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Hopf Hypersurfaces in Complex Two-Plane Grassmannians with GTW Killing Shape Operator

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In this paper, we prove that there are no Hopf hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ with Killing shape operator with respect to the generalized Tanaka–Webster connection.

Key words: Hopf hypersurface, complex 2-plane Grassmannian, Killing shape operator, generalized Tanaka–Webster connection

Mathematical Subject Classification 2010: 53C40, 53C15

1. Introduction

A complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ is defined as the set of all two-dimensional linear subspaces in $\mathbb{C}^{m+2}$ which is identified with the homogeneous space $SU(m+2)/S(U(2)\times U(m))$. Throughout this paper, we assume that $m \geq 3$. A complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ is known as a compact irreducible Hermitian symmetric space of rank two equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ with a canonical basis $\{J_1, J_2, J_3\}$ which does not contain J (see [1]). Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with N and A a unit normal vector field and the shape operator respectively, and g and ∇ be the induced metric and the corresponding Levi-Civita connection on M, respectively. In general, $\xi := -JN$ is said to be the structure or Reeb vector field. The almost contact metric 3-structure vector fields ξ_ν are defined by $\xi_\nu = -J_\nu N$ for $\nu \in \{1, 2, 3\}$. We denote by $\mathfrak{D}^\perp$ the distribution defined by $\mathfrak{D}^\perp = \operatorname{Span}\{\xi_1, \xi_2, \xi_3\}$ and $\mathfrak{D}$ its orthogonal complement distribution satisfying $T_pM = \mathfrak{D}_p \oplus \mathfrak{D}_p^\perp$ at each point $p \in M$. A real hypersurface in $G_2(\mathbb{C}^{m+2})$ is said to be Hopf if ξ is an eigenvector field of the shape operator, i.e., $A\xi = \alpha\xi$ and $\alpha = g(A\xi, \xi)$ is said to be the Hopf principal curvature.

The above definition for Hopf hypersurfaces is still valid if the ambient space is a complex space form, i.e., a complete and simply connected Kähler manifold of constant holomorphic sectional curvature c. For the first time the Hopf hypersurfaces was studied in a complex space form by T.E. Cecil and P.J. Ryan in [6]. In this paper, both local and global structures of Hopf hypersurfaces in complex space forms with constant rank of the focal map were investigated. We remark that complete description of the global structures of Hopf hypersurfaces in complex space forms without restriction on the rank of the focal map was obtained by A.A. Borisenko in [5]. Now back to our subject, classification result for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ under certain condition was obtained by J. Berndt and Y.J. Suh in [2].

Theorem 1.1 ([2])**.** *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then $\mathfrak{D}^\perp$ is invariant under the shape operator if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic quaternionic projective space $\mathbb{H}P^m$ in $G_2(\mathbb{C}^{m+2})$.*

With the help of Theorem 1.1, many characterization theorems for Hopf hypersurfaces listed above were obtained. Among others, we focus on those characterizations in terms of the extrinsic properties such as the shape operator. It is well known that there are no real hypersurfaces in nonflat complex space forms with parallel shape operators (see [7]). Similar situation was considered by Y. J. Suh in [20] in which the author proved that there are no real hypersurfaces in a complex two-plane Grassmannian with parallel shape operators. Following this step, some other generalizations of results in [20] were obtained. For example, Y. J. Suh in [21] proved that there do not exist such kinds of real hypersurfaces in complex two-plane Grassmannians with parallel second fundamental tensor on a distribution defined by $\mathfrak{F} = \xi \cup \mathfrak{D}^\perp$, where $\mathfrak{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$. Recently, Jang, Suh and Woo in [12] proved that there exist no real hypersurfaces in complex two-plane Grassmannians with Killing shape operator, i.e., $(\nabla_X A)Y + (\nabla_Y A)X = 0$ for any vector fields X, Y. Obviously, this extends Suh's results in [20] because a parallel shape operator must be of Killing type.

