Binomial distribution with parameter $p = 1/2$.

Sufficiency. The sufficient condition of this characterization is based on Theorem 4.1. Consider

$$\begin{aligned} c(x, x \oplus_q y) &= {}_qC_x^{x\oplus_q y} p^x (1 \ominus_q p)^{x\oplus_q y}(1 \ominus_q p)^{-x} \\ &= {}_qC_x^{x\oplus_q y} \left(\frac{p}{1 \ominus_q p}\right)^x (1 \ominus_q p)^{x\oplus_q y}. \end{aligned}$$

We can notice that

$$h(x) = \frac{1}{[x]_q!} \quad \text{and} \quad k(y) = \frac{\left(\frac{1\ominus_q p}{p}\right)^y}{[y]_q!}.$$

Therefore,

$$f(x) = f(0)\frac{E_q^{ax}}{[x]_q!} \quad \text{and} \quad g(y) = g(0)\left(\frac{1 \ominus_q p}{p}\right)^y \frac{E_q^{ay}}{[y]_q!}.$$

We suppose that

$$E_q^a = \lambda \quad \text{and} \quad \frac{(1 \ominus_q p)E_q^a}{p} = \mu.$$

Hence we get

$$f(x) = E_q^{-\lambda} \frac{\lambda^x}{[x]_q!} \quad \text{and} \quad g(y) = E_q^{-\mu} \frac{\mu^y}{[y]_q!}.$$

The theorem is proved. □

Conclusion

In this paper, we characterized the B-q-binomial and the q-Poisson distributions based on the q-addition operator and the conditional q-distribution. These results generalize the well-known Patil and Seshadri characterizations. The approval of the results is confirmed by comparing with the binomial distribution. The more q approaches 1, the more B-q-distribution approaches the binomial distribution. These results reveal the relationships between the discrete q-distributions and should be nominated as a primordial step in data processing and simulation studies.

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Характеризаційні теореми для B-q-біноміального і q-пуассонівского розподілів

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У цій роботі заново перевизначено і заново введено в компактній формі q-біноміальний і q-гіпергеометричний розподіли. Ці перевизначені розподіли названо B-q-біноміальним і B-q-гіпергеометричним. Крім того, узагальнення добре відомих характерізацій Патіла і Сешадрі наведено в q-аналізі. Характеризації B-q-біноміального і B-q-гіпергеометричного розподілів зображено з використанням умовного q-розподілу. Наведено необхідні і достатні умови, які визначають q-пуасонівський розподіл.

Ключові слова: q-аналіз, q-оператор додавання, характеризаційна теорема