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Trajectories of a Quadratic Differential Related to a Particular Algebraic Equation

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In this paper, we discuss the existence of a solution interpreted as the Cauchy transform of a signed measure of a particular algebraic quadratic equation of the form $z\mathcal{C}^2(z) - P(z)\mathcal{C}(z) + Q(z) = 0$ for some polynomials $P(z)$ and $Q(z)$. This issue requires the description of the critical graph of a related quadratic differential in the Riemann sphere $\overline{\mathbb{C}}$. In particular, we discuss the existence of finite critical trajectories of this quadratic differential.

Key words: quantum mechanics, WKB analysis, Cauchy transform, quadratic differentials

Mathematical Subject Classification 2010: 30C15, 31A35, 34E05

1. Introduction

Quadratic differentials have provided an important tool in the asymptotic study of some algebraic equations solutions. In quantum mechanics, trajectories of such quadratic differentials play a crucial role in the WKB analysis.

We consider the algebraic equation

$$z\mathcal{C}^2(z) - zP(z)\mathcal{C}(z) + Q(z) = 0, \tag{1.1}$$

where $P(z)$ and $Q(z)$ are two 1-degree real monic polynomials.

Notice that the above technique is the continuity to a series of papers related to the study of complex zeros of hyper-geometric polynomials, for example, [1,8].

In this paper, we discuss the existence of solutions of equation (1.1) as the Cauchy transform of compactly-supported signed measures. In Section 2, we describe the critical graphs of quadratic differentials $-\frac{q(z)}{z}\,dz^2$, where q is a polynomial of degree 3 in the Riemann sphere $\overline{\mathbb{C}}$, precisely, we discuss the existence and the number of its finite critical trajectories. In Section 3, we make the connection between the algebraic equation (1.1) and a particular quadratic differential among those studied in Section 2.

2. A quadratic differential

In the rest of this paper, we denote

$$\mathbb{C}_+ = \{z \in \mathbb{C} \mid \Im(z) > 0\}; \quad \mathbb{C}_- = \{z \in \mathbb{C} \mid \Im(z) < 0\}.$$

Below, we describe the critical graphs of the family of quadratic differentials

$$\varpi_a = -\frac{q(z)}{z}dz^2 = -\frac{(z-1)(z-a)(z-\overline{a})}{z}dz^2, \tag{2.1}$$

where $a \in \mathbb{C}_+$, and q is a monic polynomial of degree 3. We begin our investigation with some immediate observations from the theory of quadratic differentials. For more details, we refer the reader to [2,6,10].

Recall that *critical points* of a given quadratic differential $-Q(z)\,dz^2$ on the Riemann sphere $\overline{\mathbb{C}}$ are its zeros and poles; the multiplicity of a critical point is its multiplicity in the rational function Q in $\overline{\mathbb{C}}$. Zeros and simple poles are called *finite critical points*, while poles of order 2 or greater are the *infinite ones*. All other points of $\overline{\mathbb{C}}$ are called *regular points*.

Horizontal trajectories (or just trajectories) of the quadratic differential are the zero loci of the equation

$$-Q(z)\,dz^2 > 0,$$

or, equivalently,

$$\Re \int^z \sqrt{Q(t)}\,dt = \text{const}. \tag{2.2}$$

Knowing that if $z(t), t \in \mathbb{R}$, is a horizontal trajectory, we get that the function

$$t \mapsto \Im \int^t \sqrt{Q(z(u))}z'(u)\,du$$

is monotone.

The *vertical* (or *orthogonal*) trajectories are obtained by replacing $\Im$ by $\Re$ in equation (2.2). The horizontal and vertical trajectories produce two pairwise orthogonal foliations of the Riemann sphere $\overline{\mathbb{C}}$.

A trajectory passing through a critical point is called *critical*. In particular, if it starts and ends at finite critical points, it is called a *finite critical trajectory* or a *short trajectory*. Otherwise, we call it an *infinite critical trajectory*. The closure of the set of the finite and infinite critical trajectories is called the *critical graph*. A necessary condition for the existence of a short trajectory connecting two finite critical points is the existence of a Jordan arc γ connecting them such that

$$\Re \int_\gamma \sqrt{Q(t)}\,dt = 0. \tag{2.3}$$

However, this condition is not sufficient in general, see counter-examples in [11].

The local structure of such trajectories is as follows:

- At any regular point, horizontal (respectively, vertical) trajectories look locally as simple analytic arcs passing through this point, and through every regular point, a uniquely determined horizontal (respectively, vertical) trajectory passes; these horizontal and vertical trajectories are orthogonal at this point.
- From each zero of multiplicity r, there emanate $r+2$ critical trajectories spacing under equal angle $2\pi/(r+2)$.
- At a simple pole, there emanates exactly one horizontal trajectory.
- At the pole of order $r>2$, there are $r-2$ asymptotic directions (called *critical directions*) spacing under equal angle $2\pi/(r-2)$, and a neighborhood $\mathcal{U}$ such that each trajectory entering $\mathcal{U}$ stays in $\mathcal{U}$ and tends to the pole in one of the critical directions (see Figure 2.1).