The so called Tanaka–Webster connection, introduced independently by N. Tanaka in [23] and S.W. Webster in [25], is a unique affine connection on a non-degenerate pseudo-Hermitian CR-manifold. S. Tanno in [24] introduced the notion of the generalized Tanaka–Webster connection (in short, GTW connection) on a contact Riemannian manifold and such a connection is the same with the Tanaka–Webster connection when the associated CR-structure is integrable. J.T. Cho in [8, 9] introduced the generalized Tanaka–Webster connection on real hypersurfaces in Kähler manifolds, that is,

$$\widehat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi A X, Y)\xi - \eta(Y)\phi A X - k\eta(X)\phi Y \tag{1.1}$$

for any vector fields X, Y and certain non-zero constant k, where ∇ is the Levi-Civita connection of the hypersurface. The generalized Tanaka–Webster connection on real hypersurfaces coincides with the Tanaka–Webster connection when $\phi A + A\phi = 2k\phi$. Together with (1.1), we have

Theorem 1.2 ([14])**.** *There exist no Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $\alpha \neq 2k$, such that the shape operator is parallel with respect to the generalized Tanaka–Webster connection.*

In this paper, we consider Killing type shape operator with respect to the generalized Tanaka–Webster connection for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ which is defined by $(\widehat{\nabla}^{(k)}_X A)X = 0$ for any vector field X and this is also equivalent to

$$\left(\widehat{\nabla}^{(k)}_X A\right) Y + \left(\widehat{\nabla}^{(k)}_Y A\right) X = 0 \tag{1.2}$$

for any vector fields X, Y. The notion of Killing tensor fields with respect to the Levi-Civita connection was first introduced by D.E. Blair in [4]. By applying such a notion, in [4] a characterization for an almost contact metric manifold to be a cosymplectic manifold was obtained. In the present paper, as an corollary of our main result (cf. Theorem 3.8), we have

Theorem 1.3. *There exist no Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $\alpha \neq 2k$, such that the shape operator is Killing with respect to the generalized Tanaka–Webster connection.*

As a parallel shape operator with respect to the generalized Tanaka–Webster connection must be of Killing type, but in general the converse is not necessarily true. Thus we have

Remark 1.4. Theorem 1.3 extends Theorem 1.2 in [14].

Before giving detailed proof of Theorem 1.3 in Section 3, we consider a condition weaker than (1.2) and obtain another characterization for Hopf hypersurfaces of type (A) in $G_2(\mathbb{C}^{m+2})$.

2. Preliminaries

In this section, we collect some fundamental formulas shown in [1–3, 22]. Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with real codimension one and N be a unit normal vector field. On M there exists an almost contact metric structure (ϕ, ξ, η, g) induced from the Kähler structure J of $G_2(\mathbb{C}^{m+2})$. Let $\{J_1, J_2, J_3\}$ be a canonical local basis of quaternionic Kähler structure $\mathfrak{J}$ of $G_2(\mathbb{C}^{m+2})$. In this paper we put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N \tag{2.1}$$

for any vector field X, $\nu \in \{1, 2, 3\}$. From the first term of (2.1), it follows that

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(X) = g(X, \xi), \tag{2.2}$$

where the Reeb vector field ξ is determined by $\xi := -JN$. According to the condition $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$ we have an almost contact metric 3-structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ as the following

$$\begin{aligned}
&\phi_\nu^2 = -\mathrm{id} + \eta_\nu \otimes \xi_\nu, && \eta_\nu(\xi_\nu) = 1, && \phi_\nu\xi_\nu = 0,\\
&\phi_\nu\xi_{\nu+1} = \xi_{\nu+2}, && \phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, &&\\
&\phi_\nu\phi_{\nu+1} = \phi_{\nu+2} + \eta_{\nu+1} \otimes \xi_\nu, && &&
\end{aligned}$$

$$\phi_{\nu+1}\phi_\nu = -\phi_{\nu+2} + \eta_\nu \otimes \xi_{\nu+1}, \tag{2.3}$$

where the index is taken modulo three. According to condition $J_\nu J = JJ_\nu$, the relationships between two almost contact metric structures are given by

$$\begin{aligned} &\phi\phi_\nu = \phi_\nu\phi + \eta_\nu \otimes \xi - \eta \otimes \xi_\nu, \\ &\phi\xi_\nu = \phi_\nu\xi, \quad \eta_\nu(\phi X) = \eta(\phi_\nu X) \end{aligned} \tag{2.4}$$

for any vector field X. Because J is parallel with respect to the Riemannian connection of $G_2(\mathbb{C}^{m+2})$, we have

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX,Y)\xi, \quad \nabla_X\xi = \phi AX \tag{2.5}$$

for any vector fields X and Y, where we have applied the Gauss and Weingarten formulas. The Codazzi equation for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ is given by

$$\begin{aligned} (\nabla_X A)Y - (\nabla_Y A)X = {}& \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi \\ &+ \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X,Y)\xi_\nu\} \\ &+ \sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\} \\ &+ \sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\}\xi_\nu \end{aligned} \tag{2.6}$$

for any vector fields X, Y.

3. Proof of Theorem 1.3

Suppose M is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$. On M, according to (1.1), by a direct calculation we have

$$\begin{aligned} \left(\widehat{\nabla}^{(k)}_X A\right) Y = {}& (\nabla_X A)Y + g(\phi AX, AY)\xi - \eta(AY)\phi AX \\ &- k\eta(X)\phi AY - g(\phi AX, Y)A\xi + \eta(Y)A\phi AX + k\eta(X)A\phi Y \end{aligned} \tag{3.1}$$

for any vector fields X, Y. Suppose the shape operator is Killing with respect to the generalized Tanaka–Webster connection, from (1.2) we obtain $\left(\widehat{\nabla}^{(k)}_\xi A\right)\xi = 0$. Applying this on (3.1), with the help of (2.5) and $A\xi = \alpha\xi$, we obtain

$$\xi(\alpha)\xi = 0, \tag{3.2}$$

where $\alpha = \eta(A\xi)$. Taking the inner product of (3.2) with ξ gives $\xi(\alpha) = 0$.

It has been proved in [16] that for a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ the Hopf principal curvature is invariant along the Reeb flow, i.e., $\xi(\alpha) = 0$ if and only if the $\mathfrak{D}$ and $\mathfrak{D}^\perp$-components of the Reeb vector field are invariant under the shape operator. Now we have

Lemma 3.1. *If a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ has Killing shape operator with respect to the generalized Tanaka–Webster connection, then the $\mathfrak{D}$ and $\mathfrak{D}^\perp$-components of the Reeb vector field are invariant under the shape operator.*

Before showing the next key lemma, we need the following

Lemma 3.2 ([3]). *If M is a connected and oriented Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, then we have*

$$\operatorname{grad}\alpha = \xi(\alpha)\xi + 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu\xi, \tag{3.3}$$

$$\begin{aligned}2A\phi AX7 = \alpha A\phi X + \alpha\phi AX + 2\phi X + 2\sum_{\nu=1}^{3}\{&\eta_\nu(X)\phi\xi_\nu + \eta_\nu(\phi X)\xi_\nu \\ &+ \eta_\nu(\xi)\phi_\nu X - 2\eta(X)\eta_\nu(\xi)\phi\xi_\nu - 2\eta_\nu(\phi X)\eta_\nu(\xi)\xi\}\end{aligned} \tag{3.4}$$

for any vector field X, where grad *denotes the gradient operator.*

Replacing X by ξ and Y by X in (3.1) respectively, with the aid of $A\xi = \alpha\xi$, we obtain

$$\left(\widehat{\nabla}^{(k)}_\xi A\right)X = (\nabla_\xi A)X - k\phi AX + kA\phi X.$$

Similarly, replacing Y by ξ in (3.1) and using $A\xi = \alpha\xi$, we obtain

$$\left(\widehat{\nabla}^{(k)}_X A\right)\xi = (\nabla_X A)\xi - \alpha\phi AX + A\phi AX$$

for any vector field X. Form now on suppose that the shape operator is of Killing type with respect to the generalized Tanaka–Webster connection. According to (1.2) and the above two equations we get

$$(\nabla_\xi A)X + (\nabla_X A)\xi - (k+\alpha)\phi AX + kA\phi X + A\phi AX = 0$$

for any vector field X. On the other hand, replacing Y by ξ in Codazzi equation (2.6) and using (2.4) and $A\xi = \alpha\xi$ we obtain

$$(\nabla_X A)\xi - (\nabla_\xi A)X = \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu\xi - \eta_\nu(\xi)\phi_\nu X - 3\eta_\nu(\phi X)\xi_\nu\} - \phi X$$

for any vector field X. The addition of the above equation to the previous one implies