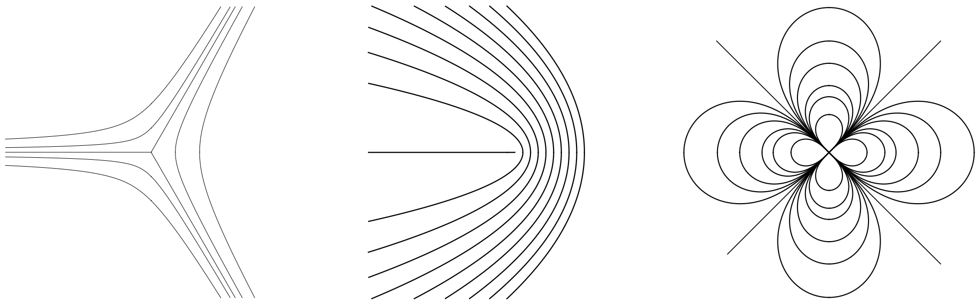

Fig. 2.1: Structure of the trajectories near a simple zero (left), a simple pole (center) and a pole of order 6 (right).

A very helpful tool that will be used in our investigation is the so-called Teichmüller Lemma (see [10, Theorem 14.1]).

Definition 2.1. A domain in $\overline{\mathbb{C}}$ bounded only by the segments of horizontal and/or vertical trajectories of ϖ_a (and their endpoints) is called a ϖ_a-polygon.

Lemma 2.2 (Teichmüller)**.** *Let Ω be a ϖ_a-polygon, and z_j be the critical points on the boundary $\partial\Omega$ of Ω, and let θ_j be the corresponding interior angles at vertices z_j, respectively. Then*

$$\sum\left(1-\frac{(n_j+2)\,\theta_j}{2\pi}\right)=2+\sum m_i, \tag{2.4}$$

where n_j are the multiplicities of z_j, and m_i the multiplicities of critical points inside Ω.

We have the following immediate observations:

- The finite critical points of ϖ_a are simple zeros $1, a, \overline{a}$ and a simple pole at 0.
- With the parametrization $u=1/z$, we get

$$\varpi_a(u)=\left(-\frac{1}{u^6}+\mathcal{O}\left(\frac{1}{u^5}\right)\right)du^2,\quad u\to 0.$$

Thus, infinity is an infinite critical point of ϖ_a,, namely a pole of order 6.

- Since the quadratic differential ϖ_a has two poles, Jenkins Three-pole Theorem (see [10, Theorem 15.2]) asserts that the situation of the so-called recurrent trajectory (whose closure might be dense in some domain in $\mathbb{C}$) cannot happen.

- Since ∞ is the only infinite critical point of ϖ_a, any critical trajectory which is not finite approaches ∞ following one of the 4 directions:

$$D_k = \left\{ z \in \mathbb{C} \mid \arg(z) = (2k+1)\frac{\pi}{4} \right\}, \quad k = 0, 1, 2, 3.$$

Similarly, for the orthogonal trajectories at ∞, though the critical directions are:

$$D_k^{\perp} = \left\{ z \in \mathbb{C} \mid \arg(z) = \frac{k\pi}{2} \right\}, \quad k = 0, 1, 2, 3.$$

Observe that if two trajectories approach ∞ in the same direction D_k, then there exists a neighborhood $\mathcal{V}$ of ∞, in which any orthogonal trajectory which traverses D_k in $\mathcal{V}$ necessarily traverses both of these two trajectories.

Lemma 2.3. *Two critical trajectories of ϖ_a emanating from the same zero cannot diverge to ∞ with the same critical direction.*

Lemma 2.4. *For any $a \in \mathbb{C}_+$, condition (2.3) is fulfilled for ϖ_a between the pairs of finite critical points $(0,1)$ and $(a, \overline{a})$. More generally, let α, β, and γ be three complex numbers $(\gamma \neq 0)$. If the quadratic differential*

$$\varpi_q = -\frac{q(z)}{z}dz^2 = -\frac{z^3 + \alpha z^2 + \beta z + \gamma}{z}dz^2$$

has two short trajectories connecting two distinct pairs of finite critical points, then

$$\Im\left(\alpha^2 - 4\beta\right) = 0.$$

In order to study the critical graph of ϖ_a, we introduce the set

$$\Sigma = \left\{ z \in \mathbb{C} \mid \Re \int_0^z \sqrt{\frac{(t-1)(t-z)(t-\overline{z})}{t}}dt = 0 \right\}. \tag{2.5}$$

Obviously, here our main focus is only in the vanishing of the real part regardless of the choice of the branch-cut of the square root in the integrand.

The following statements will be proved in Section 4.

Lemma 2.5. *The set Σ is symmetric with respect to the real axis, and it is formed by the 3 Jordan arcs:*

- *the segment $[0,1]$,*

- *two curves $\Sigma^{\pm}$ emerging from $z = 1$ and diverging respectively to infinity in $\mathbb{C}_{\pm}$.*

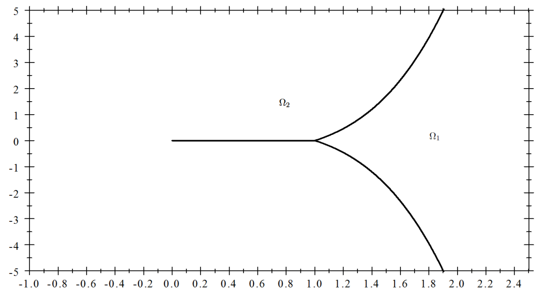

Fig. 2.2: Approximate plot of the curve Σ.

We give here the behavior of Σ at $z = 1$ and at ∞.

Lemma 2.6. *The following results hold:*

$$\lim_{\substack{z \to \infty \\ z \in \Sigma \cap \mathbb{C}^+}} \arg(z) = \frac{\pi}{2}, \qquad \lim_{\substack{z \to 1 \\ z \in \Sigma \cap \mathbb{C}^+}} \arg(z) = \frac{\pi}{3}.$$

From Lemma 2.5, Σ splits $\mathbb{C}$ into two connected domains (see Figure 2.2) :

- Ω_1 limited by $\Sigma^{\pm}$ and containing $z = 2$,
- $\Omega_2 = \mathbb{C} \setminus (\Omega_1 \cup \Sigma^{\pm} \cup [0, 1])$.

Proposition 2.7. *For any complex number $a \in \mathbb{C}_+$, the quadratic differential ϖ_a has:*

- *two short trajectories if $a \in \Omega_i$, $i = 1, 2$: the segment $[0, 1]$ and another one that connects a and $\overline{a}$ in Ω_i (see Figures 2.3,2.4);*
- *three short trajectories if $a \in \Sigma^{\pm}$: the segment $[0, 1]$ and two others that connect $z = 1$ with a and $\overline{a}$ that are symmetric with respect to the real axis (see Figure 2.5).*

Remark 2.8. The case $\varpi = -\frac{(z-a_1)(z-a_2)(z-a_2)}{z} dz^2$, with the zeros satisfying $0 < a_1 < a_2 < a_3$, is obvious. The segments $[0, a_1]$ and $[a_2, a_3]$ are the only two short trajectories (see Figure 2.6 (left)).

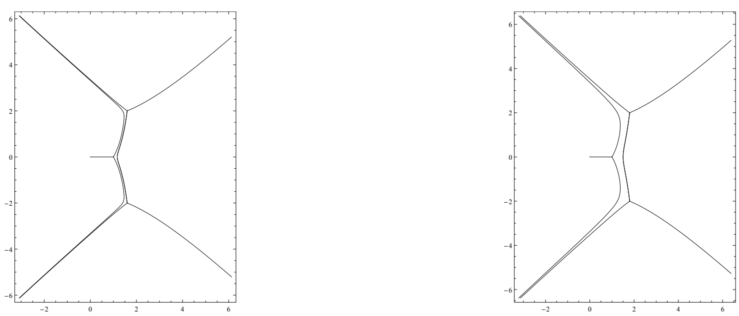

Fig. 2.3: Critical graphs when $a \in \Omega_1$, here $a = 1.6 + 2i$ (left) and $a = 1.8 + 2i$ (right).

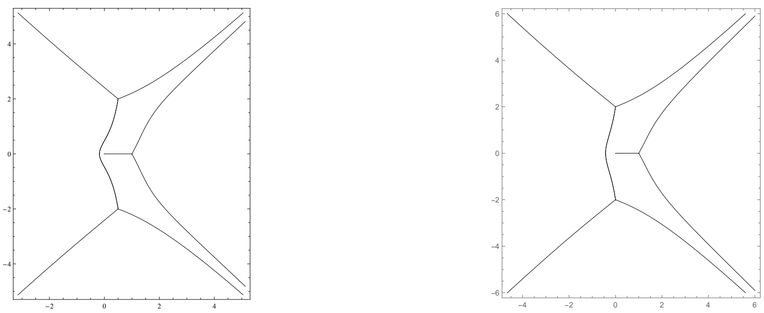

Fig. 2.4: Critical graphs when $a \in \Omega_2$, here $a = 0.5 + 2i$ (left) and $a = 2i$ (right).