$$\begin{aligned}2(\nabla_X A)\xi = &\sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu\xi - \eta_\nu(\xi)\phi_\nu X - 3\eta_\nu(\phi X)\xi_\nu\} \\ &- \phi X + (k+\alpha)\phi AX - kA\phi X - A\phi AX\end{aligned}$$

for any vector field X. Applying $A\xi = \alpha\xi$ and (2.5) in the left hand side of the previous equation we obtain

$$A\phi AX - \phi X + (k-\alpha)\phi AX - kA\phi X$$

$$-2X(\alpha)\xi + \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu\xi - \eta_\nu(\xi)\phi_\nu X - 3\eta_\nu(\phi X)\xi_\nu\} = 0 \qquad (3.5)$$

for any vector field X.

Lemma 3.3. *If a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ has Killing shape operator with respect to the generalized Tanaka–Webster connection, then either $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^\perp$.*

Proof. Without loss of generality, we assume $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ with X_0 is a unit vector field orthogonal to $\mathfrak{D}^\perp$. When either $\eta(X_0) = 0$ or $\eta(\xi_1) = 0$, Lemma 3.3 is necessarily true. In what follows, we shall suppose $\eta(X_0)\eta(\xi_1) \neq 0$. Now, applying Lemma 3.1, because the $\mathfrak{D}$ and $\mathfrak{D}^\perp$-components of the Reeb vector field are invariant under the shape operator, the action of A on $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ implies $AX_0 = \alpha X_0$. Using this and the fact $\phi X_0 \in \mathfrak{D}$, replacing X by X_0 in (3.4) we obtain $\alpha A\phi X_0 = (\alpha^2 + 4\eta^2(X_0))\phi X_0$. When $\alpha = 0$, Lemma 3.3 follows directly from (3.3). Therefore, we need to consider only such a case that $\alpha \neq 0$ holds on certain open subset. Now we write

$$AX_0 = \alpha X_0, \quad A\phi X_0 = \rho\phi X_0, \qquad (3.6)$$

where for simplicity we set $\rho = \alpha + \frac{4}{\alpha}\eta^2(X_0)$. Taking into account (3.6), with the help of (2.2), and by a direct calculation we obtain $A\phi A\phi X_0 = A\phi(\rho\phi X_0) = \rho A\phi^2 X_0 = -\alpha\rho(X_0 - \eta(X_0)\xi)$ and $\phi A\phi X_0 = \phi(\rho\phi X_0) = -\rho(X_0 - \eta(X_0)\xi)$.

According to the assumption $\xi = \eta(\xi_1)\xi_1 + \eta(X_0)X_0$, we also have $\eta^2(\xi_1) + \eta^2(X_0) = 1$. Moreover, with the aid of (3.2), from (3.3) we get

$$-2\phi X_0(\alpha)\xi = 8\eta^2(\xi_1)\eta(X_0)\xi = 8\eta^3(\xi_1)\eta(X_0)\xi_1 + 8\eta^2(\xi_1)\eta^2(X_0)X_0,$$

where we have employed the fact $\eta^2(\xi_1) + \eta^2(X_0) = 1$ and $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$.

By a direct calculation, from (3.6) we also have

$$k\phi A\phi X_0 = -k\beta\eta^2(\xi_1)X_0 + k\beta\eta(X_0)\eta(\xi_1)\xi_1$$

and

$$-kA\phi^2 X_0 = k\alpha\eta^2(\xi_1)X_0 - k\alpha\eta(X_0)\eta(\xi_1)\xi_1,$$

where we have applied again $\eta^2(\xi_1) + \eta^2(X_0) = 1$ and $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$.

Finally, by a direct calculation, we obtain

$$\sum_{\nu=1}^{3}\{\eta_\nu(\phi X_0)\phi_\nu\xi - \eta_\nu(\xi)\phi_\nu\phi X_0 - 3\eta_\nu(\phi^2 X_0)\xi_\nu\} = -\eta^2(\xi_1)X_0 - 3\eta(X_0)\eta(\xi_1)\xi_1,$$

where we have used $\eta(X_0)\phi X_0 + \eta(\xi_1)\phi_1\xi = 0$ and the fact $\phi X_0 \in \mathfrak{D}$. Replacing X by ϕX_0 in (3.5) we obtain an equation, and taking the inner product of the resulting equation with ξ_1 and X_0, respectively, we obtain

$$8\eta^2(\xi_1) + k(\rho - \alpha) - 4 = 0 \qquad (3.7)$$

and

$$8\eta^2(X_0) + k(\alpha - \rho) = 0, \tag{3.8}$$

respectively, where we have applied again the assumption $\eta(X_0)\eta(\xi_1) \neq 0$ and $\eta^2(\xi_1) + \eta^2(X_0) = 1$. Applying the previous relation again, the addition of (3.8) to (3.7) gives $4 = 0$, i.e., a contradiction. □