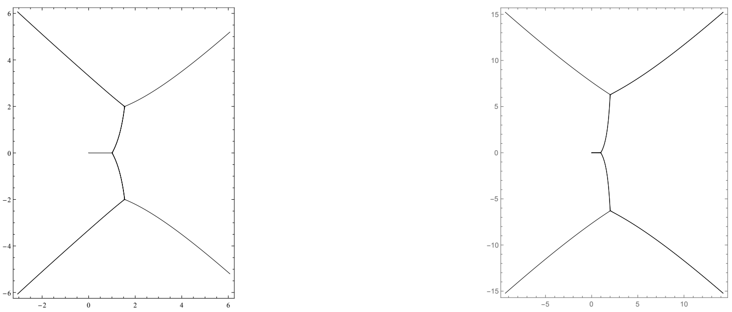

Fig. 2.5: Critical graphs when $a \in \Sigma$, here $a = 1.55 + 2i$ (left) and $a = 2 + 6.3i$ (right).

3. Connection with the algebraic equation

The Cauchy transform $\mathcal{C}_\nu$ of a compactly-supported complex Borel measure ν is defined in $\mathbb{C} \setminus \operatorname{supp}(\nu)$ by

$$\mathcal{C}_\nu(z) = \int_{\mathbb{C}} \frac{d\nu(t)}{z - t}.$$

It has the asymptotics

$$\mathcal{C}_\nu(z) = \frac{\nu(\mathbb{C})}{z} + \mathcal{O}\left(z^{-2}\right), z \to \infty,$$

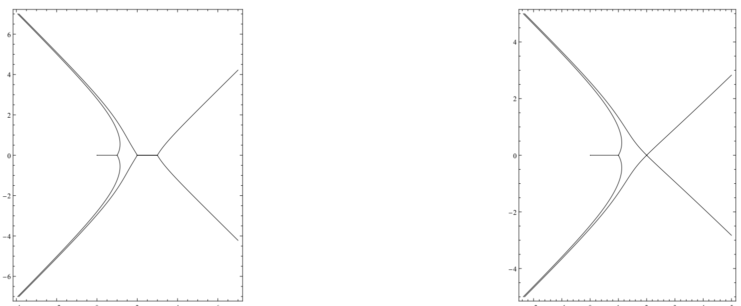

Fig. 2.6: Critical graphs for ϖ_q when $q = (z-1)(z-2)(z-3)$ (left) and $q = (z-1)(z-2)^2$ (right).

and the inversion formula (which should be understood in the distributional sense) reads as

$$\nu = \frac{1}{\pi}\frac{\partial \mathcal{C}_\nu}{\partial \overline{z}}. \tag{3.1}$$

In particular, the normalized root-counting measure $\nu_n = \nu(P_n)$ of a given complex polynomial P_n of degree n in $\mathbb{C}$ is defined by

$$\nu_n = \frac{1}{n}\sum_{P_n(a)=0} \delta_a \quad \text{(each zero is counted with its multiplicity)}.$$

Its Cauchy transform is

$$\mathcal{C}_{\nu_n}(z) = \int_{\mathbb{C}} \frac{d\nu_n(t)}{z-t} = \frac{P_n'(z)}{nP_n(z)}.$$

While going back to the algebraic equation (1.1), we are seeking a solution everywhere in $\mathbb{C}$ as a Cauchy transform $\mathcal{C}_\nu$ of a compactly-supported signed measure ν. With the choice of the square root of the discriminant

$$\Delta(z) = z\left(zP^2(z) - 4Q(z)\right)$$

of the quadratic equation (1.1) with condition

$$\sqrt{\Delta(z)} \sim z^2, \quad z \to \infty,$$

through an easy check, we have

$$\mathcal{C}(z) = \frac{zP(z) - \sqrt{\Delta(z)}}{2z} = \frac{1}{z} + \mathcal{O}\left(z^{-2}\right), \quad z \to \infty.$$

Relying on the so-called Plemelj–Sokhotsky formula, it is well known that the measure ν lies in finite critical trajectories of the quadratic differential $-\frac{\Delta(z)}{z^2}\,dz^2$. The measure ν is given explicitly by

$$d\nu(t) = \frac{1}{2i\pi}\frac{\left(\sqrt{\Delta(t)}\right)_+}{t}dt.$$

For more details, we refer the reader to [3,4,7,9].

In the sequel, for general 1-degree real monic polynomials $P(z)$ and $Q(z)$, we may assume that

$$\Delta(z) = z(z-1)(z-a)(z-\overline{a}), \quad a \in \mathbb{C}_+.$$

The following lemma gives a sufficient condition for a solution of (1.1) to be the Cauchy transform of some compactly-supported measure in $\mathbb{C}$.

Lemma 3.1 ([5, Chap. II, Theorem 1.2])**.** *Suppose $f \in L^1_{\text{loc}}(\mathbb{C})$ and that $f(z) \to 0$ as $z \to \infty$, and let μ be a compactly-supported measure in $\mathbb{C}$ such that*

$$\mu = \frac{1}{\pi}\frac{\partial f}{\partial \overline{z}}$$

in the sense of distributions. Then $f(z) = \mathcal{C}_\mu(z)$ almost everywhere in $\mathbb{C}$.

Now we announce a theorem summarising the main finding of this paper.

Theorem 3.2. *For general 1-degree real monic polynomials $P(z)$ and $Q(z)$, algebraic equation (1.1) has always a solution interpreted as a Cauchy transform of a signed measure ν, supported on the short trajectories $[0,1]$ and γ_a of the quadratic differential ϖ_a, and is given explicitly by*

$$d\nu(t) = \frac{1}{2i\pi} \frac{\left(\sqrt{\Delta(t)}\right)_+}{t} dt. \tag{3.2}$$

The measure ν is of density 1 if and only if

$$\Re a + (\Im a)^2 + \frac{15}{4} = 0.$$

If equality holds, then ν is negative on γ_a.

4. Proofs

Proof of Lemma 2.3. Suppose that γ_1 and γ_2 are two critical trajectories emanating from the zero $z_j \in \{a, 1\}$ and diverging to ∞ with the same critical direction D_k. Consider the ϖ_a-polygon with edges γ_1 and γ_2, and vertices z_j and ∞. With the notations of Lemma 2.2, we have

$$\beta_j = \begin{cases} 0 & \text{if } \theta_j = 2\pi/3, \\ -1 & \text{if } \theta_j = 4\pi/3, \end{cases}, \quad \beta_\infty = 1, \ \sum m_i \geq -1,$$

which violates (2.4). □

Proof of Lemma 2.4. Since $\frac{q(t)}{t}$ is a real rational function, then

$$\overline{\sqrt{\frac{q(t)}{t}}} = \sqrt{\frac{q(\bar{t})}{\bar{t}}}, \quad t \neq 0. \tag{4.1}$$

So, after changing the variable $u = \bar{t}$ in the second integral, we get

$$\begin{aligned} \Re\left(\int_{\bar{z}}^{z} \sqrt{\frac{q(t)}{t}} dt\right) &= \Re\left(\int_1^z \sqrt{\frac{q(t)}{t}} dt - \int_1^{\bar{z}} \sqrt{\frac{q(t)}{t}} dt\right) \\ &= \Re\left(\int_1^z \sqrt{\frac{q(t)}{t}} dt - \overline{\int_1^z \sqrt{\frac{q(t)}{t}} dt}\right) \\ &= \Re\left(2i\Im\left(\int_1^z \sqrt{\frac{q(t)}{t}} dt\right)\right) = 0. \end{aligned}$$

Let us provide a necessary condition to get two short trajectories joining two different pairs of finite critical points in the general case of the quadratic differential with simple zeros

$$\varpi_q = -\frac{q(z)}{z} dz^2 = -\frac{z^3 + \alpha z^2 + \beta z + \gamma}{z} dz^2, \quad \alpha, \beta, \gamma \in \mathbb{C}, \ \gamma \neq 0.$$

Considering two disjoint oriented Jordan arcs γ_1 and γ_2 connecting two distinct pairs of finite critical points, we define the single-valued function $\sqrt{\frac{q(z)}{z}}$ in $\mathbb{C} \setminus (\gamma_1 \cup \gamma_2)$ with the asymptotics $\sqrt{\frac{q(z)}{z}} \sim z, z \to \infty$. For $s \in \gamma_1 \cup \gamma_2$, we denote by $\left(\sqrt{\frac{q(s)}{s}}\right)_+$ and $\left(\sqrt{\frac{q(s)}{s}}\right)_-$ the limits from the $+$ and $-$ sides, respectively. (As usual, the $+$ side of an oriented curve lies to the left and the $-$ side lies to the right if one traverses the curve according to its orientation.)

From the Laurent series of $\sqrt{q(z)}$ at ∞, we obtain

$$\sqrt{\frac{q(z)}{z}} = z + \frac{\alpha}{2} - \left(\frac{\alpha^2 - 4\beta}{8z}\right) + \mathcal{O}\left(z^{-2}\right).$$

Therefore, the residue of $\sqrt{\frac{q(z)}{z}}$ at $z = \infty$ is given by

$$\operatorname{res}_{z=\infty}\left(\sqrt{\frac{q(z)}{z}}\right) = \frac{1}{8}\left(\alpha^2 - 4\beta\right).$$

Let

$$I = \int_{\gamma_1}\left(\sqrt{\frac{q(s)}{s}}\right)_+ ds + \int_{\gamma_2}\left(\sqrt{\frac{q(s)}{s}}\right)_+ ds.$$