In view of Lemma 3.3, next we consider the case $\xi \in \mathfrak{D}^\perp$. In this case, it follows that $JN \in \mathfrak{J}N$. Without loss of generality, we assume that J_1 is the almost Hermitian structure of $\mathfrak{J}$ such that $JN = J_1N$. Then, it follows from (2.3) that

$$\xi = \xi_1, \quad \phi\xi_2 = -\xi_3, \quad \phi\xi_3 = \xi_2, \quad \phi\mathfrak{D} \subset \mathfrak{D}. \tag{3.9}$$

According to the above relations, with the aid of (3.2), (3.3) becomes

$$X(\alpha) = \xi(\alpha)\xi - 4\eta_1(\phi X) = 0$$

for any vector field X, i.e., α is a constant. Substituting the above equation in (3.5) we get

$$\begin{aligned} &A\phi AX - \phi X + (k-\alpha)\phi AX - kA\phi X \\ &\qquad + \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu\xi - \eta_\nu(\xi)\phi_\nu X - 3\eta_\nu(\phi X)\xi_\nu\} = 0 \end{aligned}$$

for any vector field X. Making use of (3.9) in the above equation we obtain

$$\begin{aligned} &A\phi AX - \phi X + (k-\alpha)\phi AX \\ &\qquad - kA\phi X - \phi_1 X + 2\eta_2(X)\xi_3 - 2\eta_3(X)\xi_2 = 0 \end{aligned} \tag{3.10}$$

for any vector field X. Similarly, using (3.9) in (3.4) we obtain

$$A\phi AX = \frac{\alpha}{2}A\phi X + \frac{\alpha}{2}\phi AX + \phi X + 2\eta_3(X)\xi_2 - 2\eta_2(X)\xi_3 + \phi_1 X$$

for any vector field X, which is subtracted from (3.10) implying that

$$(2k - \alpha)(\phi AX - A\phi X) = 0$$

for any vector field X. If we suppose $\alpha \neq 2k$, it follows that A commutes with ϕ and hence we have

Lemma 3.4. *If a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ has Killing shape operator with respect to the generalized Tanaka–Webster connection and $\xi \in \mathfrak{D}^\perp$, $\alpha \neq 2k$, then $A\phi = \phi A$.*

Before stating proof of our main results, we also need Berndt and Suh's

Lemma 3.5 ([3]). *Let M be a connected orientable real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$. Then A commutes with ϕ if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

Proposition 3.6 ([2]). *Let M be a connected Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with $A\mathfrak{D} \subset \mathfrak{D}$ and ξ is tangent to $\mathfrak{D}^\perp$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1N$. Then M has three $\left(\text{if } r = \frac{\pi}{2\sqrt{8}}\right)$ or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot\left(\sqrt{8}r\right), \quad \beta = \sqrt{2}\cot\left(\sqrt{2}r\right), \quad \lambda = -\sqrt{2}\tan\left(\sqrt{2}r\right), \quad \mu = 0,$$

with some $r \in \left(0, \frac{\pi}{\sqrt{8}}\right)$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 2, \quad m(\lambda) = m(\mu) = 2m - 2$$

and the corresponding eigenspaces are

$$\begin{aligned}
&T_\alpha = \mathbb{R}\xi = \mathbb{R}\xi_1 = \mathbb{R}JN = \operatorname{Span}\{\xi\} = \operatorname{Span}\{\xi_1\},\\
&T_\beta = \mathbb{C}^\perp\xi = \mathbb{C}^\perp N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3 = \operatorname{Span}\{\xi_2, \xi_3\},\\
&T_\lambda = \{X : X \perp \mathbb{H}\xi, JX = J_1X\},\\
&T_\mu = \{X : X \perp \mathbb{H}\xi, JX = -J_1X\},
\end{aligned}$$

where $\mathbb{R}\xi$, $\mathbb{C}\xi$ and $\mathbb{H}\xi$ denote the real, complex and quaternionic span of the Reeb vector field ξ, respectively, and $\mathbb{C}^\perp\xi$ denotes the orthogonal complement of $\mathbb{C}\xi$ in $\mathbb{H}\xi$.