Since

$$\left(\sqrt{\frac{q(s)}{s}}\right)_+ = -\left(\sqrt{\frac{q(s)}{s}}\right)_-, \quad s \in \gamma_1 \cup \gamma_2,$$

we have

$$2I = \int_{\gamma_1 \cup \gamma_2}\left[\left(\sqrt{\frac{q(s)}{s}}\right)_+ - \left(\sqrt{\frac{q(s)}{s}}\right)_-\right] ds = \oint_\Gamma \sqrt{\frac{q(z)}{z}}\, dz,$$

where Γ is a closed contour encircling the curves γ_1 and γ_2. After a deformation of the contour, we pick up the residue at $z = \infty$ and get

$$I = \frac{1}{2}\oint_\Gamma \sqrt{\frac{q(z)}{z}} dz = \pm i\pi \operatorname{res}_{t=\infty}\left(\sqrt{\frac{q(z)}{z}}\right) = \pm\frac{\pi i}{8}\left(\alpha^2 - 4\beta\right).$$

A necessary condition is

$$\Im\left(\alpha^2 - 4\beta\right) = 0,$$

which is satisfied for $q = (z-1)(z-a)(z-\overline{a})$. □

Proof of Lemma 2.5. Obviously, $\Sigma \cap \mathbb{R} = [0, 1]$. The observation (4.1) shows that Σ is symmetric with respect to the real axis. In order to prove that Σ is a curve, we consider the real functions F and G (locally) defined for (x, y) in $\mathbb{C}_+$ by the formulas

$$F(x, y) = \Re\left(\int_0^x \sqrt{\frac{(u - (x + iy))(u - (x - iy))(u - 1)}{u}}\, du\right)$$

$$= \Re\left(\int_0^x \sqrt{\frac{\left((u-x)^2+y^2\right)(u-1)}{u}}\,du\right),$$

$$G(x,y) = \Re\left(\int_x^{x+iy} \sqrt{\frac{(u-(x+iy))(u-(x-iy))(u-1)}{u}}\,du\right)$$

$$= -\int_0^1 y^2\sqrt{1-t^2}\,\Im\sqrt{1-\frac{1}{x+ity}}\,dt.$$

The square roots are chosen with condition $\sqrt{X}>0$ for $X>0$.

Define

$$\Sigma = \left\{(x,y)\in\mathbb{R}^2 \mid (F+G)(x,y)=0\right\}.$$

Let us prove that

$$\Sigma\setminus[0,1]\subset\{z\in\mathbb{C}\mid \Re z>1\}.$$

Indeed, it is straightforward to check that $F(x,y)=0$ if $0\le x\le 1$ and $y>0$, and $F(x,y)\le 0$ if $x<0$ and $y>0$. On the other hand, taking the argument in $[0,2\pi[$, for $0<t\le 1$, we have

$$0<\arg(x+ity)<\arg(x-1+ity)<\pi. \tag{4.2}$$

Therefore,

$$0<\arg\left(1-\frac{1}{x+ity}\right)<\pi,$$

implying that

$$\Im\sqrt{1-\frac{1}{x+ity}}>0.$$

Thus,

$$G(x,y)<0.$$

Hence,

$$(F+G)(x,y)\le 0+G(x,y)<0,\quad x\le 1,\ y>0.$$

As a result, $(x,y)\notin\Sigma$.

Then we prove that Σ is a curve, subset of

$$\Pi=\{(x,y)\mid x>1,\ y>0\}.$$

For $x>1$, we have

$$\frac{\partial F}{\partial x}(x,y)=\sqrt{\frac{y^2(x-1)}{x}}+\int_1^x\frac{(x-u)(u-1)}{\sqrt{\left((u-x)^2+y^2\right)(u-1)u}}\,dt>0.$$

In addition, for $u_t=x+ity$, $t\in[0,1]$, we have

$$\frac{\partial G}{\partial x}(x,y)=\frac{\partial}{\partial x}\left[\Re\left(\int_0^1 iy^2\sqrt{1-t^2}\sqrt{1-\frac{1}{u_t}}\,dt\right)\right]$$

$$= -\int_0^1 \frac{y^2\sqrt{1-t^2}}{2} \Im \left(\frac{1}{u_t^2\sqrt{1-\frac{1}{u_t}}} \right) dt.$$

It suffices to check that

$$\forall t \in [0,1] \quad \Im \left(\frac{1}{u_t^2\sqrt{1-\frac{1}{u_t}}} \right) \leq 0,$$

which is equivalent to proving that

$$\forall t \in [0,1] \quad \arg \left(\frac{1}{u_t^2\sqrt{1-\frac{1}{u_t}}} \right) \in [\pi, 2\pi[\,,$$

where the argument is taken in $[0, 2\pi[$. From (4.2), for any $t \in [0,1]$, we get

$$\arg \left(\frac{1}{u_t^2\sqrt{1-\frac{1}{u_t}}} \right) = 2\pi - \left(\frac{3}{2} \arg(u_t) + \frac{1}{2} \arg(u_t - 1) \right) \in \,]\pi, 2\pi[\,.$$