Proposition 3.7 ([2]). *Let M be a connected Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with $A\mathfrak{D} \subset \mathfrak{D}$ and $\xi \in \mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r), \quad \beta = 2\cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r),$$

with some $r \in (0, \frac{\pi}{4})$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = m(\gamma) = 3, \quad m(\lambda) = m(\mu) = 4n - 4$$

and the corresponding eigenspaces are

$$\begin{aligned}
&T_\alpha = \mathbb{R}\xi = \operatorname{Span}\{\xi\}\\
&T_\beta = \mathfrak{J}J\xi = \operatorname{Span}\{\xi_1, \xi_2, \xi_3\},\\
&T_\gamma = \mathfrak{J}\xi = \operatorname{Span}\{\phi_1\xi, \phi_2\xi, \phi_3\xi\},\\
&T_\lambda, \quad T_\mu,
\end{aligned}$$

where $T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp$, $\mathfrak{J}T_\lambda = T_\lambda$, $\mathfrak{J}T_\mu = T_\mu$, $JT_\lambda = T_\mu$.

One of our main results in this paper is given as follows.

Theorem 3.8. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$. Then, the shape operator satisfies*

$$\left(\widehat{\nabla}^{(k)}_X A\right)\xi + \left(\widehat{\nabla}^{(k)}_\xi A\right)X = 0 \tag{3.11}$$

for any vector fields X with $\alpha \neq 2k$ if and only if M is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

Proof. According to all the above statements, we observe that Lemmas 3.1, 3.3 and 3.4 are true if (3.11) holds. Therefore, according to Lemmas 3.1 and 3.3, on a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$, $\alpha \neq 2k$, satisfying (3.11), either $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^\perp$. It has been proved that a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ is of type (B) when $\xi \in \mathfrak{D}$ (see [11]). Also, according to Lemmas 3.4 and 3.5 we observe that the hypersurface is of type (A) when $\xi \in \mathfrak{D}^\perp$.

First, suppose that the hypersurface is of type (A). It has been proved in [13, Remark 2] that a Hopf hypersurface of type (A) in $G_2(\mathbb{C}^{m+2})$ has Reeb parallel shape operator with respect to the generalized Tanaka–Webster connection. In this case, (3.11) becomes

$$\left(\widehat{\nabla}^{(k)}_X A\right)\xi = 0 \tag{3.12}$$

for any vector fields X. According to (3.1) and (2.4), with the aid of $\xi \in \mathfrak{D}^\perp$, (3.12) reduces to an identity. This implies that (3.11) is true if and only if M is of type (A), and it is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

Second, suppose that the hypersurface is of type (B). Next we shall show that this case cannot occur. Notice that α is a constant due to Proposition 3.7. If (3.11) holds, we see that (3.5) becomes

$$\begin{aligned} &A\phi AX - \phi X + (k-\alpha)\phi AX - kA\phi X \\ &\qquad + \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu\xi - \eta_\nu(\xi)\phi_\nu X - 3\eta_\nu(\phi X)\xi_\nu\} = 0 \end{aligned} \tag{3.13}$$

for any vector field X. Applying Proposition 3.7, replacing X by ξ_1 in (3.13) we obtain $\beta(k-\alpha)\phi\xi_1 = 0$. Because $\beta = 2\cot(2r)$ satisfying $r \in (0, \frac{\pi}{4})$ cannot be zero, it follows that $k = \alpha$, which is used in (3.13) giving

$$A\phi AX - \phi X - kA\phi X + \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu\xi - \eta_\nu(\xi)\phi_\nu X - 3\eta_\nu(\phi X)\xi_\nu\} = 0 \tag{3.14}$$

for any vector field X. Let X_λ be an eigenvector field of the shape operator with eigenvalue λ. With the aid of $k = \alpha$, replacing X by X_λ in (3.14) and applying Proposition 3.7 again, we obtain

$$\lambda\mu - 1 - \alpha\mu = 0.$$

Consequently, substituting $\alpha = -2\tan(2r)$, $\lambda = \cot(r)$ and $\mu = -\tan(r)$ into the above equation gives $1 + \tan^2(r) = 0$; a contradiction. □

With regard to the assumption $\alpha \neq 2k$ employed in Theorem 3.8, we give the following explanation.