We deduce that for any $t \in [0,1]$,

$$\Im \left(\frac{1}{u_t^2\sqrt{1-\frac{1}{u_t}}} \right) \leq 0,$$

and then

$$\frac{\partial G}{\partial x}(x,y) \geq 0.$$

We have just shown that

$$\frac{\partial (F+G)}{\partial x}(x,y) \neq 0, \quad (x,y) \in \Sigma \cap \Pi.$$

Finally, we conclude that the set Σ is a curve in $\mathbb{C}$ by applying the Implicit Function Theorem to the function $F+G$. □

Proof of Lemma 2.6. Take $z = re^{ix} \in \Sigma$, $r > 1$, $x \in \left[0, \frac{\pi}{2}\right]$. After the change of variable $t = sre^{ix}$, we get

$$\Re \left(e^{2ix} \int_0^1 \sqrt{\frac{\left(s - \frac{1}{r}e^{-ix}\right)(s-1)\left(s - e^{-2ix}\right)}{s}}\, ds \right) = 0.$$

Taking the limits when $r \to \infty$, we obtain

$$0 = \Re \int_0^1 e^{2ix} \sqrt{(s-1)\left(s - e^{-2ix}\right)}. \tag{4.3}$$

Trivially, $x \neq 0$. With the change of variable $t = \alpha u + \beta$, where

$$\beta = \frac{1+e^{-2ix}}{2}, \quad \alpha = i\frac{1-e^{-2ix}}{2},$$

equation (4.3) becomes

$$\begin{aligned} 0 &= \Re\left(\int_{\cot x}^{i} \sqrt{u^2+1}\, du\right) \\ &= \Re\left(\int_{\cot x}^{0} \sqrt{u^2+1} du + \int_0^i \sqrt{u^2+1}\, du\right) = \Re\left(\int_{\cot x}^{0} \sqrt{u^2+1}\, du\right), \end{aligned}$$

which holds if and only if $x = \frac{\pi}{2}$.

The Laurent series of $\sqrt{\frac{(t-1)(t-z)(t-\overline{z})}{t}}$ as $t \to 1$ (with the appropriate choice of the branch-cut of the square root) is

$$\sqrt{\frac{(t-1)(t-z)(t-\overline{z})}{t}} = |z-1|\sqrt{t-1} + o\left((t-1)^{\frac{1}{2}}\right).$$

We conclude that

$$0 = \lim_{z\to 1, z\in\Sigma^+} \Re\int_1^z \sqrt{\frac{(t-1)(t-z)(t-\overline{z})}{t}}\, dt = \frac{2}{3}|z-1|\,\Re(z-1)^{\frac{3}{2}},$$

and then

$$\arg(z-1)^{\frac{3}{2}} \equiv \frac{\pi}{2} \operatorname{mod}(\pi),$$

which ends the proof. □

Proof of Proposition 2.7. Clearly, the segment $[0,1]$ is always a short trajectory of ϖ_a. If $a \notin \Sigma$, then, from (2.3), there is no short trajectory connecting a to 0 or 1. By Lemma 2.3, there exist at most two critical trajectories emanating from a and approaching ∞ in the upper half-plane $\mathbb{C}^+$. Using the symmetry with respect to the real axis, at least one critical trajectory emanating from a meets a critical trajectory emanating from $\overline{a}$ somewhere at $b \in \mathbb{R} \setminus [0,1]$. Since b cannot be a zero of the quadratic differential ϖ_a, we conclude that these two critical trajectories form a short one.

If $a \in \Sigma$ and there is no short trajectory connecting a to 1, then there exist two critical trajectories γ_a and γ_1 emanating respectively from a and 1 and approaching ∞ in the same critical direction D_k. From the behavior of orthogonal trajectories at ∞, we can take an orthogonal trajectory σ that hits γ_1 and γ_a respectively in two points b and c (there are infinitely many such orthogonal trajectories σ). We consider a path γ connecting 1 and a, formed by the part of γ_1 from 1 to b, the part of σ from b to c, and the part of γ_a from c to a. Then, integrating along γ, we have

$$\Re\int_\gamma \sqrt{\frac{q(t)}{t}}\, dt = \Re\int_1^b \sqrt{\frac{q(t)}{t}}\, dt + \Re\int_b^c \sqrt{\frac{q(t)}{t}} dt + \Re\int_c^a \sqrt{\frac{q(t)}{t}}\, dt$$

$$= \Re \int_b^c \sqrt{\frac{q(t)}{t}}\, dt \neq 0,$$

which violates the fact that $a \in \Sigma$. □

Proof of Theorem 3.2. We consider the case where the discriminant of algebraic equation (1.1) is

$$\Delta(z) = z(z-1)(z-a)(z-\overline{a})$$

for some $a \in \mathbb{C}_+$. Let us suppose first that $a \notin \Sigma$ and denote by γ_a the short trajectory joining $\overline{a}$ to a. The segment $[0,1]$ and γ_a are positively oriented respectively from 0 to 1, and from $\overline{a}$ to a. As in the proof of Lemma 2.4, these orientations define the + and −-sides with respect to the curves $[0,1]$ and γ_a. We choose the square root $\sqrt{\Delta(z)}$ in $\mathbb{C} \setminus ([0,1] \cup \gamma_a)$ with asymptotics $\sqrt{\Delta(z)} \sim z^2$ as $z \to \infty$.

From the proof of Lemma 2.4, we have

$$\begin{aligned} \nu(\mathbb{C}) &= \int_{[0,1]\cup\gamma_a} d\nu(t) = \frac{1}{2i\pi} \int_{[0,1]\cup\gamma_a} \frac{\left(\sqrt{\Delta(t)}\right)_+}{t}\, dt \\ &= \frac{1}{16}\left(\alpha^2 - 4\beta\right) = -\frac{1}{4}\Re a - \frac{1}{4}(\Im a)^2 + \frac{1}{16}. \end{aligned}$$

We obtain a necessary and sufficient condition on the zero $a = x + iy$, $x \in \mathbb{R}$, $y \geq 0$, to get $\nu(\mathbb{C}) = 1$,

$$\nu(\mathbb{C}) = 1 \Longleftrightarrow -y^2 - \frac{15}{4} = x. \tag{4.4}$$

Observe that condition (4.4) cannot hold for $a \in \Sigma$. The expression of the measure ν on $[0,1]$ is

$$d\nu(t)_{|[0,1]} = \frac{1}{2i\pi} \frac{\left(\sqrt{\Delta(t)}\right)_+}{t}\, dt = \frac{1}{2\pi} \frac{\sqrt{t(1-t)(t-a)(t-\overline{a})}}{t}\, dt,$$

which obviously implies that it is positive in $[0,1]$. In order to prove that ν is a non-positive measure in γ_a, we consider the function $f(y)$ defined for $y \geq 0$ by

$$\begin{aligned} f(y) = \nu([0,1]) &= \frac{1}{2\pi} \int_0^1 \frac{\sqrt{t(1-t)(t-a)(t-\overline{a})}}{t}\, dt \\ &= \frac{1}{2\pi} \int_0^1 \frac{\sqrt{t(1-t)\left(t^2 + \left(2y^2 + \frac{15}{2}\right)t + y^4 + \frac{17}{2}y^2 + \frac{225}{16}\right)}}{t}\, dt. \end{aligned}$$

An easy study shows that $f(y)$ increases from $f(0)$ to $\lim_{y\to+\infty} f(y) = +\infty$. By the other hand,

$$f(0) = \frac{1}{2\pi} \int_0^1 \frac{\sqrt{t(1-t)\left(t^2 + \frac{15}{2}t + \frac{225}{16}\right)}}{t}\, dt = \frac{1}{2\pi} \int_0^1 \frac{\left(t + \frac{15}{4}\right)\sqrt{t(1-t)}}{t}\, dt$$

$$= \frac{1}{2\pi}\left(\mathrm{B}\left(\frac{3}{2},\frac{3}{2}\right) + \frac{15}{4}\mathrm{B}\left(\frac{1}{2},\frac{3}{2}\right)\right) = \frac{8}{\pi}\mathrm{B}\left(\frac{3}{2},\frac{3}{2}\right) = 1.$$

We conclude for every $a \in \mathbb{C}_+$ satisfying (4.4) that

$$\nu([0,1]) > 1,$$

and thus the measure ν cannot be positive on γ_a. □

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Траєкторії квадратичного диференціала, пов'язаного з деяким алгебраїчним рівнянням

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У цій статті ми обговорюємо існування розв'язку, інтерпретованого як перетворення Коші деякого заряду, алгебраїчного квадратичного рівняння вигляду $z\mathcal{C}^2(z) - P(z)\mathcal{C}(z) + Q(z) = 0$ для деяких поліномів $P(z)$ та $Q(z)$. Ця проблема потребує опису критичного графу відповідного квадратичного диференціала на сфері Рімана $\overline{\mathbb{C}}$. Зокрема, ми обговорюємо існування скінченних критичних траєкторій цього квадратичного диференціала.

Ключові слова: квантова механіка, аналіз WKB, перетворення Коші, квадратичні диференціали