Remark 3.9. Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ such that (3.11) is true. If $\xi \in \mathfrak{D}$, as mentioned before in proof of Theorem 3.8, M is of type (B) (see Main Theorem in [11]). If $\xi \in \mathfrak{D}^\perp$ and $\alpha = 2k$, as pointed out in Proposition 1 of [13], the shape operator is necessarily parallel along the Reeb vector field with respect to the generalized Tanaka–Webster connection. On the other hand, under the assumption $\alpha = 2k$, from (3.1) and the Hopf condition $A\xi = \alpha\xi$, with the help of the second term of (2.5), it is easy to check that $\left(\widehat{\nabla}^{(k)}_X A\right)\xi$ vanishes identically for any vector field X. This means that under the assumption $\alpha = 2k$, (3.11) is meaningless for $\xi \in \mathfrak{D}^{\perp}$. Because (3.11) is critical for proofs of our main theorems, then we need the assumption $\alpha \neq 2k$. The same situation occur in many literature (see for example [10, 14, 15, 17, 18]).

Finally, as an corollary of Theorem 3.8, now we give proof of Theorem 1.3.

Proof of Theorem 1.3. If the shape operator is Killing with respect to the generalized Tanaka–Webster connection, then (3.11) is necessarily true. Therefore, the proof of Theorem 1.3 for the case of $\xi \in \mathfrak{D}$ follows immediately from Theorem 3.8. Therefore, next we need only to consider the case of $\xi \in \mathfrak{D}^{\perp}$ and in this case by Theorem 3.8 the hypersurface is of type (A).

Suppose that shape operator is Killing with respect to the generalized Tanaka–Webster connection, with the aid of $A\xi = \alpha\xi$, from (3.1) and (1.2) we obtain

$$\begin{aligned}(\nabla_X A)Y + (\nabla_Y A)X - \alpha\eta(Y)\phi AX - \alpha\eta(X)\phi AY \\ - k\eta(X)\phi AY - k\eta(Y)\phi AX - \alpha g(\phi AX, Y)\xi - \alpha g(\phi AY, X)\xi \\ + \eta(Y)A\phi AX + \eta(X)A\phi AY + k\eta(X)A\phi Y + k\eta(Y)A\phi X = 0\end{aligned} \tag{3.15}$$

for any vector fields X, Y. As seen in Lemma 3.5, on a Hopf hypersurface of type (A), the shape operator commutes with ϕ. Applying this in (3.15) we obtain

$$\begin{aligned}(\nabla_X A)Y + (\nabla_Y A)X - \alpha\eta(Y)\phi AX - \alpha\eta(X)\phi AY \\ + \eta(Y)A\phi AX + \eta(X)A\phi AY = 0\end{aligned} \tag{3.16}$$

for any vector fields X, Y. Notice that the shape operator is Reeb parallel with respect to the Levi-Civita connection for any type (A) hypersurface in $G_2(\mathbb{C}^{m+2})$ (see [19, Remark 4.5]). Applying this, we observe that (3.16) is necessarily true if either X or Y was replaced by ξ because of (2.5) and $A\xi = \alpha\xi$. According to this and (3.16), if the shape operator is Killing with respect to the generalized Tanaka–Webster connection, then it is also Killing with respect to the Levi-Civita connection. However, as proved in [12, Theorem 1], this is impossible. □

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Гіперповерхні Гопфа в комплексних двоповерхневих ґрассманіанах з кілінговим оператором другої квадратичної форми відносно узагальненої зв'язності Танаки–Вебстера

Yaning Wang

У цій роботі ми доводимо, що не існує гіперповерхонь Гопфа в комплексних ґрассманіанах $G_2(\mathbb{C}^{m+2})$ з кілінговим оператором другої квадратичної форми відносно узагальненої зв'язності Танаки–Вебстера.

Ключові слова: гіперповерхня Гопфа, комплексний ґрассманіан, кілінговий оператор другої квадратичної форми, узагальнена зв'язність Танаки–Вебстера